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TWO-STAGE AND THREE-STAGE LEAST SQUARES ESTIMATION OF DISPERSION MATRIX OF DISTURBANCES IN SIMULTANEOUS EQUATIONS

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1. Introduction

Estimators of the variances and covariances of disturbances in a complete system of simultaneous linear stochastic equations are constructed with the help of residuals obtained from the estimated structural equations. Employing the k-class method of estimation, Nagar [6] considered a consistent estimator of variance and evaluated the bias to order $O_p(T^{-1})$, T being the number of observations[1]. Later, Srivastava [12] obtained the expression for the mean squared error to the same order of approximation. A similar estimator for the covariances of disturbances is presented in this paper and the bias and mean squared error, both to order $O_p(T^{-1})$, are analyzed for two-stage and three-stage least squares methods. For comparing the relative sizes of covariances, estimator for the correlation coefficient between disturbances of two equations is also proposed and its properties are investigated. Section 2 describes the system of equations and the assumptions underlying. The estimators are also presented. Section 3 introduces some notations and presents the results which are derived in Section 4.

2. Description of the system and the estimators

Consider a complete system of M linear structural equations in M jointly dependent and Λ exogenous variables:

$$YΓ+X\mathfrak{B}=U \tag{2.1}$$

where Y is a $T\times M$ matrix of observations on M jointly dependent

[1] If the sampling error of the estimator can be expressed as the sum of two components such that the first component is atmost of order $O_p(T^{-1})$ and the other is of order $O_p(T^{-\alpha})$ with $\alpha>1$, then the bias to order $O_p(T^{-1})$ refers to the expected value of the first component, neglecting terms of higher order of smallness than $O(T^{-1})$; see, e.g., Srivastava [14].

variables, X is a $T\times\Lambda$ matrix, assumed to be of full column rank, of observations on Λ exogenous variables, Γ and $\mathfrak{B}$ are matrices of coefficients associated with them and U is a $T\times M$ matrix of unobserved structural disturbances assumed to be temporally independent and normally distributed with mean zero and dispersion matrix

$$\frac{1}{T}\mathrm{E}(U'U)=\Sigma=\begin{bmatrix}\sigma_{11} & \cdots & \sigma_{1M}\\ \vdots & & \vdots\\ \sigma_{M1} & \cdots & \sigma_{MM}\end{bmatrix}. \tag{2.2}$$

The system of equation (2.1) is assumed not to contain identities, and its reduced-form is given by

$$Y=X\Pi+\bar{V} \tag{2.3}$$

where

$$\Pi=-\mathfrak{B}\Gamma^{-1} \quad \text{and} \quad \bar{V}=U\Gamma^{-1}. \tag{2.4}$$

Further, it is assumed that all the equations are identifiable through apriori information consisting of zero-one type restrictions on the elements of Γ and $\mathfrak{B}$. These restrictions when incorporated in (2.1) yields the following set of equations:

$$y_i=Y_i\gamma_i+X_i\beta_i+u_i$$

or

$$y_i=(Y_iX_i)\begin{pmatrix}\gamma_i\\ \beta_i\end{pmatrix}+u_i$$

or

$$y_i=A_i\delta_i+u_i \qquad (i=1, 2,\cdots, M) \tag{2.5}$$

where y_i is a column vector of T observations on the jointly dependent variable to be explained in the ith equation, Y_i and X_i are $T\times m_i$ and $T\times l_i$ matrices of observations on m_i $(<M)$ explanatory jointly dependent and l_i $(\leqq\Lambda)$ explanatory exogenous variables and u_i is the ith column of U. The coefficient vectors γ_i and β_i are obtained from the ith column of Γ and $\mathfrak{B}$ respectively.

If for any matrix D of full column rank we write

$$P_D=D(D'D)^{-1}D' \tag{2.6}$$

then the two-stage least squares estimator of δ_i can be expressed as

$$\hat{\delta}_{2SLS(i)}=(A_i'P_xA_i)^{-1}A_i'P_xy_i\,. \tag{2.7}$$

The disturbance vector u_i is then estimated by

$$\hat{u}_{2SLS(i)} = y_i - A_i \hat{\delta}_{2SLS(i)} \; . \tag{2.8}$$

A consistent estimator of σ_{ij}, based on two-stage least squares method, is

$$s_{ij} = \frac{\hat{u}'_{2SLS(i)} \hat{u}_{2SLS(j)}}{T} \; . \tag{2.9}$$

In order to compare the relative sizes of covariances, one may use the following estimator for ρ_{ij}, the coefficient of correlation between the disturbances of ith and jth equation,

$$r_{ij} = \frac{s_{ij}}{(s_{ii} s_{jj})^{1/2}} \; . \tag{2.10}$$

If we write all the equations (2.5) compactly

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_M \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_M \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_M \end{bmatrix}$$

or

$$y = A\delta + u \; , \tag{2.11}$$

the three-stage least squares estimator of δ is given by

$$\hat{\delta}_{3SLS} = [A'(S^{-1} \otimes I)(I \otimes P_X)A]^{-1} A'(S^{-1} \otimes I)(I \otimes P_X)y \tag{2.12}$$

where

$$S = \begin{bmatrix} s_{11} & \cdots & s_{1M} \\ \vdots & & \vdots \\ s_{M1} & \cdots & s_{MM} \end{bmatrix} \tag{2.13}$$

and $\otimes$ denotes the Kronecker product (Zellner and Theil [15]).

The disturbance vector u is estimated by

$$\hat{u}_{3SLS} = y - A\hat{\delta}_{3SLS} \tag{2.14}$$

and a consistent estimator of σ_{ij}, based on three-stage least squares method, is given by

$$\hat{\sigma}_{ij} = \frac{\hat{u}'_{3SLS(i)} \hat{u}_{3SLS(j)}}{T} \tag{2.15}$$

where $\hat{u}_{3SLS(i)}$ and $\hat{u}_{3SLS(j)}$ are the ith and jth subvectors of $\hat{u}_{3SLS}$ respectively.

The three-stage least squares estimator of ρ_{ij} is

$$\hat{\rho}_{ij}=\frac{\hat{\sigma}_{ij}}{(\hat{\sigma}_{ii}\hat{\sigma}_{jj})^{1/2}} . \tag{2.16}$$

3. Results

From (2.3) the reduced-form corresponding to the explanatory jointly dependent variables in (2.5) can be written as

$$Y_i=X\Pi_i+\bar{V}_i \tag{3.1}$$

where Π_i and $\bar{V}_i$ are submatrices of Π and $\bar{V}$ respectively.

Further, using (2.4) we can express

$$(\bar{V}_i \quad 0)=UG_i \tag{3.2}$$

where G_i is a $T\times n_i$ $(=l_i+m_i)$ matrix of constants.

Thus, we can write

$$A_i=Z_i+UG_i \tag{3.3}$$

where

$$Z_i=(X\Pi_i \quad X_i) . \tag{3.4}$$

If we denote

$$Z=\begin{bmatrix} Z_1 & 0 & \cdots & 0 \\ 0 & Z_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & Z_M \end{bmatrix} \tag{3.5}$$

and

$$G=\begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & G_M \end{bmatrix} \tag{3.6}$$

then

$$A=Z+(I\otimes U)G . \tag{3.7}$$

Now let us define

$$\begin{aligned} \Omega&=[Z'(\Sigma^{-1}\otimes I)Z]^{-1} \\ B&=\Omega Z'(\Sigma^{-1}\otimes I) \\ H&=(\Sigma^{-1}\otimes I)-(\Sigma^{-1}\otimes I)Z\Omega Z'(\Sigma^{-1}\otimes I) . \end{aligned} \tag{3.8}$$

Further, we introduce

$$
\begin{aligned}
&Q_i=(Z_i'Z_i)^{-1} \\
(3.9)\qquad &\sigma_{(i)}=\mathrm{Col}\,(\sigma_{1i}\ \sigma_{2i}\cdots\sigma_{Mi}) \\
&d_{ij}=G_i'\sigma_{(j)} \qquad (i,\ j=1,\ 2,\cdots,\ M)\ .
\end{aligned}
$$

THEOREM 1. *The bias and mean squared error, to order $O_p(T^{-1})$, of the two-stage least squares estimator s_{ij} defined by* (2.9) *are given by*[2]

$$
\begin{aligned}
(3.10)\qquad \mathrm{E}\,(s_{ij}-\sigma_{ij})=\frac{1}{T}[(\mathrm{tr}\,Q_iZ_i'Z_jQ_jZ_j'Z_i)-n_i-n_j]\sigma_{ij} \\
-(\Lambda-n_i-1)d_{ii}'Q_id_{ij}-(\Lambda-n_j-1)d_{jj}'Q_jd_{ji} \\
+(\mathrm{tr}\,G_i'\Sigma G_jQ_jZ_j'Z_iQ_i)\sigma_{ij}
\end{aligned}
$$

$$
\begin{aligned}
(3.11)\qquad \mathrm{E}\,(s_{ij}-\sigma_{ij})^2=\frac{1}{T}(\sigma_{ii}\sigma_{jj}+\sigma_{ij}^2)+\sigma_{ii}d_{ij}'Q_id_{ij}+\sigma_{jj}d_{ji}'Q_jd_{ji} \\
+2\sigma_{ij}d_{ji}'Q_jZ_j'Z_iQ_id_{ij}\ .
\end{aligned}
$$

THEOREM 2. *The bias and mean squared error, to order $O_p(T^{-1})$, of the three-stage least squares estimator $\hat{\sigma}_{ij}$ defined by* (2.15) *are*

$$
\begin{aligned}
(3.12)\qquad \mathrm{E}\,(\hat{\sigma}_{ij}-\sigma_{ij})=-\frac{1}{T}(\mathrm{tr}\,Z_j'Z_i\Omega_{ij})-d_{ji}'Q_jd_{jj}-d_{ij}'Q_id_{ii} \\
-\sum_{k,l}^{M}(d_{ji}'W_{jkl}+d_{ij}'W_{ikl})d_{kl}+(\mathrm{tr}\,G_i'\Sigma G_i\Omega_{ji})
\end{aligned}
$$

$$
(3.13)\qquad \mathrm{E}\,(\hat{\sigma}_{ij}-\sigma_{ij})^2=\frac{1}{T}(\sigma_{ii}\sigma_{jj}+\sigma_{ij}^2)+d_{ij}'\Omega_{ii}d_{ij}+d_{ji}'\Omega_{jj}d_{ji}+2d_{ji}'\Omega_{ji}d_{ij}
$$

where

$$
(3.14)\qquad W_{\mu kl}=(\mathrm{tr}\,H_{kl}P_x)\Omega_{\mu k}-2B_{\mu k}B_{kl}'+B_{\mu l}(Z_kQ_k-B_{kk}') \qquad (\mu=i,\ j)\ .
$$

Here $\Omega_{\mu k}$ is the (μ, k)th submatrix, of order $n_\mu\times n_k$, of Ω being partitioned as:

$$
(3.15)\qquad \Omega=\begin{bmatrix}\Omega_{11} & \cdots & \Omega_{1M}\\ \vdots & & \vdots \\ \Omega_{M1} & \cdots & \Omega_{MM}\end{bmatrix}.
$$

The matrices H_{kl} and $B_{\mu k}$ are similarly defined with reference to H and B respectively.

From (3.11) and (3.13), the change in mean squared error is

$$
\begin{aligned}
(3.16)\qquad &\mathrm{E}\,(s_{ij}-\sigma_{ij})^2-\mathrm{E}\,(\hat{\sigma}_{ij}-\sigma_{ij})^2 \\
&=d_{ij}'(\sigma_{ii}Q_i-\Omega_{ii})d_{ij}+d_{ji}'(\sigma_{jj}Q_j-\Omega_{jj})d_{ji} \\
&\quad+2d_{ji}'(\sigma_{ij}Q_jZ_j'Z_iQ_i-\Omega_{ji})d_{ij}\ .
\end{aligned}
$$

[2] See footnote 1.

Now let us consider the estimation of variance σ_{ii}. From Theorem 1 and Theorem 2 we find, to the order of our approximation[3],

(3.17) $$\mathrm{E}\,(s_{ii}-\sigma_{ii})=-\frac{n_i}{T}\sigma_{ii}-2(\Lambda-n_i-1)d_{ii}'Q_i d_{ii}+(\mathrm{tr}\,G_i'\Sigma G_i Q_i)\sigma_{ii}$$

(3.18) $$\mathrm{E}\,(s_{ii}-\sigma_{ii})^2=\frac{2}{T}\sigma_{ii}^2+4\sigma_{ii}d_{ii}'Q_i d_{ii}$$

(3.19) $$\mathrm{E}\,(\hat{\sigma}_{ii}-\sigma_{ii})=-\frac{1}{T}(\mathrm{tr}\,Q_i^{-1}\Omega_{ii})-2d_{ii}'Q_i d_{ii} -2\sum_{k,l}^{M} d_{ii}'W_{ikl}d_{kl}+(\mathrm{tr}\,G_i'\Sigma G_i\Omega_{ii})$$

(3.20) $$\mathrm{E}\,(\hat{\sigma}_{ii}-\sigma_{ii})^2=\frac{2}{T}\sigma_{ii}^2+4d_{ii}'\Omega_{ii}d_{ii}\,.$$

The gain in efficiency is

(3.21) $$\mathrm{E}\,(s_{ii}-\sigma_{ii})^2-\mathrm{E}\,(\hat{\sigma}_{ii}-\sigma_{ii})^2=4d_{ii}'(\sigma_{ii}Q_i-\Omega_{ii})d_{ii}$$

which is non-negative, for the matrix

(3.22) $$\sigma_{ii}Q_i-\Omega_{ii}$$

is non-negative definite (see result (V) of Appendix).

We now present the expressions for the bias and mean squared error of the estimators r_{ij} and $\hat{\rho}_{ij}$, defined by (2.10) and (2.16), of ρ_{ij}.

THEOREM 3. *The bias and mean squared error, both to order $O_p(T^{-1})$, of two-stage least squares estimator r_{ij} and three-stage least squares estimator $\hat{\rho}_{ij}$ of ρ_{ij} are*

(3.23) $$\mathrm{E}\,(r_{ij}-\rho_{ij})=-\frac{\rho_{ij}(1-\rho_{ij}^2)}{2T}+\rho_{ij}[C_{2SLS}(i,j)+C_{2SLS}(j,i)]$$

(3.24) $$\mathrm{E}\,(\hat{\rho}_{ij}-\rho_{ij})=-\frac{\rho_{ij}(1-\rho_{ij}^2)}{2T}+\rho_{ij}[C_{3SLS}(i,j)+C_{3SLS}(j,i)]$$

(3.25) $$\mathrm{E}\,(r_{ij}-\rho_{ij})^2=\frac{(1-\rho_{ij}^2)^2}{T}+\rho_{ij}^2[C_{2SLS}^*(i,j)+C_{2SLS}^*(j,i)]$$

(3.26) $$\mathrm{E}\,(\hat{\rho}_{ij}-\rho_{ij})^2=\frac{(1-\rho_{ij}^2)^2}{T}+\rho_{ij}^2[C_{3SLS}^*(i,j)+C_{3SLS}^*(j,i)]$$

where

(3.27) $$C_{2SLS}(i,j)=\frac{1}{2}\Big[(\mathrm{tr}\,G_i'\Sigma G_j Q_j Z_j' Z_i Q_i)-(\mathrm{tr}\,G_i'\Sigma G_i Q_i)$$

---

[3] Cf. Srivastava ([12], results (2.7) and (2.8), p. 439).

$$+\frac{1}{T}\{(\mathrm{tr}\, Q_i Z_i' Z_j Q_j Z_j' Z_i)-n_i\}$$
$$+\frac{2}{\sigma_{ii}}\left\{\left(\Lambda-n_i+\frac{1}{2}\right)d_{ii}'Q_i d_{ii}-d_{ji}'Q_j Z_j' Z_i Q_i d_{ii}\right\}$$
$$\left.-\frac{2(\Lambda-n_i)}{\sigma_{ij}}d_{ii}'Q_i d_{ij}+\frac{\rho_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}}d_{jj}'Q_j Z_j' Z_i Q_i d_{ii}\right]$$

(3.28)
$$C_{3\mathrm{SLS}}(i,j)=\frac{1}{2}\left[\frac{1}{\sigma_{ij}}(\mathrm{tr}\, G_i'\Sigma G_j \Omega_{ji})-\frac{1}{\sigma_{ii}}(\mathrm{tr}\, G_i'\Sigma G_i \Omega_{ii})\right.$$
$$-\frac{1}{T}\left\{\frac{1}{\sigma_{ij}}(\mathrm{tr}\, Z_j' Z_i \Omega_{ij})-\frac{1}{\sigma_{ii}}(\mathrm{tr}\, Q_i^{-1}\Omega_{ii})\right\}$$
$$-2\sum_{k,l}^{M}\left(\frac{1}{\sigma_{ij}}d_{ij}'-\frac{1}{\sigma_{ii}}d_{ii}'\right)W_{ikl}d_{kl}$$
$$-\frac{2}{\sigma_{ii}\sigma_{jj}}d_{ij}'(\sigma_{ii}Q_i+\Omega_{ii})d_{ii}+\frac{1}{\sigma_{ii}^2}d_{ii}'(2\sigma_{ii}Q_i+3\Omega_{ii})d_{ii}$$
$$\left.+\frac{1}{\sigma_{ii}}d_{ii}'\Omega_{ij}\left(\frac{1}{\sigma_{jj}}d_{jj}-\frac{2}{\sigma_{ij}}d_{ji}\right)\right]$$

(3.29)
$$C_{2\mathrm{SLS}}^*(i,j)=\frac{1}{\sigma_{ij}}d_{ij}'Q_i\left(\frac{\sigma_{ii}}{\sigma_{ij}}d_{ij}-2d_{ii}+Z_i' Z_j Q_j d_{ji}\right)$$
$$+\frac{1}{\sigma_{ii}}d_{ii}'Q_i(d_{ii}-2Z_i' Z_j Q_j d_{ji})+\frac{\rho_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}}d_{ii}'Q_i Z_i' Z_j Q_j d_{jj}$$

(3.30)
$$C_{3\mathrm{SLS}}^*(i,j)=\frac{1}{\sigma_{ij}}d_{ij}'\left[\Omega_{ii}\left(\frac{1}{\sigma_{ij}}d_{ij}-\frac{2}{\sigma_{ii}}d_{ii}\right)+\frac{1}{\sigma_{ij}}\Omega_{ij}d_{ji}\right]$$
$$+\frac{1}{\sigma_{ii}}d_{ii}'\left(\frac{1}{\sigma_{ii}}\Omega_{ii}d_{ii}-\frac{2}{\sigma_{ij}}\Omega_{ij}d_{ji}\right)+\frac{1}{\sigma_{ii}\sigma_{jj}}d_{ii}'\Omega_{ij}d_{jj}\ .$$

Here $C_{2\mathrm{SLS}}(j,i)$ is obtained from $C_{2\mathrm{SLS}}(i,j)$ by interchanging the suffixes 'i' and 'j' in the expression. The same is true for other functions.

4. Proof of the theorems

From (2.5), (2.7), (2.8) and (3.3) we have

(4.1)
$$\hat{u}_{2\mathrm{SLS}(i)}=u_i-A_i(\hat{\delta}_{2\mathrm{SLS}(i)}-\delta_i)\ .$$

If we write

(4.2)
$$\eta_{ij}=\frac{u_i' u_j}{T}-\sigma_{ij}$$

then from (A.I.a) (see Appendix) the sampling error, to order $O_p(T^{-1})$, of s_{ij} is

(4.3) $$s_{ij}-\sigma_{ij}=\Delta_{ij}=\Delta_{ij(-1/2)}+\Delta_{ij(-1)}$$

where

(4.4) $$\Delta_{ij(-1/2)}=\eta_{ij}-\frac{1}{T}(u_i'UG_jQ_jZ_j'u_j+u_i'Z_iQ_iG_i'U'u_j)$$

(4.5) $$\begin{aligned}\Delta_{ij(-1)}=\frac{1}{T}[&u_i'P_{z_i}P_{z_j}u_j-u_i'P_{z_i}u_j-u_i'P_{z_j}u_j\\ &-u_i'UG_jQ_jG_j'U'(P_X-P_{z_j})u_j+u_i'UG_jQ_jZ_j'UG_jQ_jZ_j'u_j\\ &-u_i'(P_X-P_{z_i})UG_iQ_iG_i'U'u_j+u_i'Z_iQ_iG_i'U'Z_iQ_iG_i'U'u_j\\ &+u_i'Z_iQ_iG_i'U'UG_jQ_jZ_j'u_j]\ .\end{aligned}$$

Here $\Delta_{ij(-1/2)}$ and $\Delta_{ij(-1)}$ contain terms of order $O_p(T^{-1/2})$ and $O_p(T^{-1})$ respectively.

It may be noticed that η_{ij} is of order $O_p(T^{-1/2})$ with mean zero and variance $(\sigma_{ii}\sigma_{jj}+\sigma_{ij}^2)/T$.

Owing to the normality of disturbances, we have

(4.6) $$\mathrm{E}\,[\Delta_{ij(-1/2)}]=0\ .$$

The expectations of the first three terms on the right-hand side of (4.5) are

(4.7) $$\frac{1}{T}\,\mathrm{E}\,[u_i'P_{z_i}P_{z_j}u_j]=\frac{1}{T}(\mathrm{tr}\,Q_iZ_i'Z_jQ_jZ_j'Z_i)\sigma_{ij}$$

(4.8) $$\frac{1}{T}\,\mathrm{E}\,[u_i'P_{z_i}u_j]=\frac{n_i}{T}\sigma_{ij}$$

(4.9) $$\frac{1}{T}\,\mathrm{E}\,[u_i'P_{z_j}u_j]=\frac{n_j}{T}\sigma_{ij}\ .$$

Setting $C=G_jQ_jG_j'$ and $D=(P_X-P_{z_j})$ in (A.III.a) (see Appendix), the expectation of the fourth term is equal to

(4.10) $$\frac{(\Lambda-n_j)}{T}[(T+1)d_{jj}'Q_jd_{ji}+(\mathrm{tr}\,Q_jG_j'\Sigma G_j)\sigma_{ij}]=(\Lambda-n_j)d_{jj}'Q_jd_{ji}$$

to the order of our approximation.

Similarly, the expectation of fifth term on the right-hand side of (4.5) is, to the same order of approximation, equal to

(4.11) $$d_{jj}'Q_jd_{ji}$$

where we have used the result (A.III.b).

The expectation of the sixth and seventh terms on the right-hand side of (4.5) can be derived from (4.10) and (4.11) by interchanging the suffixes 'i' and 'j':

$$\frac{1}{T}\,\mathrm{E}\,[u_i'(P_X-P_{z_i})UG_iQ_iG_i'U'u_j]=(\Lambda-n_i)d_{ii}'Q_id_{ij} \tag{4.12}$$

$$\frac{1}{T}\,\mathrm{E}\,[u_i'Z_iQ_iG_i'U'Z_iQ_iG_i'U'u_j]=d_{ii}'Q_id_{ij} \tag{4.13}$$

to the same order of approximation.

Finally, the expectation of the last term is, to the order of our approximation,

$$\frac{1}{T}\,\mathrm{E}\,[u_i'Z_iQ_iG_i'U'UG_jQ_jZ_j'u_j]=(\mathrm{tr}\,G_i'\Sigma G_jQ_jZ_j'Z_iQ_i)\sigma_{ij} \tag{4.14}$$

where use has been made of (A.III.c).

Combining (4.7)–(4.14) we get $\mathrm{E}\,[\Delta_{ij(-1)}]$ to order $O(T^{-1})$ which on using (4.3) and (4.6) leads to the expression (3.10) of Theorem 1.

For the mean squared error of s_{ij} we notice that

$$(s_{ij}-\sigma_{ij})^2=\Delta^2_{ij(-1/2)} \tag{4.15}$$

to order $O_p(T^{-1})$.

Now, we have, by virtue of normality of disturbances,

$$\mathrm{E}\,[\eta_{ij}^2]=\frac{1}{T}(\sigma_{ii}\sigma_{jj}+\sigma_{ij}^2) \tag{4.16}$$

$$\mathrm{E}\,[\eta_{ij}\cdot u_i'UG_jQ_jZ_j'u_j]=0 \tag{4.17}$$

$$\mathrm{E}\,[\eta_{ij}\cdot u_i'Z_iQ_iG_i'U'u_j]=0\,. \tag{4.18}$$

Setting $C=D=G_jQ_jZ_j'$ in (A.IV) (see Appendix), it is easy to verify that

$$\begin{aligned}\frac{1}{T^2}\,\mathrm{E}\,[u_i'UG_jQ_jZ_j'u_j]^2&=\frac{1}{T^2}\,\mathrm{E}\,[u_i'UG_jQ_jZ_j'u_j\cdot u_i'UG_jQ_jZ_j'u_j]\\&=\sigma_{jj}d_{ji}'Q_jd_{ji}\end{aligned} \tag{4.19}$$

to order $O(T^{-1})$.

Similarly, to the same order of our approximation we find, using (A.IV),

$$\frac{1}{T^2}\,\mathrm{E}\,[u_i'Z_iQ_iG_i'U'u_j]^2=\frac{1}{T^2}\,\mathrm{E}\,[u_j'UG_iQ_iZ_i'u_i]^2=\sigma_{ii}d_{ij}'Q_id_{ij} \tag{4.20}$$

and

$$\frac{1}{T^2}\,\mathrm{E}\,[u_i'UG_jQ_jZ_j'u_j\cdot u_i'Z_iQ_iG_i'U'u_j] \tag{4.21}$$

$$=\frac{1}{T^2}\,\mathrm{E}\,[u_i'UG_jQ_jZ_j'u_j\cdot u_j'UG_iQ_iZ_i'u_i]$$

$$=\sigma_{ij}d_{ji}'Q_jZ_j'Z_iQ_id_{ij}\ .$$

Utilizing (4.15)-(4.21) we obtain the result (3.11) stated in Theorem 1.

For the estimated correlation coefficient r_{ij} we have

$$r_{ij}=\frac{s_{ij}}{(s_{ii}s_{jj})^{1/2}} \tag{4.22}$$

$$=\frac{\sigma_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}}\left(1+\frac{\Delta_{ij}}{\sigma_{ij}}\right)\left[\left(1+\frac{\Delta_{ii}}{\sigma_{ii}}\right)\left(1+\frac{\Delta_{jj}}{\sigma_{jj}}\right)\right]^{-1/2}$$

$$=\frac{\sigma_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}}\left(1+\frac{\Delta_{ij}}{\sigma_{ij}}\right)\left[1+\frac{\Delta_{ii}}{\sigma_{ii}}+\frac{\Delta_{jj}}{\sigma_{jj}}+\frac{\Delta_{ii}\Delta_{jj}}{\sigma_{ii}\sigma_{jj}}\right]^{-1/2}.$$

Writing

$$\rho_{ij}=\frac{\sigma_{ij}}{(\sigma_{ii}\sigma_{jj})^{1/2}}\ , \tag{4.23}$$

expanding, using (4.3) and retaining terms to order $O_p(T^{-1})$ we find

$$\frac{r_{ij}-\rho_{ij}}{\rho_{ij}}=e_{ij(-1/2)}+e_{ij(-1)} \tag{4.24}$$

where

$$e_{ij(-1/2)}=\frac{\Delta_{ij(-1/2)}}{\sigma_{ij}}-\frac{1}{2}\left(\frac{\Delta_{ii(-1/2)}}{\sigma_{ii}}+\frac{\Delta_{jj(-1/2)}}{\sigma_{jj}}\right) \tag{4.25}$$

$$e_{ij(-1)}=\frac{\Delta_{ij(-1)}}{\sigma_{ij}}-\frac{1}{2}\left(\frac{\Delta_{ii(-1)}}{\sigma_{ii}}+\frac{\Delta_{jj(-1)}}{\sigma_{jj}}\right)+\frac{3}{8}\left(\frac{\Delta^2_{ii(-1/2)}}{\sigma^2_{ii}}+\frac{\Delta^2_{jj(-1/2)}}{\sigma^2_{jj}}\right)$$
$$+\frac{\Delta_{ii(-1/2)}\Delta_{jj(-1/2)}}{4\sigma_{ii}\sigma_{jj}}-\frac{\Delta_{ij(-1/2)}}{2\sigma_{ij}}\left(\frac{\Delta_{ii(-1/2)}}{\sigma_{ii}}+\frac{\Delta_{jj(-1/2)}}{\sigma_{jj}}\right)\ . \tag{4.26}$$

Now, from (4.6) we see that

$$\mathrm{E}\,[e_{ij(-1/2)}]=0\ . \tag{4.27}$$

The expression for the expectation of the first three terms on the right-hand side of (4.26) can be derived from (3.10) and the expectation of next two terms from (3.11). For the expectation of last three terms it is easy to verify that

$$\mathrm{E}\,[\Delta_{ii(-1/2)}\Delta_{jj(-1/2)}]=\mathrm{E}\,[\eta_{ii}\eta_{jj}]+\frac{4}{T^2}\,\mathrm{E}[u_i'UG_iQ_iZ_i'u_i\cdot u_j'UG_jQ_jZ_j'u_j] \tag{4.28}$$

$$=\frac{2}{T}\sigma^2_{ij}+4\sigma_{ij}d_{ii}'Q_iZ_i'Z_jQ_jd_{jj}$$

to order $O(T^{-1})$ where use has been made of (A.II.a) and (A.IV).

Similarly, to the same order approximation,

$$(4.29)\qquad \mathrm{E}\,[\Delta_{ij(-1/2)}\Delta_{ii(-1/2)}]=\frac{2}{T}\sigma_{ii}\sigma_{ij}+2(\sigma_{ij}d'_{ji}Q_jZ'_jZ_iQ_id_{ii}+\sigma_{ii}d'_{ij}Q_id_{ii})$$

$$(4.30)\qquad \mathrm{E}\,[\Delta_{ij(-1/2)}\Delta_{jj(-1/2)}]=\frac{2}{T}\sigma_{jj}\sigma_{ij}+2(\sigma_{ij}d'_{ij}Q_iZ'_iZ_jQ_jd_{jj}+\sigma_{jj}d'_{ji}Q_jd_{jj})\,.$$

Results (4.27)-(4.30) along with (3.10) and (3.11) yield (3.23) of Theorem 3.

Further, we have, to order $O_p(T^{-1})$,

$$(4.31)\qquad \left(\frac{r_{ij}-\rho_{ij}}{\rho_{ij}}\right)^2=e^2_{ij(-1/2)}$$
$$=\frac{\Delta^2_{ij(-1/2)}}{\sigma^2_{ij}}+\frac{1}{4}\left(\frac{\Delta^2_{ii(-1/2)}}{\sigma^2_{ii}}+\frac{\Delta^2_{jj(-1/2)}}{\sigma^2_{jj}}\right)+\frac{\Delta_{ii(-1/2)}\Delta_{jj(-1/2)}}{2\sigma_{ii}\sigma_{jj}}$$
$$-\frac{\Delta_{ij(-1/2)}}{\sigma_{ij}}\left(\frac{\Delta_{ii(-1/2)}}{\sigma_{ii}}+\frac{\Delta_{jj(-1/2)}}{\sigma_{jj}}\right).$$

The expectation of the first three terms can be obtained from (3.11). Using it along with (4.28), (4.29) and (4.30) we get (3.25) of Theorem 3.

In order to derive results for three-stage least squares estimator, let us write

$$(4.32)\qquad \hat{u}_{3\mathrm{SLS}(i)}=D_i\hat{u}_{3\mathrm{SLS}}=D_iu-D_iA(\hat{\delta}_{3\mathrm{SLS}}-\delta)=u_i-D_iA(\hat{\delta}_{3\mathrm{SLS}}-\delta)$$

where D_i is a $T\times MT$ matrix of the form

$$(4.33)\qquad D_i=[0\quad 0\cdots 0\ I\ 0\cdots 0]\,.$$

Here 0 and I are null and identity matrices of order $T\times T$ and I occurs at the ith place.

Using (4.32) and (A.I.b) we have, to order $O_p(T^{-1})$,

$$(4.34)\qquad \hat{\sigma}_{ij}-\sigma_{ij}=\nabla_{ij}=\nabla_{ij(-1/2)}+\nabla_{ij(-1)}$$

where

$$(4.35)\qquad \nabla_{ij(-1/2)}=\eta_{ij}-\frac{1}{T}[u'_iD_j(I\otimes U)G\xi_{-1/2}+\xi'_{-1/2}G'(I\otimes U')D'_iu_j]$$

$$(4.36)\qquad \nabla_{ij(-1)}=\frac{1}{T}[\xi'_{-1/2}Z'D'_iD_jZ\xi_{-1/2}+\xi'_{-1/2}G'(I\otimes U')D'_iD_j(I\otimes U)G\xi_{-1/2}$$
$$-u'_iD_jZ\xi_{-1/2}-\xi'_{-1/2}Z'D'_iu_j-u'_iD_j(I\otimes U)G\xi_{-1}$$
$$-\xi'_{-1}G'(I\otimes U')D'_iu_j]\,.$$

Now, by virtue of normality of disturbances

$$\mathrm{E}\,[V_{ij(-1/2)}]=0\ . \tag{4.37}$$

The expectation of the first term in $V_{ij(-1)}$ is

$$\begin{aligned}\frac{1}{T}\,\mathrm{E}\,[u'B'Z'D_i'D_jZBu]&=\frac{1}{T}\,\mathrm{tr}\,B'Z'D_i'D_jZB\,\mathrm{E}\,[uu']\\&=\frac{1}{T}\,\mathrm{tr}\,Z'D_i'D_jZ\Omega\\&=\frac{1}{T}(\mathrm{tr}\,Z_i'Z_j\Omega_{ji})\end{aligned} \tag{4.38}$$

where we have utilized

$$B(\Sigma\otimes I)B'=\Omega\ . \tag{4.39}$$

Similarly, partitioning B and B' as indicated in (3.15) for Ω, the expectation of the second term on the right-hand side of (4.36) is

$$\begin{aligned}&\frac{1}{T}\,\mathrm{E}\,[u'B'G'(I\otimes U')D_i'D_j(I\otimes U)GBu]\\&=\frac{1}{T}\sum_{k,l}^{M}\mathrm{E}\,[u_l'B_{il}'G_i'U'UG_jB_{jk}u_k]\\&=\sum_{k,l}^{M}\sigma_{kl}(\mathrm{tr}\,B_{il}'G_i'\Sigma G_jB_{jk})\\&=\mathrm{tr}\,G_i'\Sigma G_j\Omega_{ji}\end{aligned} \tag{4.40}$$

to order $O(T^{-1})$ where use has been made of (A.III.c) and (4.39).

Next, consider the expectation of third term:

$$\begin{aligned}\frac{1}{T}\,\mathrm{E}\,[u_i'D_jZBu]&=\frac{1}{T}\sum_{k}^{M}\mathrm{E}\,[u_i'Z_jB_{jk}u_k]\\&=\frac{1}{T}\sum_{k}^{M}\sigma_{ik}(\mathrm{tr}\,Z_jB_{jk})\\&=\frac{1}{T}(\mathrm{tr}\,Z_i'Z_j\Omega_{ji})\ .\end{aligned} \tag{4.41}$$

The expectation of the fourth term can be obtained from (4.41) by interchanging the suffixes 'i' and 'j' so that it is equal to

$$\frac{1}{T}(\mathrm{tr}\,Z_j'Z_i\Omega_{ij})\ . \tag{4.42}$$

Finally, the fifth term can be written as:

$$\begin{aligned}&\frac{1}{T}[u_i'D_j(I\otimes U)G\Omega G'(I\otimes U')H(I\otimes P_X)u\\&\quad-u_i'D_j(I\otimes U)GB(I\otimes U)GBu-u_i'D_j(I\otimes U)GB(\Delta\otimes I)Hu]\end{aligned} \tag{4.43}$$

where

$$\Delta = S - \Sigma \ . \tag{4.44}$$

Partitioning H and Ω as pointed out earlier, we can write the expectation, to order $O(T^{-1})$, of the first term in (4.43) as:

$$\frac{1}{T}\sum_{k,l}^{M} \mathrm{E}\,[u_i' U G_j \Omega_{jk} G_k' U' H_{kl} P_x u_l = \sum_{k,l}^{M} (\mathrm{tr}\, H_{kl} P_x) d_{ji}' \Omega_{jk} d_{kl} \tag{4.45}$$

which follows from the result (A.III.a).

Similarly, using (A.III.b) the expectation of the second term in (4.43) is equal to

$$\begin{aligned} \frac{1}{T}\sum_{k,l}^{M} \mathrm{E}\,[u_i' U G_j B_{jk} U G_k B_{kl} u_l] &= \sum_{k,l}^{M} d_{kl}' B_{kl} B_{jk}' d_{ji} \\ &= \sum_{k,l}^{M} d_{ji}' B_{jk} B_{kl}' d_{kl} \end{aligned} \tag{4.46}$$

to the same order of approximation.

For the expectation of third term in (4.43) we have

$$\begin{aligned} &\frac{1}{T}\sum_{k,l,m}^{M} \mathrm{E}\,[\Delta_{kl} \cdot u_i' U G_j B_{jk} H_{lm} u_m] \\ &= \frac{1}{T}\sum_{k,l,m}^{M} \mathrm{E}\,[\Delta_{kl(-1/2)} \cdot u_i' U G_j B_{jk} H_{lm} u_m] \end{aligned} \tag{4.47}$$

to the order of our approximation.

Substituting the value of $\Delta_{kl(-1/2)}$ from (4.4), dropping the terms with expectation zero and utilizing (A.IV), we find (4.47) equal to

$$-\sum_{k,l,m}^{M} [\sigma_{lm} d_{lk}' Q_l Z_l' H_{lm}' B_{jk}' d_{ji} + \sigma_{km} d_{kl}' Q_k Z_k' H_{lm}' B_{jk}' d_{ji}] \tag{4.48}$$

to the same order.

It is easy to see that

$$\sum_{m}^{M} \sigma_{lm} Q_l Z_l' H_{lm}' = Q_l Z_l' - B_{ll} \tag{4.49}$$

$$\sum_{l,m}^{M} \sigma_{km} d_{kl}' Q_k Z_k' H_{lm}' = d_{kk}' Q_k Z_k' - \sum_{l}^{M} d_{kl}' B_{lk} \ . \tag{4.50}$$

Utilizing (4.49) and (4.50), the expression (4.48) reduces to

$$\begin{aligned} &-\sum_{k,l}^{M} d_{kl}' [(Q_k Z_k' - B_{kk}) B_{jl}' - B_{kl} B_{jk}'] d_{ji} - d_{jj}' Q_j d_{ji} \\ &= -\sum_{k,l}^{M} d_{ji}' [B_{jl} (Z_k Q_k - B_{kk}') - B_{jk} B_{kl}'] d_{kl} - d_{ji}' Q_j d_{jj} \ . \end{aligned} \tag{4.51}$$

From (4.45), (4.46) and (4.51) the expectation of fifth term on the right-hand side of (4.36) is

$$-\sum_{k,l}^{M} d'_{ji}W_{jkl}d_{kl}-d'_{ji}Q_j d_{jj} \tag{4.52}$$

where W_{jkl} is defined by (3.14).

The expectation of last term in $V_{ij(-1)}$ can be obtained from (4.52) by interchanging the suffixes 'i' and 'j'

$$-\sum_{k,l}^{M} d'_{ij}W_{ikl}d_{kl}-d'_{ij}Q_i d_{ii} \,. \tag{4.53}$$

Combining (4.37), (4.38), (4.40), (4.41), (4.42), (4.52) and (4.53) we get the result (3.12) of Theorem 2.

For the mean squared error of $\hat{\sigma}_{ij}$, we observe that

$$\begin{aligned} \mathrm{E}\,(\hat{\sigma}_{ij}-\sigma_{ij})^2 &= \mathrm{E}\,[V^2_{ij(-1/2)}] \\ &= \mathrm{E}\,[\eta^2_{ij}]+\frac{1}{T^2}\,\mathrm{E}\,[u'_i D_j(I\otimes U)G\xi_{-1/2}]^2 \\ &\quad +\frac{1}{T^2}\,\mathrm{E}\,[\xi'_{-1/2}G'(I\otimes U')D'_i u_j]^2 \\ &\quad +\frac{2}{T^2}\,\mathrm{E}\,[u'_i D_j(I\otimes U)G\xi_{-1/2}\cdot\xi'_{-1/2}G'(I\otimes U')D'_i u_j] \end{aligned} \tag{4.54}$$

where terms with expectation zero have been dropped.

Now, using (A.IV) we have

$$\begin{aligned} &\frac{1}{T^2}\,[u'_i D_j(I\otimes U)G\xi_{-1/2}]^2 \\ &= \frac{1}{T^2}\,\mathrm{E}\left[\sum_{k}^{M} u'_i U G_j B_{jk} u_k\right]^2 \\ &= \frac{1}{T^2}\sum_{k,l}^{M}\mathrm{E}\,[u'_i U G_j B_{jk} u_k\cdot u'_i U G_j B_{jl} u_l] \\ &= \sum_{k,l}^{M}\sigma_{kl} d'_{ji} B_{jk} B'_{jl} d_{ji} \\ &= d'_{ji}\Omega_{jj} d_{ji} \end{aligned} \tag{4.55}$$

to the order of our approximation.

Similarly, to the same order, we find

$$\frac{1}{T^2}\,\mathrm{E}\,[\xi'_{-1/2}G'(I\otimes U')D'_i u_j]^2 = d'_{ij}\Omega_{ii} d_{ij} \tag{4.56}$$

$$\frac{1}{T^2}\,\mathrm{E}\,[u'_i D_j(I\otimes U)G\xi_{-1/2}\cdot\xi'_{-1/2}G'(I\otimes U')D'_i u_j] = d'_{ji}\Omega_{ji} d_{ij} \,. \tag{4.57}$$

Substituting (4.16), (4.55), (4.56) and (4.57) in (4.54), the result (3.13) of Theorem 2 is obtained.

For the correlation coefficient $\hat{\rho}_{ij}$ we can proceed in the same manner as in case of r_{ij} so that to order $O_p(T^{-1})$,

$$\frac{\hat{\rho}_{ij}-\rho_{ij}}{\rho_{ij}}=\varepsilon_{ij(-1/2)}+\varepsilon_{ij(-1)} \tag{4.58}$$

where

$$\varepsilon_{ij(-1/2)}=\frac{V_{ij(-1/2)}}{\sigma_{ij}}-\frac{1}{2}\left(\frac{V_{ii(-1/2)}}{\sigma_{ii}}+\frac{V_{jj(-1/2)}}{\sigma_{jj}}\right) \tag{4.59}$$

$$\begin{aligned}\varepsilon_{ij(-1)}=&\frac{V_{ij(-1)}}{\sigma_{ij}}-\frac{1}{2}\left(\frac{V_{ii(-1)}}{\sigma_{ii}}+\frac{V_{jj(-1)}}{\sigma_{jj}}\right)\\&+\frac{3}{8}\left(\frac{V^2_{ii(-1/2)}}{\sigma^2_{ii}}+\frac{V^2_{jj(-1/2)}}{\sigma^2_{jj}}\right)+\frac{V_{ii(-1/2)}V_{jj(-1/2)}}{4\sigma_{ii}\sigma_{jj}}\\&-\frac{V_{ij(-1/2)}}{2\sigma_{ij}}\left(\frac{V_{ii(-1/2)}}{\sigma_{ii}}+\frac{V_{jj(-1/2)}}{\sigma_{jj}}\right).\end{aligned} \tag{4.60}$$

The expressions (3.24) and (3.26) for the bias and mean squared error of $\hat{\rho}_{ij}$ can be derived exactly in the same way as indicated for r_{ij}.

Appendix

I. *To order* $O_p(T^{-1})$ *we have*

$$\hat{\delta}_{2SLS(i)}-\delta_i=\xi_{-1/2(i)}+\xi_{-1(i)} \tag{A.I.a}$$

$$\hat{\delta}_{3SLS}-\delta=\xi_{-1/2}+\xi_{-1} \tag{A.I.b}$$

where

$$\xi_{-1/2(i)}=Q_iZ_i'u_i$$

$$\xi_{-1(i)}=Q_i[G_i'U'(P_X-P_{z_i})u_i-Z_i'UG_iQ_iZ_i'u_i]$$

$$\xi_{-1/2}=Bu$$

$$\xi_{-1}=\Omega G'(I\otimes U')H(I\otimes P_X)u-B(I\otimes U)GBu-B(\Delta\otimes I)Hu\ .$$

Notice that

$$B=\Omega Z'(\Sigma^{-1}\otimes I)$$

$$H=(\Sigma^{-1}\otimes I)-(\Sigma^{-1}\otimes I)Z\Omega Z'(\Sigma^{-1}\otimes I)$$

$$\Delta=S-\Sigma\ .$$

PROOF. For (A.I.a), see, e.g., Srivastava ([12], equation (3.1.11) with $\phi=0$, p. 440) and for (A.I.b), see Roy and Srivastava ([8], equation (4.10) with all ϕ's and θ's equal to zero and, of course, with slight change in notations, p. 502).

II. *If D is a $T\times T$ matrix with nonstochastic elements we have*

$$\text{(A.II.a)} \qquad \mathrm{E}\,[u_i'Du_j\cdot u_g u_h'] = \sigma_{ij}\sigma_{gh}(\operatorname{tr} D)I + \sigma_{ig}\sigma_{jh}D + \sigma_{ih}\sigma_{jg}D'$$

$$\text{(A.II.b)} \qquad \mathrm{E}\,[u_i'u_j\cdot u_k'u_l\cdot u_g u_h'] = (a_0 + a_1 T + a_2 T^2)I$$

where

$$\begin{aligned} a_0 &= \sigma_{jh}(\sigma_{il}\sigma_{kg}+\sigma_{ik}\sigma_{lg}) + \sigma_{ig}(\sigma_{jl}\sigma_{kh}+\sigma_{jk}\sigma_{lh}) + \sigma_{jg}(\sigma_{il}\sigma_{kh}+\sigma_{ik}\sigma_{lh}) \\ &\quad + \sigma_{ih}(\sigma_{jl}\sigma_{kg}+\sigma_{jk}\sigma_{lg}) \\ a_1 &= \sigma_{kl}(\sigma_{ig}\sigma_{jh}+\sigma_{ih}\sigma_{jg}) + \sigma_{ij}(\sigma_{lg}\sigma_{kh}+\sigma_{lh}\sigma_{gk}) + \sigma_{gh}(\sigma_{jl}\sigma_{ik}+\sigma_{jk}\sigma_{il}) \\ a_2 &= \sigma_{ij}\sigma_{kl}\sigma_{gh} \,. \end{aligned}$$

PROOF. For (A.II.a), see Srivastava ([10], result (A.I), p. 49). The result (A.II.b) can be obtained from Srivastava ([10], Proof of (A.II), pp. 492–493].

III. *It is proved that*

$$\text{(A.III.a)} \quad \mathrm{E}\,[u_i'UCU'Du_j] = [T\sigma_{(i)}'C\sigma_{(j)} + \sigma_{ij}(\operatorname{tr}\Sigma C) + \sigma_{(j)}'C\sigma_{(i)}](\operatorname{tr} D)$$

$$\text{(A.III.b)} \quad \mathrm{E}\,[u_i'UCUDu_j] = T\sigma_{(j)}'DC'\sigma_{(i)} + \sigma_{ij}(\operatorname{tr} C'\Sigma D) + \sigma_{(i)}'DC'\sigma_{(j)}$$

$$\text{(A.III.c)} \quad \mathrm{E}\,[u_i'CU'UDu_j] = T\sigma_{ij}(\operatorname{tr} C\Sigma D) + \sigma_{(i)}'DC\sigma_{(j)} + \sigma_{(i)}'C'D'\sigma_{(j)}$$

where C and D are matrices, with nonstochastic elements, of appropriate sizes in each case.

PROOF. For the result (A.III.a), we write

$$\text{(III.1)} \qquad \mathrm{E}\,[u_i'UCU'Du_j] = \operatorname{tr}\mathrm{E}\,[u_j u_i'UCU']D \,.$$

Since $U=(u_1, u_2, \cdots, u_M)$, we have

$$\text{(III.2)} \qquad \mathrm{E}\,[u_j u_i'UCU'] = \sum_{k,l}^{M} \mathrm{E}\,[u_j u_i' u_k C_{kl} u_l'] = \sum_{k,l}^{M} C_{kl}\,\mathrm{E}\,[u_i'u_k\cdot u_j u_l']$$

where C_{kl} is the (k, l)th element of C.

Using (A.II.a), the expression (III.2) is equal to

$$\begin{aligned} \text{(III.3)} \qquad & \sum_{k,l}^{M} C_{kl}[T\sigma_{ik}\sigma_{jl} + \sigma_{ij}\sigma_{kl} + \sigma_{il}\sigma_{jk}]I \\ & = [T\sigma_{(i)}'C\sigma_{(j)} + \sigma_{ij}(\operatorname{tr}\Sigma C) + \sigma_{(j)}'C\sigma_{(i)}]I \,. \end{aligned}$$

Multiplying this result by the matrix D and then taking trace, we get (A.III.a) from (III.1).

Similarly, the results (A.III.b) and (A.III.c) can be derived.

IV. *If C and D are matrices of order $M \times T$ with nonstochastic elements, we show that*

$$\text{(A.IV)} \qquad \mathrm{E}\,[u_i'UCu_j \cdot u_k'UDu_l] = T^2\sigma_{jl}\sigma_{(i)}'CD'\sigma_{(k)}$$

where only the leading term on the right-hand side has been retained.

PROOF. Let us write

$$C = \begin{bmatrix} C_1^* \\ C_2^* \\ \vdots \\ C_M^* \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} D_1^* \\ D_2^* \\ \vdots \\ D_M^* \end{bmatrix}$$

where C_μ^* and D_μ^* ($\mu=1, 2, \cdots, M$) are μth row vector of C and D respectively.

Thus, we have

$$\begin{aligned} u_i'UCu_j \cdot u_k'UDu_l &= \sum_{m,n}^{M} u_i'u_m C_m^* u_j \cdot u_k'u_n D_n^* u_l \\ &= \sum_{m,n}^{M} C_m^*[u_i'u_m \cdot u_k'u_n \cdot u_j u_l']D_n^{*\prime} \end{aligned}$$

the expectation of which is equal to, using (A.II.b) and retaining the leading term,

$$T^2 \sum_{m,n}^{M} \sigma_{im}\sigma_{kn}\sigma_{jl} C_m^* D_n^{*\prime} = T^2\sigma_{jl}\sigma_{(i)}'CD'\sigma_{(k)} .$$

V. *The matrix $(\sigma_{11}Q_1 - \Omega_{11})$ is non-negative definite.*

PROOF. Let us partition and write

$$Z'(\Sigma^{-1} \otimes I)Z = \begin{bmatrix} \sigma^{11}Z_1'Z_1 & F_2 \\ F_2' & F_3 \end{bmatrix}$$

where F_2 and F_3 are appropriate matrices.

The submatrix Ω_{11} of $\Omega = [Z'(\Sigma^{-1} \otimes I)Z]^{-1}$ is then

$$\text{(V.1)} \qquad \begin{aligned} \Omega_{11} &= (\sigma^{11}Z_1'Z_1 - F_2'F_3^{-1}F_2)^{-1} \\ &= \frac{1}{\sigma^{11}} Q_1\left(I - \frac{1}{\sigma^{11}} F_2'F_3^{-1}F_2Q_1\right)^{-1} \qquad [Q_1 = (Z_1'Z_1)^{-1}] \end{aligned}$$

where use has been made of Rao ([7], result (2.7), p. 29).

Similarly, if

$$\Sigma=\begin{bmatrix}\sigma_{11} & f \\ f' & F_4\end{bmatrix}$$

then $(1,1)$th element σ^{11} of Σ^{-1} is

$$\sigma^{11}=\frac{1}{(\sigma_{11}-f'F_4^{-1}f)}=\frac{1}{\sigma_{11}(1-(1/\sigma_{11})f'F_4^{-1}f)}\,. \tag{V.2}$$

Utilizing (V.1) and (V.2) we find

$$\sigma_{11}Q_1-\Omega_{11}=\sigma_{11}Q_1\left[I-\left(1-\frac{1}{\sigma^{11}}f'F_4^{-1}f\right)\left(I-\frac{1}{\sigma^{11}}F_2'F_3^{-1}F_2Q_1\right)^{-1}\right]\,. \tag{V.3}$$

Since Q_1, F_3 and F_4 are positive definite, the matrix $(\sigma_{11}Q_1-\Omega_{11})$ is non-negative definite.

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