Ann. Inst. Statist. Math.
28 (1976), Part A, 403-409

PROPERTIES OF SOME TEST CRITERIA FOR COVARIANCE MATRIX

HISAO NAGAO

(Received Jan. 16, 1976; revised Jul. 28, 1976)

1. Introduction and summary

This paper is concerned with problems of testing hypotheses (i) for the equality of covariance matrix to a given matrix (ii) for sphericity (iii) for independence and (iv) for the equality of covariance matrices. For testing these hypotheses when we have a multivariate normal population, the usual test criteria are the likelihood ratio (=LR) tests. Noting that the LR tests of these hypotheses are asymptotically normal when the alternative hypotheses are true and that these limiting distributions are singular at the null hypotheses since the variances vanish, the author [3] proposed new test statistics which are essentially obtained from these variances by replacing covariance matrices by their unbiased estimates. However the proposed test statistics have the same above limiting property as the LR statistics. Thus we must consider the alternative derivation of the proposed test criteria. The purpose of this paper is therefore to give an alternative derivation. Furthermore the property of the proposed test criteria is investigated.

2. Alternative derivation of the proposed test statistics

Let the $p\times 1$ vectors $X_1, X_2, \cdots, X_N$ be a random sample from a normal distribution with mean vector μ and covariance matrix Σ. The modified LR criterion for testing the hypothesis $H_1: \Sigma=\Sigma_0$ against the alternatives $K_1: \Sigma\neq\Sigma_0$ for some given positive definite matrix Σ_0, is given by

$$(2.1)\qquad \lambda_1=\left(\frac{e}{n}\right)^{np/2}|S\Sigma_0^{-1}|^{n/2}\,\mathrm{etr}\left[-\frac{1}{2}\Sigma_0^{-1}S\right],$$

where $S=\sum_{\alpha=1}^{N}(X_\alpha-\bar{X})(X_\alpha-\bar{X})'$, $\bar{X}=N^{-1}\sum_{\alpha=1}^{N}X_\alpha$ and $n=N-1$. Then we have

$$(2.2)\qquad -2\log\lambda_1=pn\log\frac{n}{e}-n\log|S\Sigma_0^{-1}|+\mathrm{tr}\,\Sigma_0^{-1}S$$

$$= -pn - n \log \left| I + \left(\frac{1}{n} S\Sigma_0^{-1} - I \right) \right| + \operatorname{tr} \Sigma_0^{-1} S .$$

Since $((1/n)S\Sigma_0^{-1} - I) \to 0$ in probability under the null hypothesis H_1, we obtain

$$(2.3) \qquad -2 \log \lambda_1 = -pn - n \operatorname{tr} \left(\frac{1}{n} S\Sigma_0^{-1} - I \right) + \frac{n}{2} \operatorname{tr} \left(\frac{1}{n} S\Sigma_0^{-1} - I \right)^2 + \operatorname{tr} \Sigma_0^{-1} S + \cdots = T_1 + \cdots ,$$

where $T_1 = (n/2) \operatorname{tr} (S\Sigma_0^{-1}/n - I)^2$. This statistic T_1 was proposed in the previous paper [3], in which an asymptotic null distribution was derived. Later the author [5] gave the asymptotic non-null distribution of T_1.

Similarly, using the same notation as above, the modified LR statistic for testing the sphericity hypothesis H_2: $\Sigma = \sigma^2 I$ against the alternatives K_2: $\Sigma \neq \sigma^2 I$, where σ^2 is unspecified, is given by

$$(2.4) \qquad \lambda_2 = |S|^{n/2} (p^{-1} \operatorname{tr} S)^{-np/2} .$$

Then we have

$$(2.5) \qquad -2 \log \lambda_2 = -n \log \left| \frac{S}{p^{-1} \operatorname{tr} S} \right| = -n \log \left| I + \left(\frac{S}{p^{-1} \operatorname{tr} S} - I \right) \right| = T_2 + \cdots ,$$

where $T_2 = (p^2 n/2) \operatorname{tr} (S(\operatorname{tr} S)^{-1} - p^{-1} I)^2$. This statistic T_2 was proposed and an asymptotic null distribution was derived in [3]. Also the author [5] derived the asymptotic expansion of the non-null distribution of T_2.

Next we consider the test of independence. Let the $1 \times p$ vector $X' = (X^{(1)\prime}, X^{(2)\prime}, \cdots, X^{(q)\prime})$, having a normal distribution with mean vector $\mu' = (\mu^{(1)\prime}, \mu^{(2)\prime}, \cdots, \mu^{(q)\prime})$ and covariance matrix $\Sigma = (\Sigma_{ij})$ $(i, j = 1, 2, \cdots, q)$, be partitioned into q sub-vectors with components $p_1, p_2, \cdots, p_q$, respectively. Given a sample $X_1, X_2, \cdots, X_N$ of N observation on X, we wish to test the hypothesis H_3: $\Sigma_{ij} = 0$ $(i \neq j)$ against the alternatives K_3: $\Sigma_{ij} \neq 0$ for at least pair i, j $(i \neq j)$. Then the modified LR criterion is given by

$$(2.6) \qquad \lambda_3 = \frac{|S|^{n/2}}{|S_D|^{n/2}} ,$$

where $n = N - 1$ and $S_D = (S_{ii})$ with a submatrix S_{ii} of S corresponding to Σ_{ii}. Then we have

$$(2.7) \qquad -2 \log \lambda_3 = -n \log |SS_D^{-1}| = -n \log |I + (SS_D^{-1} - I)| = T_3 + \cdots ,$$

where $T_3 = (n/2) \operatorname{tr} (SS_D^{-1} - I)^2$, which was proposed and the asymptotic null distribution of T_2 was derived in [3]. Also the author [4] obtained the asymptotic non-null distribution of T_3.

Finally let the $p\times 1$ vectors $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and covariance matrix Σ_i ($i=1, 2, \cdots, k$). The modified LR statistic for testing the hypothesis H_4: $\Sigma_1=\Sigma_2=\cdots=\Sigma_k$ ($=\Sigma$) against the alternatives K_4: $\Sigma_i\neq\Sigma_j$ for some i, j ($i\neq j$), is given by

$$\lambda_4=\left\{\prod_{\alpha=1}^{k}|S_\alpha/n_\alpha|^{n_\alpha/2}\right\}\Big/|S/n|^{n/2}, \tag{2.8}$$

where $S_\alpha=\sum_{\beta=1}^{N_\alpha}(X_{\alpha\beta}-\bar{X}_\alpha)(X_{\alpha\beta}-\bar{X}_\alpha)'$, $\bar{X}_\alpha=N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha}X_{\alpha\beta}$, $S=\sum_{\alpha=1}^{k}S_\alpha$, $n_\alpha=N_\alpha-1$ and $n=\sum_{\alpha=1}^{k}n_\alpha$. Then we have

$$\begin{aligned} -2\log\lambda_4 &= -\sum_{\alpha=1}^{k}n_\alpha\log|(S_\alpha/n_\alpha)(S/n)^{-1}| \\ &= -\sum_{\alpha=1}^{k}n_\alpha\log|I+\{(S_\alpha/n_\alpha)(S/n)^{-1}-I\}| \\ &= T_4+\cdots, \end{aligned} \tag{2.9}$$

where $T_4=(1/2)\sum_{\alpha=1}^{k}n_\alpha\,\mathrm{tr}\,\{(S_\alpha/n_\alpha)(S/n)^{-1}-I\}^2$. This statistic T_4 was proposed and the asymptotic null distribution was derived in [3]. Also the author [6] derived the asymptotic expansion of the non-null distribution of T_4.

From the above results, we can see the asymptotic equivalence between the LR $-2\log\lambda_i$ and the proposed tests T_i.

3. Property of the proposed test criteria

In this section we show that the test criteria mentioned in Section 2 can be also derived from the viewpoint of testing the hypothesis in the asymptotic model. At first we state two lemmas without proof.

LEMMA 3.1. *Let S be distributed according to the Wishart distribution $W(\Sigma_n, n)$, where $\Sigma_n=\Sigma+n^{-1/2}\theta$ and Σ is a positive definite matrix. Then the $p(p+1)/2$ random variables y_{ij} ($i\leqq j$) of the statistic $Y=(y_{ij})=(\Sigma^{-1/2}S\Sigma^{-1/2}-nI)/\sqrt{2n}$ are stochastically independent as $n\to\infty$. y_{ii} converges in law to $N(\theta^*_{ii}, 1)$ and y_{ij} ($i<j$) converges in law to $N(\theta^*_{ij}, 1/2)$, where $(\theta^*_{ij})=\Sigma^{-1/2}\theta\Sigma^{-1/2}$.*

In particular, when $\theta=0$, the above lemma reduces to one in Anderson ([1], p. 75).

LEMMA 3.2. *If $P_{\delta^2}(x)$ is a noncentral χ^2 distribution with f degrees of freedom and noncentrality parameter δ^2, then the ratio $P_{\delta^2}(x)/P_0(x)$ is* *an increasing function of x for all $\delta^2>0$.*

Now we consider the problem of testing the hypothesis (H_1, K_1). By Lemma 3.1, the statistic $Y=(\Sigma_0^{-1/2}S\Sigma_0^{-1/2}-nI)/\sqrt{2n}$ under the alternative sequence K_n: $\Sigma=\Sigma_0+n^{-1/2}\theta$ is asymptotically normal. That is, let $y'=(y_{11}, y_{22},\cdots, y_{pp}, \sqrt{2}\,y_{12},\cdots, \sqrt{2}\,y_{p-1,p})$, then

$$(3.1)\qquad \begin{aligned} &y\sim N(0, I) && \text{under the hypothesis},\\ &y\sim N(\theta^*, I) && \text{under the alternatives}, \end{aligned}$$

where $\theta^{*\prime}=(\theta^*_{11},\cdots, \theta^*_{pp}, \sqrt{2}\,\theta^*_{12},\cdots, \sqrt{2}\,\theta^*_{p-1,p})$ with $(\theta^*_{ij})=\Sigma_0^{-1/2}\theta\Sigma_0^{-1/2}$. Thus we may regard as the problem of testing the hypothesis whether $\theta^*=0$ in this asymptotic model. Since the problem remains invariant under any $f\times f$ orthogonal matrix where $f=(1/2)p(p+1)$, a maximal invariant is $y'y$. Then $y'y$ has a χ^2 distribution with f degrees of freedom under the null hypothesis and a noncentral χ^2 distribution with f degrees of freedom and noncentrality parameter $\delta^2=(1/2)\theta^{*\prime}\theta^*$ under the alternatives. By Lemma 3.2, the test criterion $y'y=\mathrm{tr}\,Y^2\geqq c$ is a UMP invariant test. Hence summarizing the above result, we have the following theorem:

THEOREM 3.3. *For testing the hypothesis H_1: $\Sigma=\Sigma_0$ against the alternatives K_1: $\Sigma\neq\Sigma_0$ for unknown μ, the following criterion*

$$(3.2)\qquad T_1=\frac{n}{2}\,\mathrm{tr}\,(S\Sigma_0^{-1}/n-I)^2\geqq c$$

is an asymptotically locally UMP invariant test in the above sense.

Next we shall derive the property of the test criterion T_2 for the sphericity hypothesis. Applying Lemma 3.1, the statistic $Y=\sqrt{n/2}(pS/\mathrm{tr}\,S-I)$ is asymptotically normal under the alternative sequence K_n: $\Sigma=\sigma^2I+n^{-1/2}\theta$. That is, put $y'=(y_{11}, y_{22},\cdots, y_{pp}, \sqrt{2}\,y_{12},\cdots, \sqrt{2}\,y_{p-1,p})$, then

$$(3.3)\qquad \begin{aligned} &y\sim N(0, \tilde{\Sigma}) && \text{under the hypothesis},\\ &y\sim N(\theta^*, \tilde{\Sigma}) && \text{under the alternative}, \end{aligned}$$

where $\theta^{*\prime}=(\theta^*_{11}, \theta^*_{22},\cdots, \theta^*_{pp}, \sqrt{2}\,\theta^*_{12},\cdots, \sqrt{2}\,\theta^*_{p-1,p})$ with $(\theta^*_{ij})=\sigma^2(\theta-p^{-1}I\,\mathrm{tr}\,\theta)$ and

$$(3.4)\qquad \tilde{\Sigma}=\begin{pmatrix} I_p-(1/p)G_p & 0\\ 0 & I_{p(p-1)/2}\end{pmatrix}\quad \text{with}\quad G_p=\begin{pmatrix} 1 & 1\cdots 1\\ \cdots & \cdots\\ 1 & 1\cdots 1\end{pmatrix}.$$

Thus a random vector y is a normal distribution degenerating on $(1/2)\cdot$

$p(p+1)-1$ dimensional Euclidean space. Since $\sum_{i=1}^{p} y_{ii}=0$ in probability, the problem reduces to that of testing the hypothesis whether the mean of $y^{*\prime}=(y_{11},\cdots,y_{p-1,p-1},\sqrt{2}\,y_{12},\cdots,\sqrt{2}\,y_{p-1,p})$ is zero. Since the problem remains invariant under matrix of the following type

$$\begin{pmatrix} H_{p-1} & 0 \\ 0 & H_{p(p-1)/2} \end{pmatrix}, \tag{3.5}$$

where H_{p-1} is nonsingular and $H'_{p-1}(I_{p-1}+G_{p-1})H_{p-1}=(I_{p-1}+G_{p-1})$, and $H_{p(p-1)/2}$ is an orthogonal matrix, a maximal invariant is given by

$$z=y^{*\prime}\begin{pmatrix} I_{p-1}+G_{p-1} & 0 \\ 0 & I_{p(p-1)/2} \end{pmatrix}y^* . \tag{3.6}$$

Then z has a χ^2 distribution with $f=(1/2)p(p+1)-1$ degrees of freedom under the null hypothesis and a noncentral χ^2 with f degrees of freedom and noncentrality parameter δ^2 under the alternatives, where

$$\delta^2=\frac{1}{2}\theta^{**\prime}\begin{pmatrix} I_{p-1}+G_{p-1} & 0 \\ 0 & I_{p(p-1)/2} \end{pmatrix}\theta^{**} \tag{3.7}$$

with a subvector θ^{**} of θ^* corresponding to y^*. Using Lemma 3.2, the test criterion

$$\sum_{i=1}^{p-1} y_{ii}^2+\left(\sum_{i=1}^{p-1} y_{ii}\right)^2+2\sum_{i>j}^{p} y_{ij}^2=\sum_{i=1}^{p} y_{ii}^2+2\sum_{i>j}^{p} y_{ij}^2=\operatorname{tr} Y^2\geqq c \tag{3.8}$$

is a UMP invariant test. Hence summarizing the above result, we have the following theorem:

THEOREM 3.4. *For testing the hypothesis H_2 : $\Sigma=\sigma^2 I$ against the alternatives K_2 : $\Sigma\neq\sigma^2 I$ for unknown σ^2, the following test criterion*

$$T_2=\frac{p^2 n}{2}\operatorname{tr}\left(\frac{S}{\operatorname{tr} S}-p^{-1}I\right)^2\geqq c \tag{3.9}$$

is an asymptotically locally UMP invariant test in the above sense.

From a different standpoint, the above criterion T_2 was shown to be the locally best invariant test by John [2] and Sugiura [7].

Finally we consider the problem of testing the hypothesis (H_4, K_4) when $p=1$. Putting $y'=(\sqrt{n/2}(S_1/S-\rho_1),\cdots,\sqrt{n/2}(S_k/S-\rho_k))$, $\rho_\alpha=n_\alpha/n$, $\Sigma_\alpha=\sigma_\alpha^2$ $(\alpha=1,2,\cdots,k)$ and $\Sigma=\sigma^2$, the statistic y is an asymptotically normal under the alternative sequences K_n : $\sigma_\alpha^2=\sigma^2+n^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$. That is,

$$(3.10)\qquad \begin{aligned} &y \sim N(0, \tilde{\Sigma}) && \text{under the hypothesis},\\ &y \sim N(\theta, \tilde{\Sigma}) && \text{under the alternative}, \end{aligned}$$

where $\theta^{*\prime}=(\theta_1^*, \cdots, \theta_k^*)$ with $\theta_\alpha^*=\rho_\alpha(\theta_\alpha-\tilde{\theta})\sigma^2$ and $\tilde{\theta}=\sum_{\alpha=1}^k \rho_\alpha\theta_\alpha$, and $\tilde{\Sigma}=(\sigma_{\alpha\beta})$ with $\sigma_{\alpha\beta}=\rho_\alpha\delta_{\alpha\beta}-\rho_\alpha\rho_\beta$. Since $\sum_{i=1}^k y_i=0$ in probability, the problem reduces to that of testing the hypothesis whether the mean of $y^{*\prime}=(y_1, y_2, \cdots, y_{k-1})$ is zero. Since the problem remains invariant with respect to a group $H=\{H: \det H\neq 0$ and $H'\operatorname{diag}(\rho_1^{-1}, \cdots, \rho_{k-1}^{-1})H+\rho_k^{-1}H'G_{k-1}H=\operatorname{diag}(\rho_1^{-1}, \cdots, \rho_{k-1}^{-1})+\rho_k^{-1}G_{k-1}\}$, a maximal invariant is given by

$$(3.11)\qquad z=y^{*\prime}\{\operatorname{diag}(\rho_1^{-1}, \cdots, \rho_{k-1}^{-1})+\rho_k^{-1}G_{k-1}\}y^* .$$

Then z has a χ^2 distribution with $f=k-1$ degrees of freedom under the null hypothesis and a noncentral χ^2 with f degrees of freedom and noncentrality parameter $\delta^2=(1/2)\theta^{**\prime}\{\operatorname{diag}(\rho_1^{-1}, \cdots, \rho_{k-1}^{-1})+\rho_k^{-1}G_{k-1}\}\theta^{**}$ under the alternatives, where θ^{**} is a subvector of θ^* corresponding to y^*. Thus a test criterion

$$(3.12)\qquad \sum_{\alpha=1}^{k-1}\rho_\alpha^{-1}y_\alpha^2+\rho_k^{-1}\left(\sum_{\alpha=1}^{k-1}y_\alpha\right)^2=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}y_\alpha^2\geqq c$$

is a UMP invariant test. Hence summarizing the above result, we have the following theorem:

THEOREM 3.5. *For testing the hypothesis* $H_4: \sigma_1^2=\cdots=\sigma_k^2$ *against the alternatives* $K_4: \sigma_i^2\neq\sigma_j^2$ *for some* i, j $(i\neq j)$, *the following test criterion*

$$(3.13)\qquad T_4=\frac{1}{2}\sum_{\alpha=1}^{k} n_\alpha\{(S_\alpha/n_\alpha)(S/n)^{-1}-1\}^2\geqq c$$

is an asymptotically locally UMP invariant test in the above sense.

It is appeared that the test criteria T_3 and T_4 have the above property in the general case. But the author fails to prove them.

Acknowledgment

The author wishes to express his gratitude to the referee for his valuable comments and suggestions towards revising the paper.

KUMAMOTO UNIVERSITY

REFERENCES

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*, Wiley, New York.

[2] John, S. (1971). Some optimal multivariate tests, *Biometrika*, **58**, 123-127.

[3] Nagao, H. (1973). On some test criteria for covariance matrix, *Ann. Statist.*, **1**, 700-709.

[4] Nagao, H. (1973). Non-null distributions of two test criteria for independence under local alternatives, *J. Multivariate Anal.*, **3**, 435-444.

[5] Nagao, H. (1974). Asymptotic non-null distributions of certain test criteria for a covariance matrix, *J. Multivariate Anal.*, **4**, 409-418.

[6] Nagao, H. (1974). Asymptotic non-null distributions of two test criteria for equality of covariance matrices under local alternatives, *Ann. Inst. Statist. Math.*, **26**, 395-402.

[7] Sugiura, N. (1972). Locally best invariant test for sphericity and the limiting distributions, *Ann. Math. Statist.*, **43**, 1312-1316.