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ISOTONIC TESTS FOR SPREAD AND TAIL

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Summary

One-sample test problem for 'stochastically more (or less) spread' is defined and a family of tests with isotonic power is given. The problem is closely related to that for 'longer (or shorter) tail' in the reliability theory and the correspondence between them is shown.

To characterize the tests three spread preorders in $\mathcal{R}^n$ and corresponding tail preorders in $\mathcal{R}^n_+$ are introduced. Functions which are 'monotone' in these orders, and subsets which are 'centrifugal' or 'centripetal' with respect to these orders are studied. These notions generalize the Schur convexity.

1. Introduction

Firstly, we define the notion of 'stochastically more (or less) spread' between probability distributions on $\mathcal{R}^1$ and give a family of isotonic one-sample tests for these alternatives (Section 2). The spread order is closely related to the notion of 'longer (or shorter) tail' between probability distributions on $\mathcal{R}^1_+$, and the test problems to those in the reliability theory (Section 3).

The Shorter Tail Problem includes as a special case the test of the exponential hypothesis against the increasing failure rate average, IFRA, and the problem is well studied [11]. Our approach, however, gives deeper insight into the problem and improves previous results by Barlow and Proschan [3], and Marshall, Olkin and Proschan [9] on the subject.

The More Spread Problem is similar to that by Fraser [7], who discussed two-sample problem without much success. Our setup is new so far as the authors know.

To study the characteristics of the isotonic tests we introduce in Section 4 three spread preorders of different strictness in the sample space $\mathcal{R}^n$. The strict preorder is a kind of sample version of the spread order of distribution. The loose one is known as majorization. A function which is monotone with respect to the preorders is an unbiased test statistic and a generalization of the Schur convex function. The monotone function determines as its level sets a layer of 'centripetal or centrifugal' sets.

All the discussions of Section 4 are transferred to $\mathcal{R}_+^n$ in Section 5 with a slight change, and new results on the reliability theory are obtained as mentioned above. The intermediate of three preorders means smaller 'studentized cumulative normalized spacings,' statistics used in a life test.

Section 6 discusses some related topics: spread order which is equilibrium for sign change; two-sample tests; and the Lorenz curve. Section 7 is supplementary and states some elementary facts on order relation used in Sections 4 and 5: an upward (or downward) set as a synonym of our centrifugal (or centripetal) set; mapping which is monotone in order relation; strictness comparison of orders; and the ways to define order in $\mathcal{R}^n$.

Throughout the paper, X and Y are random variables with continuous distribution functions F and G respectively. The generalized inverse of these, F^{-1} and G^{-1} are defined as usual (see [8]). Bold lower case letters are column vectors in $\mathcal{R}^n$ or $\mathcal{R}_+^n$, the positive orthant of $\mathcal{R}^n$. Bold upper case letters are random column vectors in these sets.

2. Test for spread

In this section F and G belong to $\mathcal{F}$, the set of all continuous distribution functions on $\mathcal{R}^1$.

DEFINITION 1 (Spread order relation in $\mathcal{F}$).

$$G \succ F\,(\mathcal{S}) \qquad \text{or} \qquad Y \succ X\,(\mathcal{S})$$

iff $G^{-1}(u)-F^{-1}(u)$ is a nondecreasing function in $0<u<1$. We say 'G (or Y) is stochastically more spread than F (or X)' in this case.

The fundamental property of this definition is shown by the following theorem.

THEOREM 1. *The following conditions on F and G in $\mathcal{F}$ are equivalent.*
(1) $Y \succ X\,(\mathcal{S})$.
(2) *There exists a nondecreasing function h defined on $\mathcal{R}^1$ such that Y has the same distribution as $X+h(X)$.*
(3) *There exists a nonincreasing function k defined on $\mathcal{R}^1$ such that X has the same distribution as $Y+k(Y)$.*

PROOF. Write $G^{-1}(u)-F^{-1}(u)=H(u)$. Putting $u=U$, the uniform random variable, $Y \sim G^{-1}(U)=X+H(F(X))=X+h(X)$. Thus h is nondecreasing iff H is nondecreasing, which shows $(1)\Longleftrightarrow(2)$. $(1)\Longleftrightarrow(3)$ is similarly proved.

Remarks. Fraser [7] introduced Definition 1 in the form $G^{-1}(v)-G^{-1}(u)\geqq F^{-1}(v)-F^{-1}(u)$ for all $0<u<v<1$, and stated a sufficient condition for the two-sample unbiased test against the more spread alternative. A trouble in his theorem will be discussed in Section 6. Doksum [6] said 'G is more tail ordered than F' if $G^{-1}F(x)-x$ is nondecreasing in x, $0<F(x)<1$, and discussed the power of rank tests for stochastically largerness. Justification of Definition 1 and its comparison with other definitions were discussed elsewhere [12].

Strictly speaking $\mathcal{S}$ is a preorder relation in $\mathcal{F}$. It is easy to see that $G\succ F\,(\mathcal{S})$ and $F\succ G\,(\mathcal{S})$ iff $F(x)=G(x-a)$, for all $x\in\mathcal{R}^1$ and for some $a\in\mathcal{R}^1$. It is also easy to prove that $cX\succ X\,(\mathcal{S})$, where $c>1$.

Now we state our first test problem.

Test problem: Given a random sample from $F\in\mathcal{F}$.

H_0: $F(x)=G(x-a)$, $-\infty<x<\infty$, where $G\in\mathcal{F}$ is known but $-\infty<a<\infty$ is unknown.

H_1: $G\succ F\,(\mathcal{S})$ (Less Spread Problem)

H_2: $F\succ G\,(\mathcal{S})$ (More Spread Problem)

As the Spread Problems are invariant for the location change we make use of a maximal invariant statistic. Test functions or test statistics can be functions of $\boldsymbol{V}=(X_{(1)}-\bar{X},\cdots,X_{(n)}-\bar{X})'$, where $X_{(i)}$'s are the order statistics of a random sample $\boldsymbol{X}=(X_1,\cdots,X_n)'$ and $\bar{X}$ is its sample mean. $\boldsymbol{V}$ is a random point in the pointed convex cone $\Gamma_0^n=\{\boldsymbol{x};\ \boldsymbol{x}\in\mathcal{R}^n,\ x_1\leqq x_2\leqq\cdots\leqq x_n,\ \sum x_i=0\}$.

THEOREM 2. *Let ϕ be a test function defined on Γ_0^n. $\phi(\boldsymbol{V})$ is unbiased for the More Spread Problem and $1-\phi(\boldsymbol{V})$ is unbiased for the Less Spread Problem if ϕ is 'nondecreasing' on Γ_0^n, that is,*

$$\phi(\boldsymbol{a}+\boldsymbol{b})\geqq\phi(\boldsymbol{a})\,, \qquad \textit{for all } \boldsymbol{a},\boldsymbol{b}\in\Gamma_0^n\,.$$

PROOF. Suppose $F\succ G\,(\mathcal{S})$. $\boldsymbol{V}(\boldsymbol{X})$ has the same distribution as $\boldsymbol{V}(\boldsymbol{Y})+\boldsymbol{T}$, where $\boldsymbol{V}(\boldsymbol{Y})=(Y_{(1)}-\bar{Y},\cdots,Y_{(n)}-\bar{Y})'$ and $\boldsymbol{T}=(h(Y_{(1)})-\overline{h(Y)},\cdots,h(Y_{(n)})-\overline{h(Y)})'\in\Gamma_0$ since $X_i\sim Y_i+h(Y_i)$, where h is a nondecreasing function, by Theorem 1. $\mathrm{E}\,[\phi(\boldsymbol{V}(\boldsymbol{X}))\,|\,F]=\mathrm{E}\,[\phi(\boldsymbol{V}(\boldsymbol{Y})+\boldsymbol{T})\,|\,G]\geqq\mathrm{E}\,[\phi(\boldsymbol{V}(\boldsymbol{Y}))\,|\,G]$ if ϕ is nondecreasing. It is clear that $1-\phi(\boldsymbol{V})$ is unbiased for the Less Spread Problem iff $\phi(\boldsymbol{V})$ is so for the More Spread Problem.

Remarks. Note that if ϕ is defined by $\phi(\boldsymbol{v})=1$ if $T(\boldsymbol{v})\geqq c$ and 0 otherwise, then $\phi(\boldsymbol{V})$ is unbiased if T is nondecreasing on Γ_0^n. Let $A=$

$\{v;\ v\in\Gamma_0^n,\ \phi(v)=1\}$, the critical region for V. Using the terminology defined in Section 7, $\phi(V)$ is unbiased for the More Spread Problem if A is a 'centrifugal' set, or equivalently $A^c=\Gamma_0^n-A$ is a 'centripetal' set. Properties of the unbiased tests are studied further in Section 4.

The test has actually isotonic power for the spread semiorder in $\mathcal{F}$. If $F_2\succ F_1\succ G\,(\mathcal{S})$, then $\mathrm{E}[\phi(V)|F_2]\geqq\mathrm{E}[\phi(V)|F_1]\geqq\mathrm{E}[\phi(V)|G]$. ϕ and $1-\phi$ are 'one-sided tests' of opposite directions.

Our result does not depend on the null distribution G, but the level of test depends on G. In other words, given a level of test we have to choose the criterion c for a statistic T according to G.

3. Test for tail

Let $\mathcal{F}_+$ be the set of all continuous distribution functions on $\mathcal{R}^1_+$. In this section we assume F and G to belong to $\mathcal{F}_+$.

DEFINITION 2 (Tail semiorder relationship in $\mathcal{F}_+$).

$$G\succ F\,(\mathcal{T})\qquad\text{or}\qquad Y\succ X\,(\mathcal{T})$$

iff $G^{-1}(u)/F^{-1}(u)$ is a nondecreasing function in $0<u<1$. We say 'G (or Y) has longer tail than F (or X).'

THEOREM 3. *The following conditions on F and G in $\mathcal{F}_+$ are equivalent.*

(1) $Y\succ X\,(\mathcal{T})$.

(2) *There exists a nondecreasing function h defined on $\mathcal{R}^1_+$ such that Y has the same distribution as $X\cdot h(X)$.*

(3) *There exists a nonincreasing function k defined on $\mathcal{R}^1_+$ such that X has the same distribution as $Y\cdot k(Y)$.*

PROOF. The proof is similar to that of Theorem 1.

Remarks. Definition 2 is equivalent to $x^{-1}G^{-1}F(x)$ is nondecreasing in x, $0<F(x)<1$, or $G^{-1}F$ is a starshaped function. If Y is exponential random variable $Y\succ X\,(\mathcal{T})$ means that X has the Increasing Failure Rate Average. See [1], [2] and [5]. By Definition 2 we compare not only the length (or weight) of tails of distributions but also that of their uprising near the origin.

It is shown that $F\succ G\,(\mathcal{T})$ and $G\succ F\,(\mathcal{T})$ iff $F(x)=G(cx)$, $0<x<\infty$, for some $0<c<\infty$.

Test problem: Given a random sample from $F\in\mathcal{F}_+$.

H_{+0}: $F(x)=G(cx)$, $0<x<\infty$, where $G\in\mathcal{F}_+$ is known but $0<c<\infty$ is unknown.

H_{+1}: $G\succ F\,(\mathcal{T})$ (Shorter Tail Problem)

H_{+2}: $F \succ G\,(\mathcal{T})$ (Longer Tail Problem)

As the Tail Problems are scale invariant, we make use of a maximal invariant statistic. Test functions or test statistics can be functions of $\boldsymbol{W}=(X_{(1)}/\sum X_i, \cdots, X_{(n)}/\sum X_i)'$, a random point in $\Delta_0^n=\{\boldsymbol{x};\ \boldsymbol{x}\in \mathcal{R}_+^n,\ 0\leqq x_1\leqq x_2\leqq\cdots\leqq x_n,\ \sum x_i=1\}$.

THEOREM 4. *Let ϕ be a test function defined on Δ_0^n. $\phi(\boldsymbol{W})$ is unbiased for the Longer Tail Problem and $1-\phi(\boldsymbol{W})$ is unbiased for the Shorter Tail Problem if ϕ is 'nondecreasing' on Δ_0^n, that is,*

$$\phi(\boldsymbol{a}\cdot\boldsymbol{b}/\|\boldsymbol{a}\cdot\boldsymbol{b}\|)\geqq\phi(\boldsymbol{a})\ , \qquad \textit{for all } \boldsymbol{a}, \boldsymbol{b}\in\Delta_0^n\ ,$$

where $\boldsymbol{a}\cdot\boldsymbol{b}$ means the componentwise product of vectors and $\|\boldsymbol{a}\|$ is the l_1 norm, $\sum_{i=1}^n a_i$.

PROOF. The proof is similar to that of Theorem 2.

Remarks similar to those at the end of Section 2 can be made here. The discussions in Sections 2 and 3 are quite parallel because of the following fact.

THEOREM 5. $Y\succ X\,(\mathcal{T})$ *iff* $\log Y\succ \log X\,(\mathcal{S})$.

PROOF. Compare condition (2)'s of Theorems 1 and 3.

4. Spread order in $\mathcal{R}^n$

We write $\Gamma^n=\{\boldsymbol{x};\ \boldsymbol{x}\in\mathcal{R}^n,\ \sum_1^n x_i=0\}$, an $n-1$ dimensional subspace in $\mathcal{R}^n$ such that $\Gamma_0^n\subset\Gamma^n$. We introduce three orders in Γ_0^n, $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_\alpha)$, $\alpha=1, 2, 3$. For $\boldsymbol{a}, \boldsymbol{b}\in\Gamma^n$, replacing a_i and b_i in Definition 3 by $a_{(i)}$ and $b_{(i)}$, the ordered components of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we obtain preorders in Γ^n. For $\boldsymbol{a}, \boldsymbol{b}\in\mathcal{R}^n$, replacing a_i and b_i by $a_{(i)}-\bar{a}$ and $b_{(i)}-\bar{b}$, the ordered residuals of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, we obtain preorders in $\mathcal{R}^n$. In all cases the orders or the preorders compare spread (dispersion, variation or whatever named) of components of vectors.

DEFINITION 3. For $\boldsymbol{a}, \boldsymbol{b}\in\Gamma_0^n$, $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_\alpha)$, $\alpha=1, 2, 3$, are defined as follows.

$\mathcal{S}_3$: b_i-a_i is nondecreasing in $i=1,\cdots,n$.

$\mathcal{S}_2$: $(n-i)^{-1}\sum_{i+1}^n b_k-b_i\geqq(n-i)^{-1}\sum_{i+1}^n a_k-a_i$, $i=1,\cdots,n-1$.

$\mathcal{S}_1$: $\sum_i^n b_k\geqq\sum_i^n a_k$, $i=2,\cdots,n$.

Remarks. $(\mathcal{S}_3)$ This is equivalent to $b_{i+1}-b_i\geqq a_{i+1}-a_i$, $i=1,\cdots,$

$n-1$. Notice that $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_3)$ iff there exists $\boldsymbol{c}\in\Gamma_0^n$ such that $\boldsymbol{b}=\boldsymbol{a}+\boldsymbol{c}$, namely $\boldsymbol{b}-\boldsymbol{a}\in\Gamma_0^n$. The condition of Theorem 2 is 'ϕ or T is nondecreasing $(\mathcal{S}_3)$ in Γ_0^n.' $(\mathcal{S}_2)$ This is equivalent to $\sum_1^{i-1} b_k+(n-i+1)b_i\leqq \sum_1^{i-1} a_k+(n-i+1)a_i$, $i=1,\cdots,n-1$. The right-hand side, say, is equal to $\sum_1^{i}\{(n-k+1)(a_k-a_{k-1})\}$, which corresponds to the cumulative normalized spacings in the theory of life test. See the discussion in Section 5. $(\mathcal{S}_1)$ This is equivalent to $\sum_1^i b_k\leqq\sum_1^i a_k$ and said '$\boldsymbol{b}$ majorizes $\boldsymbol{a}$.' The theorem due to Karamata et al. says that $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_1)$ iff there exists a doubly stochastic matrix P such that $\boldsymbol{a}=P\boldsymbol{b}$. See [4]. It is shown that $\boldsymbol{b}$ majorizes $\boldsymbol{a}$ iff $\boldsymbol{b}-\boldsymbol{a}\in\{\boldsymbol{x};(\boldsymbol{x},\boldsymbol{y})\geqq 0,\ \forall\boldsymbol{y}\in\Gamma_0^n\}=(\Gamma_0^n)^+$, since the equality $(\boldsymbol{x},\boldsymbol{y})=\sum_2^n(x_i-x_{i-1})\sum_i^n y_k+x_1\sum_1^n y_k$ implies that $(\boldsymbol{x},\boldsymbol{y})\geqq 0$, $\forall\boldsymbol{x}\in\Gamma_0\Longleftrightarrow\sum_i^n y_k\geqq 0$, $i=2,\cdots,n$, or '$\boldsymbol{y}$ majorizes $\boldsymbol{0}$.' $(\Gamma_0^n)^+$ is the polar cone of Γ_0^n and includes Γ_0^n. In all orders the minimum point is $\boldsymbol{0}$, and $c\boldsymbol{a}\succ\boldsymbol{a}\,(\mathcal{S}_\alpha)$, $c\geqq 1$, for any $\boldsymbol{a}\in\Gamma_0^n$. Strictness of these orders are compared in Theorem 7.

The conditions of $\mathcal{S}_\alpha$, $\alpha=1,2,3$, are expressed in the form $L_i(\boldsymbol{b}-\boldsymbol{a})\geqq\boldsymbol{0}$, where L_i's are $(n-1)\times n$ matrices. Now the discussion in Section 7 on general orders in $\mathcal{R}^n$ is applied to Definition 3.

THEOREM 6. $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_\alpha)$, $\alpha=1,2,3$, *are expressed as* $L_i(\boldsymbol{b}-\boldsymbol{a})\geqq\boldsymbol{0}$, *which are equivalent to* $\boldsymbol{b}-\boldsymbol{a}\in K_\alpha'\mathcal{R}_+^{n-1}$, *where* L_α*'s and* K_α*'s are* $(n-1)\times n$ *matrices defined in the following. A differentiable function* H *defined on* Γ_0^n *is nondecreasing* $(\mathcal{S}_\alpha)$ *iff* $K_\alpha\,\mathrm{grad}\,H(\boldsymbol{x})\geqq\boldsymbol{0}$, $\boldsymbol{x}\in\Gamma_0^n$. *This is also true for a symmetric (invariant for a permutation of components) differentiable function* H *defined on* Γ^n.

$$L_3=\begin{bmatrix} -1 & 1 & 0\cdots & 0 & 0\\ 0 & -1 & 1\cdots & 0 & 0\\ \cdots & & & \cdots & \\ 0 & 0 & 0 & -1 & 1\end{bmatrix},$$

$$K_3=\begin{bmatrix} -(n-1) & 1 & 1 & \cdots & 1\\ -(n-2) & -(n-2) & 2 & \cdots & 2\\ -(n-3) & -(n-3) & -(n-3) & \cdots & 3\\ & \cdots & & & \cdots\\ -1 & -1 & -1 & \cdots & n-1\end{bmatrix},$$

$$L_2=\begin{bmatrix} -(n-1) & 1 & 1\cdots & 1 & 1\\ 0 & -(n-2) & 1\cdots & 1 & 1\\ & \cdots & & \cdots & \\ 0 & 0 & 0\cdots & -1 & 1\end{bmatrix},\qquad K_2=L_2,$$

$$L_1=\begin{bmatrix} 0 & 1 & 1\cdots 1 & 1 \\ 0 & 0 & 1\cdots 1 & 1 \\ \cdots & & \cdots & \\ 0 & 0 & 0\cdots 1 & 1 \\ 0 & 0 & 0\cdots 0 & 1 \end{bmatrix}, \qquad K_1=L_3 .$$

PROOF. If $\boldsymbol{a}, \boldsymbol{b} \in \Gamma_0^n$, then, $\mathbf{1}'(\boldsymbol{b}-\boldsymbol{a})=0$, where $\mathbf{1}$ is the column vector with all components one. For each $\alpha=1, 2, 3$, the matrix L_α with a row $\mathbf{1}'$ added is nonsingular. The first $n-1$ columns of their inverses form K_α' or K_α' postmultiplied by a diagonal matrix with positive diagonal elements. For example,

$$\begin{bmatrix} L_2 \\ \mathbf{1}' \end{bmatrix}^{-1}=[L_2'(L_2L_2')^{-1}, n^{-1}\mathbf{1}] ,$$

$$L_2L_2'=\begin{bmatrix} n(n-1) & & & \\ & (n-1)(n-2) & & \\ & & \cdots & \\ & & & 2\cdot 1 \end{bmatrix}.$$

Considering to premultiply the right-hand side matrix to the inequality

$$\begin{bmatrix} L_2 \\ \mathbf{1}' \end{bmatrix}(\boldsymbol{b}-\boldsymbol{a})\geqq \mathbf{0} ,$$

we can neglect the last column and this inequality is equivalent to

$$\boldsymbol{b}-\boldsymbol{a} \in L_2'(L_2L_2')^{-1}\mathcal{R}_{n-1}^{+}=L_2'\mathcal{R}_{n-1}^{+} .$$

Functions on Γ_0^n of the form $H(\boldsymbol{v})=\sum_{j>i} h_{ji}(v_j-v_i)$, where h_{ji}'s are nondecreasing function on $\mathcal{R}_{+}^1$, are nondecreasing $(\mathcal{S}_3)$. Of these typical ones like $\sum_{j>i}(v_j-v_i)^2=n\sum v_i^2$ and $\sum_{j>i} c_j(v_j-v_i)=v_n-v_1$, where $c_n=1/(n-1)$ and $c_j=1/j(j-1)$, $j=n-1,\cdots, 2$, are nondecreasing $(\mathcal{S}_1)$. The linear function $H(\boldsymbol{v})=\boldsymbol{c}'\boldsymbol{v}$, is nondecreasing $(\mathcal{S}_1)$ iff $c_1\leqq\cdots\leqq c_n$, while nondecreasing $(\mathcal{S}_2$ or $\mathcal{S}_3)$ iff $c_i\leqq(n-i)^{-1}\sum_{i+1}^{n} c_k$, $i=1,\cdots, n-1$, or equivalently

$$n^{-1}\sum_{1}^{n} c_k\leqq(n-1)^{-1}\sum_{2}^{n} c_k\leqq\cdots\leqq(c_{n-1}+c_n)/2\leqq c_n .$$

THEOREM 7.

$$\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_3)\Longrightarrow\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_2)\Longrightarrow\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_1) .$$

Therefore if ϕ, a test function on Γ_0^n, is nondecreasing $(\mathcal{S}_2$ or $\mathcal{S}_1)$, then $\phi(\boldsymbol{V})$ is unbiased for the More Spread Problem and $1-\phi(\boldsymbol{V})$ is so for the Less Spread Problem.

PROOF. $\mathcal{S}_3 \Longrightarrow \mathcal{S}_2$ will be shown later in the proof of Theorem 9. To prove $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$ put

$$M_{12}=\begin{bmatrix} n-1 & & & \\ & n-2 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}\begin{bmatrix} 1 & 1\cdots 1 & 1 \\ 0 & 1\cdots 1 & 1 \\ \cdots & & \cdots \\ 0 & 0\cdots 0 & 1 \end{bmatrix}\begin{bmatrix} n(n-1) & & \\ & (n-1)(n-2) & \\ & & \cdots \\ & & & 2\cdot 1 \end{bmatrix}$$

which is nonnegative. Then,

$$M_{12}L_2+n^{-1}(1, 2, \cdots, n-1)'1'=L_1$$

and

$$M_{12}'K_1=K_2 .$$

Each of these equations shows $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$.

A function H defined on $\mathcal{R}^n$ is called Schur convex if it satisfies

$$(x_j-x_i)\left(\frac{\partial H}{\partial x_j}-\frac{\partial H}{\partial x_i}\right)\geqq 0 .$$

If H satisfies $H(\boldsymbol{x}+s\mathbf{1})=H(\boldsymbol{x})$, $-\infty<s<\infty$, namely if H is essentially defined on Γ^n, then Schur's condition means that H is nonincreasing $(\mathcal{S}_1)$. Thus nondecreasing ($\mathcal{S}_2$ or $\mathcal{S}_3$) functions are further generalizations of the Schur convex function, and $K_\alpha \operatorname{grad} H(\boldsymbol{x})\geqq \mathbf{0}$, $\alpha=2, 3$, are generalizations of Schur's condition. The functions defined on Γ^n or $\mathcal{R}^n$ and nondecreasing $(\mathcal{S}_\alpha)$, $\alpha=1, 2, 3$, are symmetric. All the nondecreasing $(\mathcal{S}_\alpha)$ functions take their minimum value at $\mathbf{0}$.

Let A be a symmetric (with respect to permutation of coordinates) convex set of Γ^n. Then A is centripetal $(\mathcal{S}_1)$ and vice versa. The Birkoff-Neumann theorem states that every stochastic matrix is a convex linear form of permutation matrices. Combining with the theorem by Karamata et al., we see that the smallest centripetal $(\mathcal{S}_1)$ set containing a fixed point $\boldsymbol{b}\in\Gamma^n$, that is $\{\boldsymbol{a};\ \boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{S}_1)\}$ is nothing but the convex hull of $n!$ points which are obtained by permuting the components of $\boldsymbol{b}$, or the smallest symmetric convex set of Γ^n containing $\boldsymbol{b}$. The Γ_0^n part of such a set A is a critical region for $\boldsymbol{V}$ of an unbiased test for the Less Spread Problem. Refer to [4], [10] for majorization and the Schur convexity.

5. Tail order in $\mathcal{R}_+^n$

We write $\Delta^n=\{\boldsymbol{x};\ \boldsymbol{x}\in\mathcal{R}_+^n,\ \sum_1^n x_i=1\}$, the unit simplex. Δ_0^n is one part of $n!$ symmetric partition of Δ^n. We introduce three orders in

Δ_0^n, $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_\alpha)$, $\alpha=1, 2, 3$. For $\boldsymbol{a}, \boldsymbol{b} \in \Delta^n$, replacing a_i and b_i in Definition 4 by $a_{(i)}$ and $b_{(i)}$, we obtain preorders in Γ^n. For $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}_+^n$, replacing a_i and b_i by $a_{(i)}\Big/\sum_1^n a_k$ and $b_{(i)}\Big/\sum_1^n b_k$, we obtain preorders in $\mathcal{R}_+^n$. In all cases the orders or the preorders compare spread of tail components of nonnegative vectors. By analogy with the distribution case we use the symbols of 'tail order.'

DEFINITION 4. $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_\alpha)$, $\boldsymbol{a}, \boldsymbol{b} \in \Delta_0^n$.

$\mathcal{T}_3$: b_i/a_i is nondecreasing in $i=1,\cdots,n$. (If $a_i=0$, then $b_i=0$ and 0/0 is regarded as zero.)

$\mathcal{T}_2$: The same as $\mathcal{S}_2$.

$\mathcal{T}_1$: The same as $\mathcal{S}_1$.

Remarks. ($\mathcal{T}_3$) This is equivalent to $a_{i+1}/a_i \leqq b_{i+1}/b_i$, $i=1, 2, \cdots, n-1$. Notice that $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_3)$ iff there exists $\boldsymbol{c} \in \Delta_0^n$ such that $\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{c}/\|\boldsymbol{a}\cdot\boldsymbol{c}\|$. Thus the condition of Theorem 4 is 'ϕ or T is nondecreasing ($\mathcal{T}_3$) in Δ_0^n.' $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_3)$ and $a_1>0$ is equivalent to $\log \boldsymbol{b} \succ \log \boldsymbol{a}\,(\mathcal{S}_3)$, where the logarithm is operated componentwise. This fact corresponds to Theorem 5. ($\mathcal{T}_2$) If $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}_+^n$, $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_2)$ means that $\boldsymbol{b}$ has smaller 'Studentized cumulative normalized spacings,' the statistics used in the theory of life test for indicating shorter tail than the exponential distribution. ($\mathcal{T}_2$ and $\mathcal{T}_1$) Even if $\boldsymbol{a}, \boldsymbol{b} \in \Delta_0^n$, $\boldsymbol{b}-\boldsymbol{a}$ belongs to Γ^n, then the definitions for Γ_0^n can be applied directly. In all orders $n^{-1}\mathbf{1}$ is the minimum point.

We could define tail orders just by $\log \boldsymbol{b} \succ \log \boldsymbol{a}\,(\mathcal{S}_\alpha)$, and $\mathcal{T}_3$ is actually equivalent to this. Our definitions are justified by some well established statistical procedures. One reason for preferring these definitions is that Δ_0^n or Δ^n is a simplex in a hyperplane of $\mathcal{R}^n$, but the exponential transformation of Γ_0^n or Γ^n is not. In other words, normalizing a vector in $\mathcal{R}_+^n$ we prefer the sum to the product of its component being one.

THEOREM 8. *$\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_3)$ is expressed as $L_3(\boldsymbol{a})(\boldsymbol{b}-\boldsymbol{a}) \geqq \mathbf{0}$ or as $(\boldsymbol{b}-\boldsymbol{a}) \in K_3'(\boldsymbol{a})\mathcal{R}_+^{n-1}$, where $L_3(\boldsymbol{a})$ and $K_3(\boldsymbol{a})$ are $(n-1)\times n$ matrices depending on $\boldsymbol{a}$ defined below. A differentiable function defined on Δ_0^n is nondecreasing ($\mathcal{T}_3$) iff $K_3(\boldsymbol{x})\operatorname{grad} H(\boldsymbol{x}) \geqq \mathbf{0}$, $\boldsymbol{x} \in \Delta_0^n$. For $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{T}_\alpha)$, $\alpha=1, 2$, the corresponding expressions are the same as Theorem* 6

$$L_3(\boldsymbol{a})=\begin{bmatrix} -a_2 & a_1 & 0 \cdots & 0 & 0 \\ 0 & -a_3 & a_2 \cdots & 0 & 0 \\ & \cdots & & \cdots & \\ 0 & 0 & 0 \cdots & -a_n & a_{n-1} \end{bmatrix},$$

$$K_3(\boldsymbol{a})=\frac{1}{\sum a_i}\begin{bmatrix} \frac{1}{a_1a_3}\sum_2^n a_k & & & \\ & \frac{1}{a_2a_3}\sum_3^n a_k & & \\ & & \ddots & \\ & & & \frac{1}{a_{n-1}a_n}a_n \end{bmatrix}\tilde{K}_3(\boldsymbol{a}),$$

where $\tilde{K}_3(\boldsymbol{a})$ *is an* $(n-1)\times n$ *matrix with the ith row* $\left(-a_1,\cdots,-a_i,\ a_{i+1}\sum_1^i a_k\Big/\sum_{i+1}^n a_k,\cdots,a_n\sum_1^i a_k\Big/\sum_{i+1}^n a_k\right)$.

PROOF. It can be shown that

$$\begin{bmatrix} L_3(\boldsymbol{a}) \\ \boldsymbol{1}' \end{bmatrix}^{-1}=[K_3(\boldsymbol{a}),\boldsymbol{a}]\,.$$

Remarks. $K_3(\boldsymbol{a})$ in the condition of the theorem can be replaced by $\tilde{K}_3(\boldsymbol{a})$. The theorem assumes that $\boldsymbol{a},\boldsymbol{b}\in\Delta_0^n$. An elaboration of the condition is as follows. For $\boldsymbol{a}\in\Delta_0^n$, $\boldsymbol{b}$ belongs to Δ_0^n and $\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{T}_3)$ iff $\boldsymbol{b}=\boldsymbol{a}+K_3(\boldsymbol{a})\boldsymbol{p}$, where $\boldsymbol{p}$ is a positive vector such that $\boldsymbol{1}'\boldsymbol{p}\leqq 1$. It should be remarked that $L_3(\boldsymbol{1})=L_3$ and $K_3(\boldsymbol{1})=K_3$. (The vector $\boldsymbol{1}$ does not belong to Δ_0^n but $n^{-1}\boldsymbol{1}$.)

THEOREM 9.

$$\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{T}_3)\Longrightarrow\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{T}_2)\Longrightarrow\boldsymbol{b}\succ\boldsymbol{a}\,(\mathcal{T}_1)\,.$$

Therefore if ϕ, *a test function defined on* Δ_0^n, *is nondecreasing* ($\mathcal{T}_2$ *or* $\mathcal{T}_1$), *then* $\phi(\boldsymbol{w})$ *is unbiased for the Longer Tail Problem and* $1-\phi(\boldsymbol{w})$ *is so for the Shorter Tail Problem.*

PROOF. $\mathcal{T}_3\Longrightarrow\mathcal{T}_2$ Let $M_{23}(\boldsymbol{a})$, $\boldsymbol{a}\in\Delta_0^n$, be an $(n-1)\times(n-1)$ nonnegative matrix with elements $m_{ij}(\boldsymbol{a})$ defined by

$$m_{ij}(\boldsymbol{a})=\begin{cases} na_1\,, & i=1\,,\\ \sum_1^{i-1}a_k+(n-i+1)a_i\,, & 1<i\leqq j\,,\\ \left[-(n-i)a_i+\sum_{i+1}^n a_k\right]\sum_i^j a_k\Big/\sum_{j+1}^n a_k\,, & j<i\,. \end{cases}$$

For example, when $n=4$, $M_{23}(\boldsymbol{a})$ is

$$\begin{bmatrix} 4a_1 & 4a_1 & 4a_1 \\ \dfrac{(-2a_2+a_3+a_4)a_1}{a_2+a_3+a_4} & a_1+3a_2 & a_1+3a_2 \\ \dfrac{(-a_3+a_4)a_1}{a_2+a_3+a_4} & \dfrac{(-a_3+a_4)(a_1+a_2)}{a_3+a_4} & a_1+a_2+2a_3 \end{bmatrix} .$$

Then it can be shown that

$$\frac{1}{\sum a_i} M_{23}(\boldsymbol{a}) \begin{bmatrix} \dfrac{1}{a_1a_2}\sum_2^n a_k & & & \\ & \dfrac{1}{a_2a_3}\sum_3^n a_k & & \\ & & \ddots & \\ & & & \dfrac{1}{a_{n-1}a_n} a_n \end{bmatrix} L_3(\boldsymbol{a})$$

$$+\left(\sum_2^n a_k-(n-1)a_1, \cdots, a_n-a_{n-1}\right)' \mathbf{1}' = L_2 ,$$

and

$$M_{23}(\boldsymbol{a})'K_2=\tilde{K}_3(\boldsymbol{a}) .$$

$\mathcal{T}_2 \Longrightarrow \mathcal{T}_1$ is equivalent to $\mathcal{S}_2 \Longrightarrow \mathcal{S}_1$.

Remarks. This proves $\mathcal{S}_3 \Longrightarrow \mathcal{S}_2$ since $L_3(\boldsymbol{a})$ and $K_3(\boldsymbol{a})$ include as limits L_3 and K_3. Barlow and Proschan [3] proved $\mathcal{T}_3 \Longrightarrow \mathcal{T}_2$ and $\mathcal{T}_3 \Longrightarrow \mathcal{T}_1$.

The discussion on $\mathcal{S}_1$ at the end of Section 4 can be applied for $\mathcal{T}_1$. For example, a centripetal ($\mathcal{T}_1$) set in Δ^n is nothing but a symmetric convex set, and a differentiable nondecreasing ($\mathcal{T}_1$) function on Δ^n satisfies Schur's condition.

An example shows the difference between $\mathcal{T}_3$ and $\mathcal{T}_2$. Put

$$\begin{aligned} A(p) &= \{\boldsymbol{w};\ \boldsymbol{w} \in \Delta_0^n,\ w_1 \leqq pw_2 \leqq \cdots \leqq p^{n-1}w_n,\ 0<p<1\} \\ &= \bigcap_{i=1}^{n-1} \{\boldsymbol{w};\ H^{(i)}(\boldsymbol{w}) \leqq 0\} , \end{aligned}$$

where

$$H^{(i)}(\boldsymbol{w}) = -p^{i-1}w_i + p^i w_{i+1} , \qquad \boldsymbol{w} \in \Delta_0^n .$$

$K_2 \operatorname{grad} H^{(i)}(\boldsymbol{w}) \geqq 0$ is not satisfied, since $\operatorname{grad} H^{(i)}(\boldsymbol{w}) = (0, \cdots, -p^{i-1}, p_i, \cdots, 0)'$ and $-p^{i-1}+p^i<0$. However $K_3(\boldsymbol{w}) \operatorname{grad} H^{(i)}(\boldsymbol{w}) \geqq \mathbf{0}$, $\boldsymbol{w} \in A(p)$. Therefore $A(p)$ is centripetal ($\mathcal{T}_3$) but not centripetal ($\mathcal{T}_2$).

6. Some additional observations

One may think $\boldsymbol{a}$ and $-\boldsymbol{a}$ should have the same spread. In our definition $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{S}_a) \Longleftrightarrow -\boldsymbol{b} \succ -\boldsymbol{a}\,(\mathcal{S}_a)$, but we cannot always compare $\boldsymbol{b}$ and $-\boldsymbol{a}$. A looser 'equilibrium' definition is '$\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{S}_a)$ or $\boldsymbol{b} \succ -\boldsymbol{a}\,(\mathcal{S}_a)$' to compare $\boldsymbol{a}$ and $\boldsymbol{b}$. In this case we have always '$\boldsymbol{a} \succ -\boldsymbol{a}$ and $-\boldsymbol{a} \succ \boldsymbol{a}$.' A stricter 'equilibrium' definition is to require '$\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{S}_a)$ and $\boldsymbol{b} \succ -\boldsymbol{a}\,(\mathcal{S}_a)$.' In this case, however, $\boldsymbol{a} \succ \boldsymbol{a}$ iff $\boldsymbol{a}=-\boldsymbol{a}$. If $\boldsymbol{b} \succ \boldsymbol{a}\,(\mathcal{S}_1)$ and $\boldsymbol{b} \succ -\boldsymbol{a}\,(\mathcal{S}_1)$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}^n$, then $\|\boldsymbol{b}-\bar{b}\boldsymbol{1}\| \geqq \|\boldsymbol{a}-\bar{a}\boldsymbol{1}\|$ for any norm in Γ^n. In $\mathcal{R}^n_+$ this discussion regards $\boldsymbol{a}$ and $1/\boldsymbol{a}$ (componentwise reciprocal) to have the same tail.

Our results on one-sample problems are beautiful. Then what on two-sample problems. Fraser [7] states a sufficient condition for a two-sample test to be unbiased for the Spread Problem. There is a difficulty, however, in the problem. The two-sample Spread Problem is invariant only for linear transformation allowing different shift for each sample. A maximal invariant statistic is $(\boldsymbol{V}, \boldsymbol{W})=(X_{(1)}-\bar{X}, \cdots, X_{(m)}-\bar{X};\ Y_{(1)}-\bar{Y}, \cdots, Y_{(n)}-\bar{Y})$. Test functions $\phi(\boldsymbol{v}, \boldsymbol{w})$ such that

$$\phi(\boldsymbol{v}, \tilde{\boldsymbol{w}}) \geqq \phi(\tilde{\boldsymbol{v}}, \boldsymbol{w})\,, \quad \text{for } \tilde{\boldsymbol{v}} \geqq \boldsymbol{v}\,(\mathcal{S}_3) \text{ and } \tilde{\boldsymbol{w}} \geqq \boldsymbol{w}\,(\mathcal{S}_3)$$

are unbiased for $G \succ F\,(\mathcal{S})$. A similar statement holds for the Tail Problem. The tests, however, are not similar for the null hypothesis H_0. That is, $\mathrm{P}\,[H_t | H_0]$ depends on the distribution F or G. To specify the null distribution is to break down the problem into one-sample problem. Thus the similarity condition contradicts with the invariance requirement. If the locations of F and G are known, then the situation is quite different.

There is a tight relation between spread order in $\mathcal{F}$ and $\mathcal{S}_1$, and between tail order in $\mathcal{F}_+$ and $\mathcal{T}_1$. Regard $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n$ as sample points and consider empirical distributions F^x_n and F^y_n. If we extend Definition 1 to discontinuous distribution functions, then $\boldsymbol{y} \succ \boldsymbol{x}\,(\mathcal{S}_1)$ iff $F^y_n \succ F^x_n\,(\mathcal{S})$. This is also true for tail. If we regard $F^{-1}(u)$ as the limit of $x_{(i)}$, $n \to \infty$, we get from $\boldsymbol{y} \succ \boldsymbol{x}\,(\mathcal{T}_1)$, $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}^n_+$, a tail order defined by

$$\int_v^1 G^{-1}(u)du \Big/ \int_0^1 G^{-1}(u)du \geqq \int_v^1 F^{-1}(u)du \Big/ \int_0^1 F^{-1}(u)du\,.$$

This is equivalent to

$$\frac{1}{\mu_G} \int_0^{G^{-1}(t)} s\,dG(s) \leqq \frac{1}{\mu_F} \int_0^{F^{-1}(t)} s\,dF(s)\,,$$

that is 'G has the uniformly smaller Lorenz curve than F's.'

7. Some notes on order relation

Let E be an ordered set with the relation $b \geqq a$. (Here and in the following order can be just preorder.)

DEFINITION 5. A set $A \subset E$ is 'upward' iff $a \in A$ and $b \geqq a$ imply $b \in A$. Similarly it is 'downward' iff $b \in A$ and $b \geqq a$ imply $a \in A$.

A set A is upward iff its complement A^c is downward. If A_i's are upward (or downward) sets $\bigcup_i A_i$ and $\bigcap_i A_i$ are upward.

For spread and tail order we call 'centrifugal' or 'centripetal' rather than upward or downward.

DEFINITION 6. A mapping h from an ordered set E to another ordered set F is 'nondecreasing' iff $h(b) \geqq h(a)$ for any $b \geqq a$. A 'nonincreasing' mapping is similarly defined.

DEFINITION 7. For a nondecreasing h, $H(h, \lambda) = \{a;\ h(a) \geqq \lambda\}$ is a 'high-level set' and $K(h, \lambda) = \{a;\ h(a) \leqq \lambda\}$ is a 'low-level set.'

THEOREM 10. *A mapping h from E to F is nondecreasing iff high-level sets $H(h, \lambda)$ are upward for all $\lambda \in F$.*

PROOF. For a fixed $a \in E$ let $H(a)$ be the meet of all $H(h, \lambda)$'s such that $a \in H(h, \lambda)$. $H(a)$ is an upward set if $H(h, \lambda)$'s are all upward and $H(a) = \{b;\ h(b) \geqq h(a)\}$. From the definition of an upward set $h(b) \geqq h(a)$ for all $b \geqq a$. Thus h is nondecreasing. The necessity is obvious.

Now assume that two orders α and β are defined in E and α is stronger than β: $b \geqq a\,(\alpha) \Longrightarrow b \geqq a\,(\beta)$. If a set A is upward (or downward) with respect to the weaker β, then it is so with respect to the stronger α. If a mapping h is nondecreasing (or nonincreasing) with respect to β, then it is so with respect to α.

As well known an order relation in $\mathcal{R}^n$ is defined by a pointed convex cone C, that is, $\boldsymbol{b} \succ \boldsymbol{a}$ if $\boldsymbol{b} - \boldsymbol{a} \in C$. For any $m \times n$ matrix L with rank n, $L(\boldsymbol{b} - \boldsymbol{a}) \geqq \boldsymbol{0}$ defines also $\boldsymbol{b} \succ \boldsymbol{a}$. (The inequality means componentwise inequalities here and in the following.) If L is square and nonsingular, then this condition is equivalent to $\boldsymbol{b} - \boldsymbol{a} \in L^{-1}\mathcal{R}^m_+$. In general, however, a computational process is necessary to obtain an $m \times n$ matrix K such that $L(\boldsymbol{b} - \boldsymbol{a}) \geqq \boldsymbol{0} \Longleftrightarrow \boldsymbol{b} - \boldsymbol{a} \in K'\mathcal{R}^m_+$.

THEOREM 11. *$H(\boldsymbol{x})$ is a differentiable function defined on $\mathcal{R}^n$. H is nondecreasing with respect to order relation $\boldsymbol{b} \succ \boldsymbol{a} \Longleftrightarrow \boldsymbol{b} - \boldsymbol{a} \in K'\mathcal{R}^m_+$ iff $K \operatorname{grad} H(\boldsymbol{x}) \geqq \boldsymbol{0}$, $\boldsymbol{x} \in \mathcal{R}^n$.*

PROOF. Let $\boldsymbol{t}^{(i)}$, $i = 1, \cdots, m$, be the column vectors of K'. They are extreme rays of the cone $K'\mathcal{R}_+^m$. H is nondecreasing iff $H(\boldsymbol{a}+\sum p_k \boldsymbol{t}^{(k)}) \geqq H(\boldsymbol{a})$ for any $\boldsymbol{a} \in \mathcal{R}^n$ and for any small nonnegative p_k's. Therefore the derivative in the directions of $\boldsymbol{t}^{(k)}$, $\sum t_i^{(k)} \cdot \partial H/\partial x_i$, should be nonnegative for $k=1,\cdots,m$. This is equivalent to $K \operatorname{grad} H(\boldsymbol{a}) \geqq \boldsymbol{0}$ for any $\boldsymbol{a} \in \mathcal{R}^n$.

Suppose that two orders α and β are defined in $\mathcal{R}^n$ using the cones C_α and C_β, the matrices L_α and L_β, or the matrices K_α and K_β as above. Then α is stronger than β iff $C_\alpha \subset C_\beta$, there exists a nonnegative matrix N such that $L_\beta = NL_\alpha$, or there exists a nonnegative matrix M such that $K_\alpha = MK_\beta$.

In the definition of $\boldsymbol{b} \succ \boldsymbol{a}$ by $\boldsymbol{b}-\boldsymbol{a} \in C$, the pointed convex cone C may depend on $\boldsymbol{a}$ provided that the cone satisfies the condition $\boldsymbol{b}+C(\boldsymbol{b}) \subset \boldsymbol{a}+C(\boldsymbol{a})$ for any $\boldsymbol{b} \succ \boldsymbol{a}$. This condition is satisfied if $C(\boldsymbol{b}) \subset C(\boldsymbol{a})$, which is expressed in terms of the matrices L or K as the last paragraph. An order relation determined by varying cone appears in Section 5.

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