A NOTE ON DISTRIBUTION-FREE CONFIDENCE BOUNDS FOR $\mathrm{P}(X<Y)$ WHEN X AND Y ARE DEPENDENT

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If X and Y are two independent and continuous random variables, Govindarajulu [1] has considered the problem of obtaining distribution-free confidence bounds for $\mathrm{P}(X<Y)$ based on random samples of sizes m and n from the X and Y populations respectively. It is of interest to consider the case when X and Y are dependent. Let $(X_1, Y_1), \cdots, (X_n, Y_n)$ be a random sample from the joint distribution of X and Y. Let H denote the unknown distribution of $X-Y$. Then $p=\mathrm{P}(X<Y)=H(0)$. A point estimate of p is given by

$$(1) \qquad \hat{p}=H_n(0), \qquad \text{where } H_n(0)=(\text{number of } Z_i=X_i-Y_i\leqq 0)/n .$$

Then, $n^{1/2}(\hat{p}-p)=n^{1/2}[H_n(0)-H(0)]$ is asymptotically normal with mean 0 and variance $\sigma^2=H(0)[1-H(0)]$.

It can easily be verified that an unbiased estimate of σ^2 is $\hat{\sigma}^2=[n/(n-1)]H_n(0)[1-H_n(0)]$. Hence a consistent estimate of σ^2 is $H_n(0)[1-H_n(0)]$. Also since $\sigma^2\leqq 1/4$ a solution of the equation $\mathrm{P}(p\leqq\hat{p}+\varepsilon)\geqq\gamma$ is given by

$$(2) \qquad \varepsilon\geqq(4n)^{-1/2}\Phi^{-1}(\gamma)$$

and the solution of the equation $\mathrm{P}(|\hat{p}-p|\leqq\varepsilon)\geqq\gamma$, $0<\gamma<1$ is given by

$$(3) \qquad \varepsilon\geqq(4n)^{-1/2}\Phi^{-1}((1+\gamma)/2) ,$$

when Φ denotes the standard normal distribution function. If n is random and there exists a positive integer N such that n/N converges to a positive constant λ in pr., then the factor $(4n)^{-1/2}$ occurring in (2) and (3) should be replaced by $(4N\lambda)^{-1/2}$. Using an unbiased (a consistent) estimate for σ^2 we obtain

$$(2)' \qquad \varepsilon\geqq(\hat{\sigma}^2/n)^{1/2}\Phi^{-1}(\gamma) ,$$

$$(3)' \qquad \varepsilon\geqq(\hat{\sigma}^2/n)^{1/2}\Phi^{-1}((1+\gamma)/2)$$

for the one- and two-sided cases respectively, where $\hat{\sigma}^2$ denotes either the unbiased or the consistent estimate of σ^2. The bounds in (2) and (3) ((2)′ and (3)′) are distribution-free (asymptotically distribution-free). Further, the bounds given by (2)′ and (3)′ would be shorter than those in (2) and (3) respectively. It should be of interest to note that the bounds in (2) and (3) coincide with the upper bounds obtained earlier by the author when X and Y are independent.

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REFERENCE

[1] Govindarajulu, Zakkula (1968). Distribution-free confidence bounds for $P(X<Y)$, *Ann. Inst. Statist. Math.*, **20**, 229–238.