Ann. Inst. Statist. Math.
28 (1976), Part A, 309-328

THE NEW TEST CRITERION FOR THE HOMOGENEITY OF PARAMETERS OF SEVERAL POPULATIONS

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(Received Oct. 30, 1975; revised Sept. 18, 1976)

Summary

The new test criterion for testing the homogeneity of parameters of several populations is proposed and the test properties of it is discussed. The asymptotic expansions of the distributions of test criterion are discussed under (i) null hypothesis, (ii) fixed alternative hypothesis and (iii) local alternative hypothesis converging to the null hypothesis with appropriate rate of convergence as the sample size increases. As a particular case the asymptotic theory of a statistic for a homogeneity of variances of normal populations is also discussed and the exact moments of it under a null hypothesis can be used to obtain a percentage point by a Pearsonian curve fitting.

1. Introduction

Let $\boldsymbol{x}_i=(x_{i1}, x_{i2}, \cdots, x_{in_i})$, $i=1, 2, \cdots, k$ be a random sample from the ith population with probability density function (p.d.f.) $f(x|\theta_i)$ which depends on an unknown parameter θ_i. The problem considered here is that of testing the hypothesis

$$H: \quad \theta_1=\theta_2=\cdots=\theta_k \ (=\theta, \text{ say}) ,$$

against

$$K: \quad \text{violation of at least one equality} .$$

Let $\hat{\theta}_i$ be a maximum likelihood estimator (m.l.e.) of θ_i in terms of $\boldsymbol{x}_i$ under the alternative hypothesis K, and $\bar{\theta}$ also an m.l.e. of θ in terms of $n=\sum_{i=1}^{k} n_i$ observations $\boldsymbol{x}=(\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k)$ under H.

Hayakawa [4] has studied the asymptotic expansions of the distribution of the likelihood ratio criterion λ defined as

$$\lambda=\prod_{i=1}^{k}\prod_{\alpha=1}^{n_i}\frac{f(x_{i\alpha}|\hat{\theta}_i)}{f(x_{i\alpha}|\bar{\theta})} .$$

For the class of density functions satisfying the condition such that $\bar{\theta}=\sum_{i=1}^{k}\rho_i\hat{\theta}_i$, $\rho_i=n_i/n$ (>0), $i=1,\cdots,k$, he showed that the limiting distribution of $-2\log\lambda$ under a fixed alternative hypothesis K_ϵ: $\theta_i=\theta+\varepsilon_i$, $i=1,\cdots,k$ is normally distributed and it has singularity at the null hypothesis since the variance of its limiting one vanishes. He also showed that the limiting distribution of $-2\log\lambda$ becomes a non-central chi-square with $k-1$ degrees of freedom with non-centrality parameter $\sigma^2=(1/2)\sigma^2(\theta)\sum_{i=1}^{k}\rho_i(\varphi_i-\tilde{\varphi})^2$ under the sequence of Pitman's alternative K_n: $\theta_i=\theta+\varphi_i/\sqrt{n}$, $i=1,\cdots,k$, where

$$\sigma^2(\theta)=\mathrm{E}\left[\left\{\frac{\partial\log f(x|\theta)}{\partial\theta}\right\}^2\middle|H\right], \qquad \tilde{\varphi}=\sum_{i=1}^{k}\rho_i\varphi_i .$$

Due to a method suggested by Wald [17] for a construction of a statistic expressing a measure of departure from a null hypothesis, we propose a statistic for testing a homogeneity of parameters as follows:

$$T=n\sigma^2(\bar{\theta})\sum_{\alpha=1}^{k}\rho_\alpha(\hat{\theta}_\alpha-\bar{\theta})^2 . \tag{1}$$

For the case of means of normal populations $N(\theta_i,1)$, $i=1,\cdots,k$, T becomes as

$$T_m=n\sum_{i=1}^{k}\rho_i(\bar{x}_i-\bar{x})^2 , \tag{2}$$

where

$$\bar{x}_i=\sum_{\alpha=1}^{n_i}x_{i\alpha}/n_i , \qquad \bar{x}=\sum_{i=1}^{k}\rho_i\bar{x}_i .$$

For a fixed sample size n, T_m is distributed as follows:

(i) central chi-square with $k-1$ degrees of freedom under H: $\theta_1=\cdots=\theta_k$,

(ii) non-central chi-square with $k-1$ degrees of freedom and a non-centrality parameter $\delta^2=n\sum_{i=1}^{k}\rho_i(\varepsilon_i-\tilde{\varepsilon})^2$, $\tilde{\varepsilon}=\sum_{i=1}^{k}\rho_i\varepsilon_i$ under K_ϵ: $\theta_i=\theta+\varepsilon_i$, $i=1,\cdots,k$,

(iii) non-central chi-square with $k-1$ degrees of freedom and a non-centrality parameter $\delta^2=\sum_{i=1}^{k}\rho_i(\varphi_i-\tilde{\varphi})^2$ under K_n: $\theta_i=\theta+\varphi_i/\sqrt{n}$, $i=1,\cdots,k$,

(iv) T gives a uniformly most powerful invariance test.

For the case of variances of normal populations $N(0,\theta_i)$, $i=1,\cdots,k$, T becomes as

$$(3) \qquad T_\sigma = \frac{n}{2}\sum_{i=1}^{k} \rho_i (s_i - \bar{s})^2/\bar{s}^2$$

where

$$s_i = \sum_{\alpha=1}^{n_i} x_{i\alpha}^2/n_i\ , \qquad \bar{s} = \sum_{i=1}^{k} \rho_i s_i\ .$$

Nagao [8] has studied the testing a homogeneity of variance and co-variance matrices of k p-dimensional multivariate normal populations $H: \Sigma_1 = \cdots = \Sigma_k$ and proposed a test statistic T_4. T_4 agrees with T_σ when $p=1$. Nagao gave the asymptotic expansions of the distribution of T_4 under various hypothesis [8], [9], [10].

Rao ([15], p. 389) studied this problem and proposed a test statistic R such that

$$(4) \qquad R = n\sum_{i=1}^{k} \frac{\rho_i}{s_i^2(T_i)}(T_i - \hat{\theta})^2\ , \qquad \hat{\theta} = \sum_{i=1}^{k} \frac{n_i T_i}{s_i^2(T_i)} \Big/ \sum_{i=1}^{k} \frac{n_i}{s_i^2(T_i)}\ ,$$

where T_i's are consistency estimators of θ_i's enjoying that $\sqrt{n_i}(T_i - \theta_i)$ has an asymptotic normal distribution with mean zero and variance $s_i^2(\theta_i)$ as n_i goes to infinity. He gave the limiting distribution of R under a null hypothesis H.

The following notations and convensions will be adopted.

Defining the log-likelihood function by

$$L_i(\theta_i) = \sum_{\alpha=1}^{n_i} \log f(x_{i\alpha} | \theta_i)\ , \qquad i = 1, \cdots, k\ ,$$

we require for $L_i(\theta_i)$ to be regular with respect to θ_i-derivatives.

(i) $y_i^{(l)} = n_i^{-l/2} \dfrac{\partial^l L_i(\theta_i)}{\partial \theta_i^l}\ , \qquad l = 1, 2, \cdots\ ,$

$v_i = \sqrt{n_i}(\hat{\theta}_i - \theta_i)\ .$

(ii) $m(\theta_i) = \mathrm{E}\,[\log f(x|\theta_i)] = \int \log f(x|\theta_i) f(x|\theta_i) dx\ ,$

$m_{(1)}(\theta_i) = \mathrm{E}\left[\dfrac{\partial \log f}{\partial \theta_i}\right] = 0\ ,$

$m_{(k^\alpha, l^\beta)}(\theta_i) = \mathrm{E}\,[(\partial^k \log f_i)^\alpha (\partial^l \log f_i)^\beta]\ ,$

$\partial^k \log f_i = \dfrac{\partial^k \log f(x|\theta_i)}{\partial \theta_i^k}$

$k, l, \alpha, \beta = 0, 1, 2, 3, 4\ .$

The following equalities hold for $m_{(\cdot)}$'s by the regularity conditions

$$
\begin{aligned}
&m_{(2)}(\theta_i)+m_{(1^2)}(\theta_i)=0\\
(5)\qquad &m_{(3)}(\theta_i)+3m_{(21)}(\theta_i)+m_{(1^3)}(\theta_i)=0\\
&m_{(4)}(\theta_i)+4m_{(31)}(\theta_i)+3m_{(2^2)}(\theta_i)+6m_{(21^2)}(\theta_i)+m_{(1^4)}(\theta_i)=0\ .
\end{aligned}
$$

(iii) Any function evaluated at $\theta=\hat{\theta}$ will be denoted by the addition of a circumflex.

(iv) o_p and O_p will denote orders of magnitude in probabilitic sense.

2. Asymptotically uniformly most powerful invariance of T

The statistic T_m given by (2) for testing the equality of means of normal populations with known variance is same form as (21) in Lehmann ([5], p. 275), which implies that the critical region $\{T_m \geqq c\}$ gives the uniformly most powerful invariant test, where c is the upper $100\alpha\%$ critical point of a central chi-square random variable with $k-1$ degrees of freedom. Nagao [11] considered the asymptotically uniformly most powerful invariance of the test statistics T_1 and T_2 proposed for testing the covariance matrix of normal population.

We show in this section that the test statistic T gives asymptotically uniformly most powerful invariant test.

Put $u_i=\sqrt{n_i}\sigma(\bar{\theta})(\hat{\theta}_i-\bar{\theta})$, $\sigma(\bar{\theta})=\sqrt{\sigma^2(\bar{\theta})}=\sqrt{\sigma_i^2(\bar{\theta})}$, $i=1,2,\cdots,k$, then

$$T=\sum_{i=1}^{k} u_i^2\ .$$

Expanding u_i, $i=1,2,\cdots,k$ at $\theta_1=\cdots=\theta_k=\theta$ in Taylor series, we have

$$\boldsymbol{u}=\sigma\{(I-\sqrt{\boldsymbol{\rho}}\sqrt{\boldsymbol{\rho}}')\boldsymbol{v}+\tilde{\Phi}\sqrt{\boldsymbol{\rho}}\}+o_p(1)\ ,$$

where

$$
\begin{aligned}
&\boldsymbol{u}'=(u_1,u_2,\cdots,u_k)\ ,\qquad \sigma=\sigma(\theta)\ ,\\
&\sqrt{\boldsymbol{\rho}}'=(\sqrt{\rho_1},\sqrt{\rho_2},\cdots,\sqrt{\rho_k})\ ,\qquad \boldsymbol{v}'=(v_1,\cdots,v_k)\ ,\\
&\tilde{\Phi}=\mathrm{diag}\,(\varphi_1-\tilde{\varphi},\cdots,\varphi_k-\tilde{\varphi})\ ,\\
&\varphi_i=\sqrt{n}(\theta_i-\theta)\ ,\qquad \tilde{\varphi}=\sum_{i=1}^{k}\rho_i\varphi_i\ .
\end{aligned}
$$

It is easy to show that $\boldsymbol{u}$ is asymptotically normally distributed with mean $\sigma\tilde{\Phi}\sqrt{\boldsymbol{\rho}}$ and variance-covariance matrix $I-\sqrt{\boldsymbol{\rho}}\sqrt{\boldsymbol{\rho}}'$. Since the rank of $I-\sqrt{\boldsymbol{\rho}}\sqrt{\boldsymbol{\rho}}'$ is only $k-1$, that is, $\sqrt{\boldsymbol{\rho}}'\boldsymbol{u}=0$ with probability one, the problem should be considered by $\boldsymbol{u}^{*\prime}=(u_1,\cdots,u_{k-1})$ where $\boldsymbol{u}^*$ is distributed as

$$u^* \sim N(0,\ I_{k-1}-\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}') \qquad \text{under } H\,,$$
$$u^* \sim N(\sigma\tilde{\Phi}_{k-1}\sqrt{\boldsymbol{\rho}_{k-1}},\ I_{k-1}-\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}') \qquad \text{under } K\,,$$

where

$$\sqrt{\boldsymbol{\rho}_{k-1}}'=(\sqrt{\rho_1}\,,\cdots,\sqrt{\rho_{k-1}})\,, \qquad \tilde{\Phi}_{k-1}=\mathrm{diag}\,(\varphi_1-\tilde{\varphi},\cdots,\varphi_{k-1}-\tilde{\varphi})\,.$$

As

$$(I_{k-1}-\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}')^{-1}=I_{k-1}+\frac{1}{\rho_k}\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}'\,,$$

the problem is invariant under the transformation L_{k-1} on $\boldsymbol{u}^*$ such that

$$L_{k-1}\Big(I_{k-1}+\frac{1}{\rho_k}\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}'\Big)L'_{k-1}=I_{k-1}+\frac{1}{\rho_k}\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}'\,. \tag{6}$$

The construction, for example, of this matrix L_{k-1} is as follows. Let the last column of $(k-1)\times(k-1)$ orthogonal matrix $A=[\boldsymbol{a}_1, \boldsymbol{a}_2,\cdots, \boldsymbol{a}_{k-1}]$ be defined by $\boldsymbol{a}_{k-1}=\sqrt{\boldsymbol{\rho}_{k-1}}/\sqrt{1-\rho_k}$. The other $(k-2)$ column vectors can be defined by some arbitrary rule. Let H be an orthogonal matrix of order $(k-1)$ such that $H=\mathrm{diag}\,(H_{k-2}, 1)$ where H_{k-2} is also an arbitrary matrix of order $(k-2)$. Putting $L_{k-1}=AHA'$, L_{k-1} enjoys the following properties.

$$L_{k-1}L'_{k-1}=I_{k-1}\,, \qquad L_{k-1}\sqrt{\boldsymbol{\rho}_{k-1}}=\sqrt{\boldsymbol{\rho}_{k-1}}\,,$$

which implies that (6) holds.

Thus, the maximal invariant is expressed as

$$z=\boldsymbol{u}^{*\prime}\Big(I_{k-1}+\frac{1}{\rho_k}\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}'\Big)\boldsymbol{u}^*\,,$$

and z is asymptotically distributed as non-central χ^2 with $k-1$ d.f. and non-centrality parameter δ^2

$$\begin{aligned}\delta^2&=\frac{1}{2}(\sigma\tilde{\Phi}_{k-1}\sqrt{\boldsymbol{\rho}_{k-1}})'\Big(I_{k-1}+\frac{1}{\rho_k}\sqrt{\boldsymbol{\rho}_{k-1}}\sqrt{\boldsymbol{\rho}_{k-1}}'\Big)(\sigma\tilde{\Phi}_{k-1}\sqrt{\boldsymbol{\rho}_{k-1}})\\&=\frac{\sigma^2}{2}\sum_{i=1}^{k}\rho_i(\varphi_i-\tilde{\varphi})^2=\frac{\sigma^2}{2}\nu_2\,.\end{aligned}$$

z can also be rewritten as

$$z=\sum_{i=1}^{k-1}(u_i^*)^2+\frac{1}{\rho_k}(\sqrt{\boldsymbol{\rho}_{k-1}}'\boldsymbol{u}^*)^2=\sum_{i=1}^{k}u_i^2=T\,.$$

These facts give the following theorem.

THEOREM 1. *For the testing H against K, $\{T \geqq c\}$ is the asymptotically uniformly most powerful invariant test in above sense.*

3. Asymptotic distribution of T

In this section we consider the asymptotic expansions of the distribution of T under various hypothesis. The method used here is same as in Hayakawa [4]. The similar type of argument is applied in the distribution theory of a likelihood ratio criterion for a composite hypothesis of parameters of a single population, Hayakawa [3] and Peers [12].

3.1. *Null hypothesis*

First we consider the asymptotic expansion of a distribution of T under a null hypothesis H: $\theta_1=\theta_2=\cdots=\theta_k=\theta$.

Expanding T at $\theta_1=\cdots=\theta_k=\theta$ in Taylor series, we have

$$(7)\qquad T=\sigma^2 v'(I-\sqrt{\rho}\sqrt{\rho}')v+\frac{1}{\sqrt{n}}\left[\frac{d\sigma^2}{d\theta}\right]_\theta \sqrt{\rho}'v\cdot v'(I-\sqrt{\rho}\sqrt{\rho}')v$$
$$+\frac{1}{2n}\left[\frac{d^2\sigma^2}{d\theta^2}\right]_\theta (v'\sqrt{\rho})^2 v'(I-\sqrt{\rho}\sqrt{\rho}')v+o_p(1/n)\ .$$

The equation satisfied by v may be written as

$$0=\hat{z}=z+Y_2 v+\frac{1}{2}Y_3 v^{(2)}+\frac{1}{3!}Y_4 v^{(3)}+o_p(1/n)\ ,$$

where

$$z'=(z_1, z_2,\cdots, z_k)=(y_1^{(1)},\cdots, y_k^{(1)})$$
$$v^{(l)'}=(v_1^l, v_2^l,\cdots, v_k^l)\ ,\qquad l=2, 3$$
$$Y_l=\mathrm{diag}\,(y_1^{(l)}, y_2^{(l)},\cdots, y_k^{(l)})\ ,\qquad l=2, 3, 4\ .$$

We have by solving the above equation with respect to v,

$$(8)\qquad v=-Y_2^{-1}z-\frac{1}{2}Y_2^{-3}Y_3 z^{(2)}-\frac{1}{2}Y_2^{-5}Y_3^2 z^{(3)}+\frac{1}{6}Y_2^{-4}Y_4 z^{(3)}+o_p(1/n)\ ,$$

where

$$z^{(l)'}=(z_1^l, z_2^l,\cdots, z_k^l)\ ,\qquad l=2, 3\ .$$

Finally, inserting (8) into (7), we have with a little algebra the asymptotic expansion of T in terms of θ-derivatives $y_i^{(l)}$'s under the null hypothesis up to order $O_p(1/n)$ as follows:

$$(9)\qquad T=l_0+l_1+l_2+o_p(1/n)\ ,$$

where

$$l_0=\sigma^2 z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-1}z$$

$$l_1=\sigma^2 z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-3}Y_3 z^{(2)} - \frac{1}{\sqrt{n}}\left[\frac{d\sigma^2}{d\theta}\right]_\theta \sqrt{\rho}'Y_2^{-1}z\cdot z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-1}z$$

$$\begin{aligned} l_2=&\sigma^2 zY_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-5}Y_3^2 z^{(3)}\\ &-\frac{\sigma^3}{3}z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-4}Y_4 z^{(3)}\\ &+\frac{\sigma^2}{4}z^{(2)\prime}Y_3Y_2^{-3}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-3}Y_3 z^{(2)}\\ &-\frac{1}{\sqrt{n}}\left[\frac{d\sigma^2}{d\theta}\right]_\theta \sqrt{\rho}'Y_2^{-1}z\cdot z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-3}Y_3 z^{(2)}\\ &-\frac{1}{2\sqrt{n}}\left[\frac{d\sigma^2}{d\theta}\right]\sqrt{\rho}'Y_2^{-3}Y_3 z^{(2)}\cdot z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-1}z\\ &+\frac{1}{2n}\left[\frac{d^2\sigma^2}{d\theta^2}\right]z'Y_2^{-1}\sqrt{\rho}\sqrt{\rho}'Y_2^{-1}z\cdot z'Y_2^{-1}(I-\sqrt{\rho}\sqrt{\rho}')Y_2^{-1}z\,. \end{aligned}$$

To find the moment generating function (MGF) of T under H, we need the Edgeworth type A series expansion for z, Y_2, Y_3 and Y_4 to the required order, which is stated as following lemma.

LEMMA. *The joint probability density function of z, Y_2, Y_3 and Y_4 is expressed up to order $O(n^{-1})$ as*

$$f_1=f_0\left\{1+\frac{A}{\sqrt{n}}+\frac{B}{n}\right\}+o\left(\frac{1}{n}\right)\,, \tag{10}$$

where

$$f_0=\prod_{i=1}^{k}(2\pi m_{(1^2)}(\theta_i))^{-1/2}\exp\{-z_i^2/2m_{(1^2)}(\theta_i)\}\prod_{l=2}^{4}\delta(y_i^{(l)}-m_{(l)}(\theta_i)/n_i^{(l-2)/2})$$

$$A=-\sum_{i=1}^{k}m_{(1^3)}(\theta_i)H_3(z_i)/6\sqrt{\rho_i}+\sum_{i=1}^{k}m_{(21)}(\theta_i)H_1(z_i)\delta_{2i}^{(1)}/\sqrt{\rho_i}\,,$$

$$\begin{aligned} B=&\sum_{i=1}^{k}\{m_{(2^2)}(\theta_i)-(m_{(2)}(\theta_i))^2\}\delta_{2i}^{(2)}/\rho_i\\ &-\sum_{i=1}^{k}\{m_{(21^2)}(\theta_i)-m_{(2)}(\theta_i)m_{(1^2)}(\theta_i)\}H_2(z_i)\delta_{2i}^{(1)}/\rho_i\\ &+\sum_{i=1}^{k}m_{(31)}(\theta_i)H_1(z_i)\delta_{3i}^{(1)}/\rho_i+\sum_{i=1}^{k}\{m_{(1^4)}(\theta_i)-3(m_{(1^2)}(\theta_i))^2\}H_4(z_i)/24\\ &+\sum_{i=1}^{k}\{m_{(21)}(\theta_i)\}^2H_2(z_i)\delta_{2i}^{(2)}/2\rho_i \end{aligned}$$

$$+\sum_{i\neq j} m_{(21)}(\theta_i)m_{(21)}(\theta_j)H_1(z_i)H_1(z_j)\delta_{2i}^{(1)}\delta_{2j}^{(1)}/2\sqrt{\rho_i}\sqrt{\rho_j}$$

$$-\sum_{i=1}^{k} m_{(21)}(\theta_i)m_{(1^3)}(\theta_i)H_4(z_i)\delta_{2i}^{(1)}/6\rho_i$$

$$-\sum_{i\neq j} m_{(21)}(\theta_i)m_{(1^3)}(\theta_j)H_1(z_i)H_3(z_j)\delta_{2i}^{(1)}/6\sqrt{\rho_i}\sqrt{\rho_j}$$

$$+\sum_{i=1}^{k} \{m_{(1^3)}(\theta_i)\}^2H_6(z_i)/72\rho_i$$

$$+\sum_{i\neq j} m_{(1^3)}(\theta_i)m_{(1^3)}(\theta_j)H_3(z_i)H_3(z_j)/72\sqrt{\rho_i}\sqrt{\rho_j} .$$

$$\delta_{li}^{(r)}\equiv\delta^{(r)}(y_i^{(l)}-m_{(l)}(\theta_i)/n_i^{(l-2)/2})/\delta(y_i^{(l)}-m_{(l)}(\theta_i)/n_i^{(l-2)/2}) ,$$

and $\delta^{(r)}$ *is the rth derivative of Dirac* δ*-function.* δ *enjoies the following properties.*

$$\delta(y_i^{(2)}-m_{(2)}(\theta_i))=0 , \qquad y_i^{(2)}\neq m_{(2)}(\theta_i) ,$$

$$\int \delta(y_i^{(2)}-m_{(2)}(\theta_i))dy_i^{(2)}=1 ,$$

$$\int h(\cdot\cdot, y_i^{(2)}, \cdot\cdot)\delta(y_i^{(2)}-m_{(2)}(\theta_i))dy_i^{(2)}=h(\cdot\cdot, m_{(2)}(\theta_i), \cdot\cdot) ,$$

$$\int h(\cdot\cdot, y_i^{(2)}, \cdot\cdot)\delta^{(r)}(y_i^{(2)}-m_{(2)}(\theta_i))dy_i^{(2)}=(-1)^r\left[\frac{\partial^r h}{\partial(y_i^{(2)})^r}\right]_{y_i=m_{(2)}(\theta_i)} .$$

$H_r(z)$ *is defined by*

$$\exp\{-z^2/2\sigma^2\}H_r(z)=\frac{d^r}{dz^r}\exp\{-z^2/2\sigma^2\} .$$

The MGF of T under H is expressed as

$$M(t)=\mathrm{E}[\exp(tT)\,|\,H]=\int\cdots\int \exp(tT)f_1 dz\prod_{i=2}^{4}dY_i+o(1/n) .$$

Carrying out the integration with respect to z, Y_2, Y_3 and Y_4, and noting the regularity conditions, we have after some lengthy algebra,

$$(11)\quad M(t)=(1-2t)^{-(k-1)/2}\left[1+\frac{1}{n}\left\{\frac{a_3}{(1-2t)^3}+\frac{a_2}{(1-2t)^2}+\frac{a_1}{1-2t}+a_0\right\}\right]+o(1/n) ,$$

where

$$a_3=\frac{1}{72\sigma^6}(3m_{(21)}+2m_{(1^3)})^2(-9k^2-18k+12+15\tilde{\rho})$$

$$a_2=\frac{1}{8\sigma^6}[(5m_{(3)}^2+14m_{(3)}m_{(21)}+10m_{(21)}^2)k^2+6m_{(21)}^2k$$
$$-(m_{(3)}^2+4m_{(3)}m_{(21)}+7m_{(21)}^2)-(4m_{(3)}^2+10m_{(3)}m_{(21)}+9m_{(21)}^2)\tilde{\rho}]$$

$$-\frac{1}{8\sigma^4}(\tilde{\rho}-2k+1)(4m_{(31)}+12m_{(21^2)}+3m_{(1^4)}+3m^2_{(1^2)})$$

$$a_1=\frac{1}{8\sigma^6}[-(m_{(3)}+m_{(21)})^2k^2+2(4m^2_{(3)}+13m_{(3)}m_{(21)}+7m^2_{(21)})k$$
$$-(6m^2_{(3)}+20m_{(3)}m_{(21)}+10m^2_{(21)})-(m^2_{(3)}+4m_{(3)}m_{(21)}+3m^2_{(21)})\tilde{\rho}]$$
$$+\frac{1}{4\sigma^4}[(2m_{(4)}+4m_{(31)}+8m_{(2^2)}+m_{(21^2)}-m_{(1^4)}-4m^2_{(1^2)})k$$
$$-(2m_{(4)}+6m_{(31)}+8m_{(2^2)}+5m_{(21^2)}-3m^2_{(1^2)})$$
$$-(m_{(4)}+2m_{(31)}+3m_{(2^2)}+2m_{(21^2)}-m^2_{(1^2)})\tilde{\rho}]$$

$$a_0=\frac{1}{8\sigma^6}\Big[-(2m_{(3)}m_{(21)}+2m^2_{(21)})k+\frac{5}{3}m^2_{(3)}+8m_{(3)}m_{(21)}+5m^2_{(21)}$$
$$-\left(\frac{5}{3}m^2_{(3)}+6m_{(3)}m_{(21)}+3m^2_{(21)}\right)\tilde{\rho}\Big]$$
$$+\frac{1}{8\sigma^4}[-(4m_{(2^2)}+2m_{(21^2)}-2m^2_{(1^2)})k$$
$$+(4m_{(2^2)}-2m_{(21^2)}-m_{(1^4)}-3m^2_{(1^2)})+(4m_{(21^2)}+m_{(1^4)}+m^2_{(1^2)})\tilde{\rho}]\ ,$$

$$\tilde{\rho}=\sum_{i=1}^{k}\frac{1}{\rho_i}\ .$$

Inversion of this moment generating function gives the following theorem.

THEOREM 2. *The null distribution of the statistic T is expanded asymptotically in terms of χ^2-distributions for large n with fixed $\rho_i=n_i/n$ (>0) as*

$$\mathrm{P}\{T\leqq x\}=P_{k-1}+\frac{1}{n}\sum_{l=0}^{3}a_lP_{k+2l-1}+o(1/n)\ , \tag{12}$$

where a_0, a_1, a_2, a_3 are given in (11) *and $P_f=\mathrm{P}\{\chi^2_f\leqq x\}$.*

Applying the general inversion expansion formula due to Hill and Davis [5], we have a following asymptotic formula for the $100\alpha\%$ point of T

$$u+\frac{1}{n}\Big[\frac{2a_3u}{f(f+2)(f+4)}\{u^2+(f+4)u+(f+2)(f+4)\}$$
$$+\frac{2a_2u}{f(f+2)}(u+f+2)+\frac{2a_1u}{f}\Big]+O(1/n^2)\ , \tag{13}$$

where $\mathrm{P}\{\chi^2_f\geqq u\}=\alpha$ and $f=k-1$.

From the MGF of T, the asymptotic expectation of T is expressed as

$$\mathrm{E}(T|H)=k-1+\frac{C}{n}+o(1/n)\ ,$$

where

$$\begin{aligned}C=\frac{1}{2\sigma^4}[&(4m_{(31)}+2m_{(2^2)}+13m_{(21^2)}+3m_{(1^4)}+2m^2_{(1^2)})k\\&-(2m_{(31)}+2m_{(2^2)}+5m_{(21^2)}+m_{(1^4)})\\&-2(m_{(31)}+4m_{(21^2)}+m_{(1^4)}+m^2_{(1^2)})\tilde{\rho}]\\-\frac{1}{4\sigma^6}[&\{3m^2_{(3)}+10m_{(3)}m_{(21)}+8m^2_{(21)}\}k^2\\&+2\{8m^2_{(3)}+23m_{(3)}m_{(21)}+14m^2_{(21)}\}k\\&-4\{2m^2_{(3)}+5m_{(3)}m_{(21)}+3m^2_{(21)}\}\\&-\{11m^2_{(3)}+36m_{(3)}m_{(21)}+24m^2_{(21)}\}\tilde{\rho}]\ .\end{aligned}$$

A correction factor c which makes the term of order $1/n$ in $\mathrm{E}(T|H)$ vanish can be obtained.

$$c=1-\frac{C}{n(k-1)}+o(1/n)\ .$$

We are able to give the correction factor c^* such that the term of order $1/n$ of the asymptotic expansion of the distribution of $-2\log\lambda$, λ the likelihood ratio criterion of H against K, and the term of order $1/n$ of $\mathrm{E}[-2\log\lambda]$ vanish simultaneously for a class of a continuous density of Koopman-Pitman type such that

$$f(x|\theta)=\begin{cases}\exp\{p(\theta)K(x)+S(x)+q(\theta)\} & a<x<b,\ \alpha<\theta<\beta\ ,\\ 0\ , & \text{otherwise}\ ,\end{cases}$$

where $p(\theta)$ is nontrivial continuous functions of θ, and $S(x)$ and $K'(x)\neq 0$ are continuous function of x. We assume that $f(x|\theta)$ satisfies the conditions

$$H(\theta)=\lambda\theta G(\theta)\quad\text{and}\quad\lambda G(\theta)<0\ ,\qquad\text{for all }\theta\ ,$$

where

$$G(\theta)=\frac{dp(\theta)}{d\theta}\qquad\text{and}\qquad H(\theta)=\frac{dq(\theta)}{d\theta}\ .$$

The correction factor c^* is given by

$$c^*=1+2(\tilde{\rho}-1)D/(k-1)n+o(1/n)\ ,$$

where

$$D=\left(\frac{dG}{d\theta}\right)^2\Big/6G^3-\frac{d^2G}{d\theta^2}\Big/8G^2 .$$

Unfortunately, even though the same family of distribution it is hard to find similar c^* for T.

Example 1. Under the null hypothesis H: $\sigma_1^2=\cdots=\sigma_k^2$ for normal population, the asymptotic expansion of the distribution of T_σ expressed by (3) is given as follows,

$$\mathrm{P}\{T_\sigma\leqq x\}=P_{k-1}+\frac{1}{n}\{a_3P_{k+5}+a_2P_{k+3}+a_1P_{k+1}+a_0P_{k-1}\}+o(1/n)$$

where

$$a_3=\frac{1}{3}(-3k^2-6k+4+5\tilde{\rho})$$

$$a_2=\frac{1}{2}(4k^2+6k-3-7\tilde{\rho})$$

$$a_1=-k^2-k+2\tilde{\rho}$$

$$a_0=\frac{1}{6}(1-\tilde{\rho}) .$$

This expression agrees, in the case $p=1$, with (7.6) of Nagao [8].

Example 2. For the exponential distribution having the pdf

$$f(x\mid\theta)=\begin{cases}\exp(-x/\theta)/\theta , & x\geqq 0\\ 0 , & x<0 ,\end{cases}$$

the statistic T becomes

$$T_e=n\sum_{i=1}^{k}\rho_i(\hat{\theta}_i-\bar{\theta})^2/\bar{\theta}^2 , \tag{14}$$

and the asymptotic expansion of the distribution of T_e is expressed as

$$\mathrm{P}\{T_e\leqq x\}=P_{k-1}+\frac{1}{n}\sum_{i=0}^{3}b_iP_{k-1+2i}+o(1/n) ,$$

where

$$b_i=a_i/2 , \qquad i=0, 1, 2, 3 .$$

This expression implies that the convergence of T_e to a central chi-square is more rapid than the one of T_σ.

3.2. *Fixed alternative hypothesis*

We shall consider the asymptotic expansion of the distribution of T under a fixed alternative K: $\exists i, j, \theta_i \neq \theta_j$. Expanding $\sigma_i^2(\bar{\theta})$ at $\theta = \tilde{\theta} = \sum_{\alpha=1}^{k} \rho_\alpha \theta_\alpha$, we have

$$\sigma_i^2(\bar{\theta}) = \sigma^2(\tilde{\theta}) + \frac{1}{\sqrt{n}} \sqrt{\boldsymbol{\rho}}' \boldsymbol{v} \left[\frac{d\sigma^2}{d\tilde{\theta}}\right] + \frac{1}{2n} (\sqrt{\boldsymbol{\rho}}' \boldsymbol{v})^2 \left[\frac{d^2\sigma^2}{d\tilde{\theta}^2}\right] + o_p(1/n) ,$$

and noting

$$\hat{\theta}_\alpha - \bar{\theta} = -\left\{e_\alpha + \frac{1}{\sqrt{n}} \boldsymbol{a}_\alpha' \boldsymbol{v}\right\} ,$$

$$e_\alpha = \tilde{\theta} - \theta_\alpha ,$$

$$\boldsymbol{a}_\alpha = \sqrt{\boldsymbol{\rho}} - \frac{1}{\sqrt{\rho_\alpha}} \boldsymbol{\varepsilon}_\alpha , \qquad \boldsymbol{\varepsilon}_\alpha' = (0, \cdots, 0, \overset{\alpha\text{th}}{1}, 0, \cdots, 0) ,$$

T will be expanded as following form.

$$T = n \sum_{\alpha=1}^{k} \rho_\alpha \sigma^2(\tilde{\theta})(\tilde{\theta} - \theta_\alpha)^2 + \sqrt{n} \sum_{\alpha=1}^{k} \rho_\alpha \boldsymbol{l}_\alpha' \boldsymbol{v} + \sum_{\alpha=1}^{k} \rho_\alpha \boldsymbol{v}' W_\alpha \boldsymbol{v} + O(1/n) ,$$

where

$$\boldsymbol{l}_\alpha = 2\sigma^2(\tilde{\theta}) e_\alpha \boldsymbol{a}_\alpha + \left[\frac{d\sigma^2}{d\tilde{\theta}}\right] e_\alpha^2 \sqrt{\boldsymbol{\rho}}$$

$$W_\alpha = \sigma^2(\tilde{\theta}) \boldsymbol{a}_\alpha \boldsymbol{a}_\alpha' + 2\left[\frac{d\sigma^2}{d\tilde{\theta}}\right] e_\alpha \sqrt{\boldsymbol{\rho}}\, \boldsymbol{a}_\alpha' + \frac{1}{2}\left[\frac{d^2\sigma^2}{d\tilde{\theta}^2}\right] e_\alpha^2 \sqrt{\boldsymbol{\rho}} \sqrt{\boldsymbol{\rho}}' .$$

Setting $\boldsymbol{l} = \sum_{\alpha=1}^{k} \rho_\alpha \boldsymbol{l}_\alpha$ and $W = \sum_{\alpha=1}^{k} \rho_\alpha W_\alpha$, we obtain the asymptotic expansion of T up to order $O_p(1/\sqrt{n})$ in the following form:

$$\begin{aligned} T_K &= \left\{T - n \sum_{\alpha=1}^{k} \rho_\alpha \sigma^2(\tilde{\theta})(\tilde{\theta} - \theta_\alpha)^2\right\} \Big/ \sqrt{n} \\ &= \boldsymbol{l}' \boldsymbol{v} + \frac{1}{\sqrt{n}} \boldsymbol{v}' W \boldsymbol{v} + o_p(1/\sqrt{n}) \\ &= -\boldsymbol{l}' Y_2^{-1} \boldsymbol{z} - \frac{1}{2} \boldsymbol{l}' Y_2^{-3} Y_3 \boldsymbol{z}^{(2)} + \frac{1}{\sqrt{n}} \boldsymbol{z}' Y_2^{-1} W Y_2^{-1} \boldsymbol{z} + o_p(1/\sqrt{n}) . \end{aligned}$$

Since all cummulants of $y_i^{(3)}$ except the first one are $O(n^{-1})$, we may replace $y_i^{(3)}$ by $m_{(3)}^{(i)}(\theta_i)/\sqrt{n_i}$ in the above expression and still retain the expansion correct to $O_p(1/\sqrt{n})$.

The MGF of T_K under K can be obtained by the similar way as (11) by the use of (10) up to order $1/\sqrt{n}$.

$$M_K(t)=\exp\left\{\frac{t^2}{2}\boldsymbol{l}'K_2^{-1}\boldsymbol{l}\right\}\left[1+\frac{1}{\sqrt{n}}\{tb_1+t^3b_3\}+o(1/\sqrt{n})\right], \tag{15}$$

where

$$b_1=\mathrm{tr}\, K_2^{-1}W+\frac{1}{2}\sum_{i=1}^{k}\frac{m_{(3)}(\theta_i)+2m_{(21)}(\theta_i)}{(\sigma_i^2(\theta_i))^2}\frac{l_i}{\sqrt{\rho_i}},$$

$$b_3=\boldsymbol{l}'K_2^{-1}WK_2^{-1}\boldsymbol{l}+\frac{1}{6}\sum_{i=1}^{k}\frac{2m_{(3)}(\theta_i)+3m_{(21)}(\theta_i)}{(\sigma_i^2(\theta_i))^3}\frac{l_i^3}{\sqrt{\rho_i}},$$

and

$$K_2=\mathrm{diag}\,(\sigma_1^2(\theta_1),\cdots,\sigma_k^2(\theta_k)).$$

Putting $\tau_K^2=\boldsymbol{l}'K_2^{-1}\boldsymbol{l}$, the inversion of this MGF gives the following theorem.

THEOREM 3. *Under the fixed alternative K, the distribution of the statistic* $T_K=\{T-n\sigma^2(\tilde\theta)\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde\theta)^2\}/\sqrt{n}$ *can be expanded asymptotically for large n as*

$$\mathrm{P}\{T_K/\tau_K\leqq x\}=\Phi(x)-\frac{1}{\sqrt{n}}\{\Phi^{(1)}(x)b_1/\tau_K+\Phi^{(3)}(x)b_3/\tau_K^3\}+o(1/\sqrt{n}), \tag{16}$$

where b_1 *and* b_3 *are given in* (15), *and* $\Phi^{(j)}(x)$ *is the jth derivative of the standard normal distribution function.*

Under the null hypothesis H: $\theta_1=\cdots=\theta_k$, $\tau_K^2=0$, because $\boldsymbol{l}_\alpha=\boldsymbol{0}$, for all α, which implies that T has also singularity at the null hypothesis.

Example 3. T_σ has the asymptotic expansion of the distribution under the fixed alternative hypothesis as follows;

$$\mathrm{P}\{T_{k\sigma}/\tau_K\leqq x\}=\Phi(x)-\frac{1}{\sqrt{n}}\{\Phi^{(1)}(x)b_1/\tau_K+\Phi^{(3)}(x)b_3/\tau_K^3\}+o(1/n)$$

where

$$T_{k\sigma}=\left\{T_\sigma-\frac{n}{2}(A_2-1)\right\}\Big/\sqrt{n},$$

$$\tau_K^2=2\{A_4-2A_2A_3+A_2^3\}$$

$$b_1=\sum_{i=1}^{k}a_i^2-4A_3+3A_2^2$$

$$b_3=\frac{10}{3}A_6-8A_3A_4-A_2(8A_5-14A_3^2)+A_2^2\left(14A_4-\frac{64}{3}A_2A_3\right)+6A_2^5$$

$$a_i=\theta_i/\tilde{\theta}\ , \qquad A_l=\sum_{i=1}^{k}\rho_i\left(\frac{\theta_i}{\tilde{\theta}}\right)^l\ , \quad l=2, 3, 4, 5, 6\ .$$

3.3. *Local alternative hypothesis*

In 3.2 we have studied that the limiting distribution of T has the singularity at the null hypothesis so that we have to know the asymptotic behavior near the null hypothesis. In this section we consider the asymptotic behavior of T under the sequence of alternative hypothesis converging to the null hypothesis with arbitrary rate of convergence as the sample sizes tend to infinity. The sequence of alternatives K_ψ is specified as, K_ψ: $\theta_i=\theta+\varepsilon_i/\psi(n_i)$. Especially, three cases are studied, (i) $\psi(n_i)=n^{1/2}\sqrt{\rho_i}$, (ii) $\psi(n_i)=n^{3/4}\sqrt{\rho_i}$, (iii) $\psi(n_i)=n^{1/4}\sqrt{\rho_i}$.

(i) Case of $\psi(n_i)=n^{1/2}\sqrt{\rho_i}$

Under the sequence of alternatives K_ψ: $\theta_i=\theta+\varphi_i/\sqrt{n}$, $\varphi_i=\varepsilon_i/\sqrt{\rho_i}$, T is expanded asymptotically as,

$$\begin{aligned}T=&\sigma^2(v+\tilde{\Phi}\sqrt{\rho}\,)'(I-\sqrt{\rho}\,\sqrt{\rho}\,')(v+\tilde{\Phi}\sqrt{\rho}\,)\\&+\frac{1}{\sqrt{n}}\left[\frac{d\sigma^2}{d\theta}\right](\sqrt{\rho}\,'v+\tilde{\varphi})(v+\tilde{\Phi}\sqrt{\rho}\,)'(I-\sqrt{\rho}\,\sqrt{\rho}\,')(v+\tilde{\Phi}\sqrt{\rho}\,)\\&+o_p(1/\sqrt{n})\ .\end{aligned}$$

Replacing v by (8) and setting $y_i^{(3)}(\theta_i)$ with $m_{(3)}^{(i)}(\theta_i)/\sqrt{n_i}$, we have the following asymptotic expression of T;

$$\begin{aligned}T=&\sigma^2(z-Y_2\tilde{\Phi}\sqrt{\rho}\,)'Y_2^{-1}(I-\sqrt{\rho}\,\sqrt{\rho}\,')Y_2^{-1}(z-Y_2\tilde{\Phi}\sqrt{\rho}\,)\\&+\frac{1}{\sqrt{n}}\Big\{\sigma^2 m_{(3)}(D_\sigma^{-1}z+\tilde{\Phi}\sqrt{\rho}\,)'(I-\sqrt{\rho}\,\sqrt{\rho}\,')D_\sigma^{-3}P^{-1/2}z^{(2)}\\&+\left[\frac{d\sigma^2}{d\theta}\right](\sqrt{\rho}\,'D_\sigma^{-1}z+\tilde{\varphi})(D_\sigma^{-1}z+\tilde{\Phi}\sqrt{\rho}\,)'(I-\sqrt{\rho}\,\sqrt{\rho}\,')\\&\cdot(D_\sigma^{-1}z+\tilde{\Phi}\sqrt{\rho}\,)\Big\}+o_p(1/\sqrt{n})\ .\end{aligned}$$

Using Lemma 1 and expanding $\sigma^2(\theta_i)$ at $\theta_i=\theta$ in Taylor series,

$$D_\sigma^{-1}=\frac{1}{\sigma^2}I-\frac{1}{\sqrt{n}}\frac{1}{\sigma^4}\left[\frac{d\sigma^2}{d\theta}\right]\Phi+o\left(\frac{1}{\sqrt{n}}\right)\ , \qquad \Phi=\mathrm{diag}\,(\varphi_1, \varphi_2, \cdots, \varphi_k)\ ,$$

the MGF of T will be expressed after some lengthy algebra as follows;

$$(17)\quad M_3(t)=(1-2t)^{-(k-1)/2}\exp\left\{\frac{\sigma^2 t}{1-2t}\nu_2\right\}\left[1+\frac{1}{\sqrt{n}}\sum_{j=0}^{3}\tilde{a}_j(1-2t)^{-j}+o\left(\frac{1}{\sqrt{n}}\right)\right]\ ,$$

where

$$\tilde{a}_3=\frac{\nu_3}{6}(m_{(3)}-m_{(1^3)})\ ,$$

$$\tilde{a}_2=\frac{\nu_3}{2}(m_{(21)}+m_{(1^3)})+\frac{1}{2\sigma^2}(m_{(3)}-m_{(1^3)})(\bar{\varphi}-k\tilde{\varphi})$$

$$\tilde{a}_1=\frac{1}{2}m_{(21)}\nu_3-\frac{1}{2}(m_{(3)}+m_{(21)})\tilde{\varphi}\nu_2-\frac{1}{2\sigma^2}(m_{(3)}-m_{(1^3)})(\bar{\varphi}-k\tilde{\varphi})$$

$$a_0=\frac{\nu_3}{6}m_{(3)}+\frac{1}{2}(m_{(3)}+m_{(21)})\tilde{\varphi}\nu_2 ,$$

$$\bar{\varphi}=\sum_{i=1}^{k}\varphi_i , \qquad \nu_l=\sum_{\alpha=1}^{k}\rho_\alpha(\varphi_\alpha-\tilde{\varphi})^l ,$$

Inverting this MGF, we have the following theorem.

THEOREM 4. *Under the sequence of alternatives* K_ϕ : $\theta_i=\theta+\varphi_i/\sqrt{n}$, $i=1, 2,\cdots, k$, *the distribution of* T *is expanded asymptotically for large* n *as*

$$\text{(18)} \qquad P\{T\leqq x\}=P_{k-1}(\lambda^2)+\frac{1}{\sqrt{n}}\sum_{j=0}^{3}\tilde{a}_jP_{k-1+2j}(\lambda^2)+o(1/\sqrt{n}) ,$$

where $\lambda^2=m_{(1^2)}\nu_2/2$, $P_f(\lambda^2)=P\{\chi_f^2(\lambda^2)\leqq x\}$ *and* $\chi_f^2(\lambda^2)$ *is a non-central* χ^2-*random variable with* f *degrees of freedom and with non-centrality parameter* λ^2. $\tilde{a}_j$*'s are given in* (17), *respectively.*

(ii) Case of $\phi=n^{3/4}\sqrt{\rho_i}$

We are able to give the asymptotic expansion of MGF of $M_4(t)$ of T for the case $\phi=n^{3/4}\sqrt{\rho_i}$ by the use of $M_3(t)$ for $\phi=n^{1/4}\sqrt{\rho_i}$. $M_3(t)$ is expanded up to order $1/\sqrt{n}$ and the next term should be order $1/n$, of which fact can be checked from the derivation of $M_3(t)$. This implies that we obtain the asymptotic expansion of the MGF of T up to order $n^{3/4}$ if we replace φ_i with $\varphi_i/\sqrt[4]{n}$ in $M_3(t)$. Therefore, $M_4(t)$ of T for $\phi=n^{3/4}\sqrt{\rho_i}$ should be expanded as the following form:

$$M_4(t)=(1-2t)^{-(k-1)/2}\left\{1+\frac{m_{(1^2)}\nu_2}{2\sqrt{n}}\{(1-2t)^{-1}-1\}+o(1/\sqrt{n})\right\}$$

since $\tilde{a}_j$'s in $M_3(t)$ are homogeneous polynomials of degree 3 with respect to φ_i, which implies that order $\tilde{a}_j$'s are $O(1/\sqrt[4]{n^3})$ if we replace φ_i with $\varphi_i/\sqrt[4]{n}$.

THEOREM 5. *Under the sequence of the alternatives* K_ϕ : $\theta_i=\theta_i+\varphi_i/\sqrt[4]{n^3}$, $i=1, 2,\cdots, k$, *the distribution of* T *is expanded asymptotically for large* n *as*

$$\text{(19)} \qquad P\{T\leqq x\}=P_{k-1}+\frac{m_{(1^2)}\nu_2}{2\sqrt{n}}\{P_{k+1}-P_{k-1}\}+o(1/\sqrt{n}) ,$$

where $P_f=P(\chi_f^2\leqq x)$ *and* χ_f^2 *is a central* χ^2 *random variable with* f *de-*

grees of freedom.

Note. It is interesting to note that the expression (19) is same form as the expression (37) in Hayakawa [4]. This means that the likelihood ratio criterion for testing the homogeneity of parameters has the same behavior up to order $1/\sqrt{n}$ when the rate of convergence is $n^{3/4}$.

(iii) Case of $\phi=n^{1/4}\sqrt{\rho_i}$

Under the sequence of alternatives K_ϕ: $\theta_i=\theta+\varphi_i/\sqrt[4]{n}$, T can be expanded asymptotically as following form:

$$(20)\qquad \begin{aligned} T^* &= n^{-1/4}\left\{T-\sqrt{n}\,\sigma^2\nu_2-\sqrt[4]{n}\,\tilde{\varphi}\nu_2\left[\frac{d\sigma^2}{d\theta}\right]\right\} \\ &= -2\sigma^2\sqrt{\rho}'Y_2^{-1}z+\frac{1}{\sqrt[4]{n}}\left\{\sigma^2 z'Y_2^{-1}(I-\sqrt{\rho}\,\sqrt{\rho}')Y_2^{-1}z\right. \\ &\quad \left.-\left[\frac{d\sigma^2}{d\theta}\right](\nu_2\sqrt{\rho}'Y_2^{-1}z+2\tilde{\varphi}\sqrt{\rho}'\Phi Y_2^{-1}z)+\frac{1}{2}\left[\frac{d^2\sigma^2}{d\theta^2}\right]\tilde{\varphi}^2\nu_2\right\} \\ &\quad +o_p(1/\sqrt{n})\,. \end{aligned}$$

Using Lemma 1 and

$$D_\sigma^{-1}=\frac{I}{\sigma^2}-\frac{1}{\sqrt[4]{n}}\left[\frac{d\sigma^2}{d\theta}\right]\frac{\Phi}{\sigma^4}+o\left(\frac{1}{\sqrt[4]{n}}\right),$$

we have the moment generating function of T^* after some lengthy algebra as follows:

$$(21)\qquad \begin{aligned} M_5(t) &= \mathrm{E}\left[\exp\left(tT^*\right)\right] \\ &= \exp\left\{2\sigma^2\nu_2 t^2\right\}\left[1+\frac{1}{\sqrt[4]{n}}\sum_{j=1}^{3}\bar{a}_j t^j+o(1/\sqrt[4]{n})\right], \end{aligned}$$

where

$$\bar{a}_1=k-1+\frac{1}{2}\tilde{\varphi}^2\nu_2\left[\frac{d^2\sigma^2}{d\theta^2}\right],$$

$$\bar{a}_2=-2\left[\frac{d\sigma^2}{d\theta}\right](\nu_3-\tilde{\varphi}\nu_2)$$

$$\bar{a}_3=4\sigma^4\nu_2\,.$$

Putting $\tau^2=4\sigma^2\nu_2$, the inversion of $M_5(t)$ gives the following theorem.

THEOREM 6. *Under the sequence of alternatives* K_φ: $\theta_0=\theta+\varphi_i/\sqrt[4]{n}$, $i=1,2,\cdots,k$, *the distribution of the statistic* T^* *given in* (20) *can be expanded asymptotically for large* n *as*

$$\mathrm{P}\{T^*/\tau\leqq x\}=\Phi(x)-\frac{1}{\sqrt[4]{n}}\sum_{j=1}^{3}\bar{a}_j\Phi^{(j)}(x)/\tau^j+o\left(\frac{1}{\sqrt{n}}\right)$$

and $\bar{a}_j$*'s are in* (21).

4. Moments and percentage point of T_σ

We have studied the asymptotic expansions of the distribution of T under various hypothesis in a previous section. In this section we discuss the percentage point of T_σ. Under the null hypothesis $H: \sigma_1=\cdots=\sigma_k$, the statistic T_σ can be represented as

$$T_\sigma=\frac{n}{2}\left\{\sum_{i=1}^{k}\frac{z_i^2}{\rho_i}-1\right\}$$

where $\{z_1, z_2, \cdots, z_k\}$ are Dirichlet random variables with pdf

$$\frac{\Gamma(n/2)}{\prod_{i=1}^{k}\Gamma(n_i/2)}\prod_{i=1}^{k}z_i^{n_i/2-1}\left(1-\sum_{j=1}^{k-1}z_j\right)^{n_k/2-1}, \qquad 0\leqq z_i\leqq 1\,, \quad \sum_{i=1}^{k}z_i=1\,.$$

It seems to be difficult to have an exact distribution function of T_σ for arbitrary k, however, we are able to give an exact expression of a distribution of T_σ for $k=2$.

$$\mathrm{P}\{T_\sigma\leqq x\}=\mathrm{P}\left\{|z_1-\rho_1|\leqq\sqrt{\frac{2\rho_1\rho_2 x}{n}}\right\}$$

where z_1 is a Beta random variable with pdf

$$\Gamma\left(\frac{n_1+n_2}{2}\right)\Big/\left\{\Gamma\left(\frac{n_1}{2}\right)\Gamma\left(\frac{n_2}{2}\right)\right\}\cdot z_1^{n_1/2-1}(1-z_1)^{n_2/2-1}\,.$$

The exact moments of $\sum_{i=1}^{k}z_i^2/\rho_i$ can be obtained as follows.

$$a_1=\mathrm{E}\left[\sum z_i^2/\rho_i\right]=\frac{n+2k}{n+2}\,,$$

$$a_2=\mathrm{E}\left[\{\sum z_i^2/\rho_i\}^2\right]=\left[n^3+4(k+2)n^2+4k(k+10)n+48\tilde{\rho}_1\right]\Big/\prod_{i=1}^{3}(n+2i)\,,$$

$$\begin{aligned}a_3=\mathrm{E}\left[\{\sum z_i^2/\rho_i\}^3\right]=\prod_{i=1}^{5}(n+2i)^{-1}[&n^5+6(k+4)n^4\\&+(12k^2+168k+160)n^3+(8k^3+240k^2+1408k+144\tilde{\rho}_1)n^2\\&+(288k+4096)\tilde{\rho}_1 n+3840\tilde{\rho}_2]\end{aligned}$$

$$a_4=\mathrm{E}\left[\{\sum z_i^2/\rho_i\}^4\right]=\prod_{i=1}^{7}(n+2i)^{-1}[n^7+8(k+6)n^6$$

$$+(24k^2+432k+832)n^5+(32k^3+1152k^2+8832k+5376$$
$$+288\tilde{\rho}_1)n^4+\{16k^4+960k^3+16064k^2+71424k$$
$$+(1152k+18688)\tilde{\rho}_1\}n^3+\{(1152k^2+44288k+359424)\tilde{\rho}_1$$
$$+15360\tilde{\rho}_2\}n^2+\{(30720k+798720)\tilde{\rho}_2+6912\tilde{\rho}_1^2\}n$$
$$+645120\tilde{\rho}_3] ,$$

where

$$\tilde{\rho}_l=\sum_{\alpha=1}^{k}(1/\rho_\alpha)^l , \qquad l=1, 2, 3 .$$

The four central moments μ_1, μ_2, μ_3, μ_4 of T_σ are expressed in terms of a_i's, $i=1, 2, 3, 4$ as follows.

$$\mu_1=\frac{n}{2}(a_1-1)$$

$$\mu_2=\frac{n^2}{4}(a_2-a_1^2)$$

$$\mu_3=\frac{n^3}{8}(a_3-3a_1a_2+2a_1^3)$$

$$\mu_4=\frac{n^4}{16}(a_4-4a_1a_3+6a_2a_1^2-3a_1^4) .$$

Using these moments $\beta_1=\mu_3^2/\mu_2^3$ and $\beta_2=\mu_4/\mu_2^2$ are calculated and we can approximate a percentage point of T_σ by the use of Table A given by Johnson, Nixon, Amos and Pearson [6].

Example 4. When $k=5$, $n_1=n_2=\cdots=n_5=50$, using (13) of the generalized inversion formula we have the following approximate 5% point of T_σ.

First term	Term of order $1/n$	Approximate value
9.4877	−0.1164	9.371

From the exact values of moments we have

$$\mu_1=3.96825 , \quad \sqrt{\mu_2}=2.8054 , \quad \sqrt{\beta_1}=1.4752 , \quad \beta_2=6.5078$$

Table A in [6] gives 1.9276, which implies the 5% point of T_σ is

$$3.9682+2.8054\times 1.9276=9.3759 ,$$

which shows remarkable agreement with the value due to the asymptotic expansion of a percentage point.

Example 5. When $k=2$, $n_1=n_2=12$, the exact distribution of T_σ can be evaluated by a Beta distribution (23). Using a table of the percentage points of the Beta-distribution, Pearson and Hartley, Table 16, [13], and (22), 5% point of T_σ is obtained as 3.401. Since $\sqrt{\beta_1}=2.0461$, $\beta_2=7.9876$, the tables given in [6] does not cover the case $\sqrt{\beta_1}=2.0461$ which is close to the extreme value 2.0 in [6], we extrapolate the value for $(\sqrt{\beta_1}, \beta_2)=(2.0461, 7.9876)$ and obtain it as 3.400. This value also shows an agreement with the value due to incomplete Beta function.

Table of 5 percentage points of T_σ is given for $k=2(1)10$ and $\nu=1(1)10$ in the case of equal sample sizes. For $k=2$, the incomplete Beta tables are used and for $k\geqq 2$ the tables of Johnson et al. [6] are used.

Acknowledgements

The author wishes to express his gratitude to the referee for his helpful comments in revising paper. Thanks are due to Mrs. Y. Ogata for her help to performing the tabulations.

Upper 5 percentage points of T_σ

$k \backslash \nu$	1	2	3	4	5	6	7	8	9	10
2	0.99	1.80	2.31	2.63	2.84	2.99	3.10	3.19	3.26	3.31
3	2.68	3.84	4.38	4.70	4.90	5.05	5.16	5.25	5.31	5.37
4	4.50	5.81	6.36	6.66	6.85	6.98	7.08	7.16	7.22	7.27
5	6.36	7.72	8.25	8.52	8.69	8.80	8.88	8.95	9.00	9.04
6	8.23	9.58	10.06	10.30	10.44	10.53	10.60	10.65	10.69	10.72
7	10.09	11.39	11.81	12.01	12.12	12.20	12.25	12.28	12.31	12.34
8	11.92	13.15	13.51	13.67	13.75	13.81	13.84	13.87	13.89	13.91
9	13.73	14.86	15.16	15.28	15.34	15.37	15.40	15.41	15.42	15.43
10	15.52	16.53	16.77	16.85	16.89	16.91	16.91	16.92	16.92	16.92

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