SOME INEQUALITIES BASED ON INVERSE FACTORIAL SERIES

T. MATSUNAWA

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Summary

Fairly sharp bounds (lower and upper) of the quantities $\log \Gamma(x+1)$, $\log(1 \pm x) \sum_{i=1}^{p} 1/(x+i)$ and $\sum_{i=1}^{p} 1/(x+i)^2$ are given by evaluating the corresponding series of inverse factorials. These results are useful in the asymptotic theory of order statistics and record value statistics and also in the elementary analytic number theory, with which the quantities frequently concerned.

1. Introduction

Let $g(x)$ be a real valued function in real x. Consider a situation in which the functional form of $g(x)$ is known but is not so easy to handle it exactly. In such a case we must usually make an effort to search a good approximation for $g(x)$. For example, as the approximation, we may use a few first terms of a certain asymptotic expansion for $g(x)$. There often occurs, however, that we are not satisfied with the asymptotic expression for $g(x)$, and then our desire gives rise to find some accurate upper and lower bounds for the function.

The purpose of this article is to give some exact evaluations for the familiar quantities mentioned before, by means of estimating the following convenient but less known infinite series (cf. [4], [5]).

Suppose that $g(x)$ has the convergent series of the form

$$g(x) = \sum_{i=i_0}^{\infty} \frac{a_i}{(x+1)(x+2)\cdots(x+i)}, \qquad x \in D, \tag{1.1}$$

where D denotes the domain of convergence for $g(x)$ and a_i's $(i=i_0, i_0+1, \cdots)$ are real constants independent of x. In such a case we shall say that $g(x)$ is expressible in *the inverse factorial series*. This type of series for certain quantities is useful for deriving exact lower and upper bounds of them in the later sections.

The main steps to obtain the bounds for $g(x)$ are

(i) to give the corresponding inverse factorial expansion to the quantity $g(x)$ under consideration,

(ii) to evaluate the coefficients a_i's by a double inequality of the form

$$\underline{a}_i \leq a_i \leq \bar{a}_i\,, \qquad (i \geq i_0) \tag{1.2}$$

and finally,

(iii) to calculate the majorant (the minorant) of the $g(x)$ with the coefficient $\bar{a}_i(\underline{a}_i)$ $(i \geq i_0)$ by the aids of Lemma 3.2 below, and hence we obtain

$$\underline{g}(x) \leq g(x) \leq \bar{g}(x)\,, \qquad x \in D\,. \tag{1.3}$$

So far as the author is aware of, the inequalities presented below have not appeared yet in literature. The use of our results in this article may be recommended, from the view point of their accuracy and passable easiness of computations, in various approximation problems (e.g. [2]).

The first result in Theorem 2.1 sharpens the well-known Stirling asymptotic formula for natural numbers by presenting a double inequality, which also become an improvement of H. Robbins' evaluation [3]. The second and the third inequality for the quantity $\log(1 \pm x)$ in the theorem are useful, too, because of their sharper bounds. Thus the results are sometimes more convenient than those obtained by using the usual Maclaurin's formula. Other inequalities obtained in the theorem are related to the sum of reciprocals, which have also wide applicabilities in the theory of record values and that of analytic number theory. The main results are presented firstly in the following section. In Section 3 some necessary lemmas are stated. The proofs of the main inequalities are carried out in the final section.

2. The main results

We list up the inequalities in the following theorem based on the inverse factorial expansions introduced in the preceding section. The proof of the theorem will be given in Section 4.

THEOREM 2.1.

(i) *For positive number $x \geq 2$, it holds that*

$$\log \Gamma(x+1) = \frac{1}{2} \log 2\pi + \left(x + \frac{1}{2}\right) \log x - x + \frac{1}{12x} - R(x)\,, \tag{2.1}$$

with

(2.2) $$\underline{R}(x) < R(x) < \bar{R}(x)\,,$$

where

(2.3) $$R(x) = \sum_{i=2}^{\infty} \frac{a_{i+1}}{x(x+1)(x+2)\cdots(x+i)}\,,$$

(2.4) $$a_r = \frac{1}{r}\int_0^1 t(1-t)(2-t)\cdots(r-1-t)\left(\frac{1}{2}-t\right)dt\,, \qquad (r \geq 2),$$

(2.5) $$\underline{R}(x) = \frac{1}{360x(x-1)(x+1)} - \frac{1}{120x^2(x-1)(x+1)}\,,$$

and

(2.6) $$\bar{R}(x) = \frac{1}{360x(x-1)(x+1)} + \frac{11}{480x^2(x-1)(x+1)}\,.$$

(ii) *For real number* $u > 0$, *it holds that*

(2.7) $$\log\left(1+\frac{1}{u}\right) = \frac{1}{u} - S(u)$$

with

(2.8) $$\underline{S}(u) < S(u) < \bar{S}(u)\,,$$

where

(2.9) $$S(u) = \sum_{i=1}^{\infty} \frac{b_i}{u(u+1)(u+2)\cdots(u+i)}\,,$$

(2.10) $$b_1 = \frac{1}{2}\,, \quad b_r = \int_0^1 t(1-t)(2-t)\cdots(r-1-t)dt\,, \quad (r \geq 2)\,,$$

(2.11) $$\underline{S}(u) = \frac{1}{2u(u+1)} + \frac{1}{6u(u+1)^2}$$

and

(2.12) $$\bar{S}(u) = \frac{1}{2u(u+1)} + \frac{1}{6u^2(u+1)}\,.$$

(iii) *For real number* $v > 1$, *it holds that*

(2.13) $$\log\left(1-\frac{1}{v}\right) = -\frac{1}{v-1} + S^*(v)\,.$$

with

(2.14) $$\underline{S}^*(v) < S^*(v) < \bar{S}^*(v)\,,$$

where

$$S^*(v)=\sum_{i=1}^{\infty}\frac{b_i}{(v-1)v(v+1)(v+2)\cdots(v+i-1)}\,, \tag{2.15}$$

b_r's being the same as in (2.10),

$$\underline{S}^*(v)=\frac{1}{2v(v-1)}+\frac{1}{6v^2(v-1)} \tag{2.16}$$

and

$$\bar{S}^*(v)=\frac{1}{2v(v-1)}+\frac{1}{6v(v-1)^2}\,. \tag{2.17}$$

(iv) *For real number $x\geq 0$ and positive integer $p\geq 2$, it holds that*

$$\sum_{i=1}^{p}\frac{1}{x+i}=\log\frac{x+p}{x+1}+\frac{1}{2}\left(\frac{1}{x+p}+\frac{1}{x+1}\right)-\frac{T(x+p)}{x+p}+\frac{T(x+1)}{x+1} \tag{2.18}$$

with

$$\bar{T}(s)/6<\underline{T}(s)<T(s)<\bar{T}(s)\,, \tag{2.19}$$

where

$$T(s)=\sum_{i=1}^{\infty}\frac{c_{i+1}}{(s+1)(s+2)\cdots(s+i)}\,,\qquad (s>1), \tag{2.20}$$

$$c_r=\frac{1}{r}\int_0^1 t(1-t)(2-t)\cdots(r-1-t)dt\,,\qquad (r\geq 2), \tag{2.21}$$

$$\underline{T}(s)=\frac{1}{12(s-1)}-\frac{1}{6(s-1)(s+1)}-\frac{1}{6(s-1)(s+1)(s+2)} \tag{2.22}$$

and

$$\bar{T}(s)=\frac{1}{12(s-1)}\,. \tag{2.23}$$

(v) *For real number $x\geq 0$ and positive integer $p\geq 2$, it holds that*

$$\sum_{i=1}^{p}\frac{1}{(x+i)^2}=U(x+1)-U(x+p+1) \tag{2.24}$$

with

$$\underline{U}(s)<U(s)<\bar{U}(s)\,, \tag{2.25}$$

where

$$U(s)=\sum_{i=1}^{\infty}\frac{d_i}{s(s+1)(s+2)\cdots(s+i-1)}\,, \qquad (s>1), \tag{2.26}$$

$$d_r=\Gamma(r)/r\,, \qquad r=1, 2, \cdots, \tag{2.27}$$

$$\underline{U}(s)=\frac{1}{s}+\frac{1}{2s(s+1)}+\frac{2}{3s^2(s+1)} \tag{2.28}$$

and

$$\bar{U}(s)=\frac{1}{s}+\frac{1}{2s(s+1)}+\frac{2}{3s(s-1)(s+1)}\,. \tag{2.29}$$

From (iv) and (v) we can easily obtain the following result given by Schlömilch [4]:

COROLLARY 2.1.
(i) *For positive integer $p\geq 2$, it holds that*

$$\sum_{i=1}^{p}\frac{1}{i}=C+\log p+\frac{1}{2p}-\frac{1}{p}T(p) \tag{2.30}$$

where C denotes the Euler constant $(=0.5772157\cdots)$.
(ii) *For positive integer $p\geq 2$, it holds that*

$$\sum_{i=1}^{p}\frac{1}{i^2}=\frac{\pi^2}{6}-U(p+1)\,. \tag{2.31}$$

3. Necessary lemmas

In this section we state some lemmas which play fundamental roles for the calculations in the proof of the theorem. First, we prove the following.

LEMMA 3.1. *Let $g(x; m, n)$ be a real-valued function of the form*

$$g(x; m, n)=\int_0^{\infty} e^{-xt}(1-e^{-t})^{-m}t^{-n}dt\,, \tag{3.1}$$

where x is positive numbers belonging to the domain D, m any real number, and n takes value 1 or 2. Then, we can rewrite the integral (3.1) into a more convenient form:

$$g(x; m, n)=\frac{(-1)^n}{(n-1)!}\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}\int_0^1(1-y)^{x-1}y^{-m}h_i(y; n)dy \tag{3.2}$$

where $\gamma_0=-1$, $\gamma_1=1/2$ and

$$\gamma_i=\int_0^1 t(1-t)\cdots(i-1-t)dt\,, \qquad i\geq 2\,, \tag{3.3}$$

and where

$$h_i(y;1)=y^{i-1} \tag{3.4}$$
$$h_i(y;2)=(i-1)y^{i-2}(1-y)\,.$$

Remark. The above lemma is closely related to the Schlömilch theorem on the expansions of the series of inverse factorials for a certain analytic function (Cf. [5], pp. 142), but unfortunately we cannot directly follow the theorem in our real-valued situation, chiefly because of the non-existence of the inequality like Cauchy's one for the derivatives of analytic functions. Conversely, our proof below can be applicable to the complex statement corresponding to this lemma.

PROOF. Making the following transformation for $0\le y\le 1$,

$$e^{-t}=1-y, \text{ and hence } t=-\ln(1-y), \tag{3.5}$$

we have from (3.1)

$$g(x;m,n)=\lim_{\varepsilon\to+0}\int_0^{1-\varepsilon}(1-y)^{x-1}y^{-m}[-\ln(1-y)]^{-n}dy\,. \tag{3.6}$$

Now, let us modify the term $[-\ln(1-y)]^{-n}$ in the integrand. Note here that for any real $u\in[0,1]$ and $y\in[0,1-\varepsilon]$

$$\int_0^1(1-y)^u du=\frac{-y}{\ln(1-y)}\,, \tag{3.7}$$

and that, since $0\le y<1$, by the generalized binomial theorem we can represent the integrand in (3.7) as

$$(1-y)^u=\sum_{i=0}^{\infty}(-1)^i\binom{u}{y}y^i\equiv\sum_{i=0}^{\infty}\phi_i(u;y)\,. \tag{3.8}$$

Since, for each i and any temporarily fixed y, $\phi_i(u;y)$ is a continuous function of u over the closed interval $[0,1]$ and the infinite series $\sum_{i=0}^{\infty}\phi_i(u;y)$ converges uniformly over the interval to the continuous function of u, $(1-y)^u$, by (3.8), we may carry out the integral term-by-term;

$$\int_0^1(1-y)^u du=\sum_{i=0}^{\infty}\left[\int_0^1(-1)^i\binom{u}{i}du\right]y^i\equiv-\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}y^i\,, \tag{3.9}$$

where $\gamma_0=-1$, $\gamma_1=1/2$ and

$$\gamma_i=\int_0^1 t(1-t)\cdots(i-1-t)dt\,,\qquad i\ge 2\,. \tag{3.10}$$

Then, from (3.7) and (3.9), we have

$$(3.11)\qquad [-\ln(1-y)]^{-1}=-\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}y^{i-1}\,,\qquad y\in[0,1-\varepsilon]\,.$$

Well, since $\varphi_i(y)=(\gamma_i/i!)y^{i-1}$ is continuously differentiable function of y over $[0,1-\varepsilon]$ and $\sum_{i=0}^{\infty}\varphi_i'(y)$ $(n=1,2)$ converges uniformly over $[0,1-\varepsilon]$ by the Weierstrass theorem with noticing the inequality $|\gamma_i|\leq\Gamma(i)/6$ $(i\geq 2)$ proved later, the series $\sum_{i=0}^{\infty}\varphi_i(y)$ is also differentiable and its derivative coincides with $\sum_{i=0}^{\infty}\varphi_i'(y)$. Thus, we have for $y\in[0,1-\varepsilon]$,

$$(3.12)\qquad [-\ln(1-y)]^{-2}=\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}(1-y)\frac{d}{dy}y^{i-1}=\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}(i-1)y^{i-2}(1-y)\,.$$

Hence, noticing the fact that the series in (3.12) is also uniformly convergent, we immediately obtain the formula (3.2) from (3.6), (3.11) and (3.12), which completes the proof of the lemma.

Secondly, we show the inequalities for a_r, b_r and c_r defined in (2.4), (2.10) and (2.21), respectively:

LEMMA 3.2.
(i) *For $r\geq 3$, it holds that*

$$(3.13)\qquad \frac{\Gamma(r-1)}{120r}\leq a_r\leq\frac{\Gamma(r-1)}{64}\left(1-\frac{37}{15r}\right).$$

(ii) *For $r\geq 2$, it holds that*

$$(3.14)\qquad \frac{\Gamma(r-1)}{6}\leq b_r\leq\frac{\Gamma(r)}{6}\,,$$

and hence for $r\geq 2$

$$(3.15)\qquad \frac{\Gamma(r-1)}{6r}\leq c_r\leq\frac{\Gamma(r)}{6r}\,.$$

PROOF. For each $r\geq 3$, decomposing the integral in (2.4) in such a way as

$$a_r=\frac{1}{r}\left[\int_0^{1/2}x(1-x)(2-x)\cdots(r-1-x)\left(\frac{1}{2}-x\right)dx\right.$$
$$\left.-\int_{1/2}^{1}x(1-x)\cdots(r-1-x)\left(x-\frac{1}{2}\right)dx\right],$$

and making the substitution $y=1-x$ in the second part of the RHS of the above, we can express a_r as

$$(3.16)\quad a_r=\frac{1}{r}\int_0^{1/2} y(1-y)\left(\frac{1}{2}-y\right)[(2-y)(3-y)\cdots(r-1-y) \\ -(1+y)(2+y)\cdots(r-2+y)]dy\,.$$

Then, it can be easily seen that

$$a_r>\frac{1}{r}\left[\frac{3}{2}\cdot\frac{5}{2}\cdot\ \cdots\ \cdot\frac{2(r-1)-1}{2}-\frac{3}{2}\cdot\frac{5}{2}\cdot\ \cdots\ \cdot\frac{2(r-2)+1}{2}\right] \\ \cdot\int_0^{1/2} y(1-y)\left(\frac{1}{2}-y\right)dy=0\,,$$

and that

$$a_r<\frac{1}{r}[2\cdot 3\cdot\ \cdots\ \cdot(r-1)-1\cdot 2\cdot\ \cdots\ \cdot(r-2)] \\ \cdot\int_0^{1/2} y(1-y)\left(\frac{1}{2}-y\right)dy=\frac{(r-2)\Gamma(r-1)}{64r}\,.$$

Thus, it follows that

$$(3.17)\qquad 0\le a_r\le\frac{(r-2)\Gamma(r-1)}{64r}\equiv a_r^*\,,\qquad r\ge 3\,.$$

The above lower bound zero, however, is not desirable to our purpose; we try to find a better bound than the above. Let us assume temporarily that $r\ge 4$, then we have

$$(3.18)\quad a_r>\frac{1}{r}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\ \cdots\ \cdot\frac{2r-3}{2}\int_0^{1/2} y(1-y)\left(\frac{1}{2}-y\right)[(2-y)-(1+y)]dy \\ =\frac{1}{120r}\left(\frac{5}{2}\cdot\frac{7}{2}\cdot\ \cdots\ \cdot\frac{2r-3}{2}\right)>\frac{\Gamma(r-1)}{120r}\,.$$

On the other hand, we have for $r\ge 4$

$$(3.19)\quad a_r\le\frac{1}{r}\cdot 2\cdot 3\cdot\ \cdots\ \cdot(r-2)\int_0^{1/2} y(1-y)\left(\frac{1}{2}-y\right)[(r-1-y)-(1+y)]dy \\ =\frac{\Gamma(r-1)}{64}\left(1-\frac{37}{15r}\right)\ (<a_r^*).$$

It should be noted that the bounds in (3.18) and (3.19) can be applied even when $r=3$. In fact it is easily calculated that $a_3=1/360$, which coincides with both the lower bound in (3.18) and the upper bound in (3.19) when putting formally $r=3$ in both bounds. Thus, we obtained the statement (i) of the lemma.

The proof of the case (ii) can be accomplished by the same manner and will be omitted here.

Finally we list up two formulae which are useful in subsequent discussions (cf. Lemma A.1 in [1]).

LEMMA 3.3.

(i) *For real number* $x>0$, *it holds that*

$$\sum_{i=1}^{\infty} \frac{\Gamma(i)}{x(x+1)\cdots(x+i)}=\frac{1}{x^2}. \tag{3.20}$$

(ii) *For real number* $x\geq 2$, *it holds that*

$$\sum_{i=0}^{\infty} \frac{\Gamma(i+1)}{\Gamma(x+i+1)}=\frac{1}{(x-1)\Gamma(x)}. \tag{3.21}$$

PROOF.

(i) By induction it is easily verified that for natural number n

$$\frac{1}{x^2}-\sum_{i=1}^{n} \frac{\Gamma(i)}{x(x+1)\cdots(x+i)}=\frac{1}{x^2}\cdot\frac{\Gamma(n+1)}{(x+1)(x+2)\cdots(x+n)}. \tag{3.22}$$

Thus, letting $n\to\infty$ we immediately obtain (3.20).

(ii) For natural number n we can also see the following equality

$$\begin{aligned}\sum_{i=0}^{n} \frac{\Gamma(i+1)}{\Gamma(x+i+1)}&=\sum_{i=0}^{n} \frac{\Gamma(i+1)}{(x+i)\Gamma(x+i)}\\ &=\frac{1}{x-1}\sum_{j=1}^{n+1}\left\{\frac{\Gamma(j)}{(x+j-2)\Gamma(x+j-2)}\right.\\ &\qquad\left.-\frac{\Gamma(j+1)}{(x+j-1)\Gamma(x+j-1)}\right\}\\ &=\frac{1}{x-1}\left\{\frac{1}{\Gamma(x)}-\frac{1}{x+n}\,\frac{\Gamma(n+2)}{\Gamma(n+x)}\right\}.\end{aligned} \tag{3.23}$$

Accordingly, by letting $n\to\infty$, we obtain (3.21).

4. The proof of Theorem 2.1

In this section we prove the inequalities of Theorem 2.1 according to the steps described in the introductory section.

4.1. *The proof of the inequality* (2.2)

The formula (2.1) with (2.3) was given by Schlömilch [4], when x is any positive integer greater than or equal to 2.

By Binet's first expression for $\log\Gamma(\cdot)$ in terms of an infinite integral (cf. [5], p. 248), we may express

$$\log\Gamma(x+1)=\frac{1}{2}\log 2\pi+\left(x+\frac{1}{2}\right)\log x-x+R^*(x), \tag{4.1}$$

where

$$R^*(x) \equiv \int_0^\infty \left(\frac{1}{2} - \frac{1}{t} + \frac{1}{e^t - 1} \right) \frac{e^{-tx}}{t} dt . \tag{4.2}$$

Then, our problem is to evaluate the remainder term $R^*(x)$, which is easily reduced to

$$R^*(x) = \frac{1}{2} g(x; 0, 1) - g(x; 0, 2) + g(x+1; 1, 1) . \tag{4.3}$$

Here, in view of Lemma 3.1, we have

$$g(x; 0, 1) = \int_0^1 \frac{(1-y)^{x-1}}{y} dy - \frac{1}{2x} - \sum_{i=2}^\infty \frac{\gamma_i}{ix(x+1)\cdots(x+i-1)} , \tag{4.4}$$

$$\begin{aligned} g(x; 0, 2) = &\int_0^1 \frac{(1-y)^{x-1}}{y^2} dy - \int_0^1 \frac{(1-y)^{x-1}}{y} dy \\ &+ \sum_{i=2}^\infty \frac{\gamma_i}{ix(x+1)\cdots(x+i-2)} \left(1 - \frac{i-1}{x+i-1} \right) , \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} g(x+1; 1, 1) = &\int_0^1 \frac{(1-y)^{x-1}}{y^2} dy - \frac{3}{2} \int_0^1 \frac{(1-y)^{x-1}}{y} dy + \frac{1}{3x} \\ &+ \sum_{i=2}^\infty \frac{1}{ix(x+1)\cdots(x+i-2)} \left(\gamma_i - \frac{\gamma_{i+1} - \gamma_i}{x+i-1} \right) . \end{aligned} \tag{4.6}$$

From (4.3)–(4.6), it follows that

$$R^*(x) = \frac{1}{12x} - \sum_{i=2}^\infty \frac{a_{i+1}}{x(x+1)\cdots(x+i)} , \tag{4.7}$$

where for $r \geq 3$

$$a_r = \frac{1}{r} \left\{ \gamma_{r+1} - \left(r - \frac{1}{2} \right) \gamma_r \right\} = \frac{1}{r} \int_0^1 t(1-t)\cdots(r-1-t) \left(\frac{1}{2} - t \right) dt . \tag{4.8}$$

Thus, by (4.1) and (4.7), the formula (2.1) with (2.3) immediately follows.

We are now in a position to prove the inequality (2.2). By (i) of Lemma 3.2 and (i) of Lemma 3.3,

$$\begin{aligned} R(x) &\geq \frac{1}{120} \sum_{i=3}^\infty \frac{\Gamma(i)}{x(x+1)\cdots(x+i-1)i} \\ &= \frac{x-1}{120} \left\{ \sum_{i=3}^\infty \frac{\Gamma(i)}{(x-1)x(x+1)\cdots(x+i-1)(i-1)} \right. \end{aligned} \tag{4.9}$$

$$-\sum_{i=3}^{\infty}\frac{\Gamma(i)}{(x-1)x(x+1)\cdots(x+i-1)i}\Bigg\}$$

$$\geq\frac{1}{120}\sum_{i=2}^{\infty}\frac{\Gamma(i)}{x(x+1)\cdots(x+i)}-\frac{1}{360}\sum_{i=3}^{\infty}\frac{\Gamma(i)}{x(x+1)\cdots(x+i-1)},$$

by (i) of Lemma 3.3,

$$=\frac{1}{120}\left[\frac{1}{x^2}-\frac{1}{x(x+1)}\right]-\frac{1}{360}\left[\frac{1}{x-1}-\frac{1}{x}-\frac{1}{x(x+1)}\right]$$

$$=\frac{1}{360x(x-1)(x+1)}-\frac{1}{120}\left(\frac{1}{x^2-1}-\frac{1}{x^2}\right)=\underline{R}(x).$$

On the other hand, by (i) of Lemma 3.2 and (i) of Lemma 3.3, we have

$$R(x)\leq\frac{1}{64}\sum_{i=3}^{\infty}\frac{\Gamma(i-1)}{x(x+1)\cdots(x+i-1)}-\frac{37}{960}\sum_{i=3}^{\infty}\frac{\Gamma(i-1)}{x(x+1)\cdots(x+i-1)i} \tag{4.10}$$

$$\leq\frac{1}{64}\left[\frac{1}{x^2}-\frac{1}{x(x+1)}\right]-\frac{37}{960}\cdot120\underline{R}(x)$$

$$=\frac{1}{360x(x-1)(x+1)}+\frac{11}{480x^2(x-1)(x+1)}=\bar{R}(x),$$

which completes the proof of (i) of Theorem 2.1.

Remark 4.1. For any positive *integer* n, H. Robbins [3] gave the following inequality

$$0<R(n)<\frac{1}{12n}-\frac{1}{12n+1}. \tag{4.11}$$

Comparing this bounds with ours in (2.2) with (2.5) and (2.6), the latter ones are more accurate than the former, except for $n\leqq 2$. Note further that our bounds in (2.2) are also applicable for any *real* number $x>1$.

4.2. *The proof of the inequality* (2.8)

Since, for $b>a>0$ and an arbitrary $\varepsilon>0$

$$\int_{\varepsilon}^{\infty}\frac{e^{-at}-e^{-bt}}{t}dt=\int_{a\varepsilon}^{b\varepsilon}\frac{e^{-u}}{u}du=e^{-\theta}\int_{a\varepsilon}^{b\varepsilon}\frac{1}{u}du=e^{-\theta}\log\frac{b}{a},\qquad(a\varepsilon<\theta<b\varepsilon),$$

then by letting $\varepsilon\to+0$, it follows that

$$\log\frac{b}{a}=\int_0^{\infty}(e^{-at}-e^{-bt})\frac{dt}{t},\qquad(a>0,\ b>0). \tag{4.12}$$

Making use of the above formula, we have for $u>0$

$$\text{(4.13)} \qquad \log\left(1+\frac{1}{u}\right)=\log\frac{u+1}{u}=\int_0^\infty e^{-ut}(1-e^{-t})\frac{dt}{t}$$

here using Lemma 3.1

$$\begin{aligned}&=g(u;\,-1,1)\\&=-\sum_{i=0}^{\infty}\frac{\gamma_i}{i!}\int_0^1(1-y)^{x-1}y\cdot h_i(y;1)dy\\&=-\sum_{i=0}^{\infty}\frac{\gamma_i}{u(u+1)\cdots(u+i)}.\end{aligned}$$

Noticing the fact that $\gamma_0=-1$, $\gamma_i=b_i$ $(i\geq 1)$, we thus have proved the formula (2.7).

Now, we shall prove the inequality (2.8). By (ii) of Lemma 3.2 and Lemma 3.3, we can evaluate $S(u)$ in (2.9) as

$$\text{(4.14)} \qquad \begin{aligned}S(u)&>\frac{1}{2u(u+1)}+\frac{1}{6}\sum_{i=2}^{\infty}\frac{\Gamma(i-1)}{u(u+1)\cdots(u+i)}\\&=\frac{1}{2u(u+1)}+\frac{1}{6u(u+1)^2}=\underline{S}(u).\end{aligned}$$

On the other hand,

$$\text{(4.15)} \qquad \begin{aligned}S(u)&<\frac{1}{2u(u+1)}+\frac{1}{6}\sum_{i=2}^{\infty}\frac{\Gamma(i)}{u(u+1)\cdots(u+i)}\\&=\frac{1}{2u(u+1)}+\frac{1}{6u^2(u+1)}=\bar{S}(u).\end{aligned}$$

Combining (4.14) and (4.15) we get the inequality (2.8), and thus we have completes the proof of (ii) of Theorem 2.1.

In almost the same manner as the above proof we can prove the inequality (2.14).

4.3. *The proof of the inequality* (2.19)

Since for $a>0$

$$\text{(4.16)} \qquad \frac{1}{a}=\int_0^\infty e^{-at}dt,$$

we have for real number $x\geq 0$ and positive integer $p\geq 2$

$$\text{(4.17)} \qquad \begin{aligned}\sum_{i=1}^{p-1}\frac{1}{x+i}&=\int_0^\infty e^{-(x+1)t}\cdot\frac{1-e^{-(p-1)t}}{1-e^{-t}}dt\\&=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-(x+p)t}}{1-e^{-t}}\right)dt-\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-(x+1)t}}{1-e^{-t}}\right)dt\end{aligned}$$

$$=\psi(x+p)-\psi(x+1),$$

where for $s>1$

$$\psi(s)\equiv\frac{d}{dx}\log\Gamma(s)=\int_0^\infty\left(\frac{e^{-t}}{t}-\frac{e^{-st}}{1-e^{-t}}\right)dt,$$

(cf. [5], p. 247). Further, by using (4.12) with $a=1$ and $b=s$, we can rewrite $\psi(s)$ as

$$\psi(s)=\log s+\int_0^\infty\left(\frac{1}{t}-\frac{1}{1-e^{-t}}\right)e^{-st}dt, \tag{4.18}$$

and making the transformation (3.5) and using (3.11), it follows that

$$\begin{aligned}\psi(s)&=\log s-\sum_{i=1}^{\infty}\frac{\gamma_i}{is(s+1)\cdots(s+i-1)}\\&=\log s-\frac{1}{2s}-\frac{1}{s}T(s).\end{aligned} \tag{4.19}$$

where $T(s)$ is the infinite series defined in (2.20).

From (4.17) and (4.19) we immediately obtain (2.18).

Next, we shall show the inequality (2.19). By (2.20) and (3.15) and applying Lemma 3.3 it follows that for $s>1$

$$\begin{aligned}T(s)&<\frac{1}{6}\sum_{i=1}^{\infty}\frac{\Gamma(i+1)}{(s+1)(s+2)\cdots(s+i)(i+1)},\\&\le\frac{\Gamma(s+1)}{12}\sum_{i=1}^{\infty}\frac{\Gamma(i+1)}{\Gamma(s+i+1)}=\frac{\Gamma(s+1)}{12}\left\{\frac{1}{(s-1)\Gamma(s)}-\frac{1}{\Gamma(s+1)}\right\}\\&=\frac{1}{12(s-1)}=\bar{T}(s).\end{aligned} \tag{4.20}$$

On the other hand we can evaluate $T(s)$ as

$$\begin{aligned}T(s)&>\frac{1}{6}\sum_{i=1}^{\infty}\frac{\Gamma(i)}{(s+1)(s+2)\cdots(s+i)(i+1)}\\&=\frac{s}{6}\sum_{i=1}^{\infty}\frac{1}{i+1}\cdot\frac{\Gamma(i)}{(s+1)\cdots(s+i+1)}+\frac{1}{6}\sum_{i=1}^{\infty}\frac{\Gamma(i)}{(s+1)\cdots(s+i+1)}\\&=\frac{1}{6}(s+1)\sum_{i=1}^{\infty}\frac{\Gamma(i)}{(s+1)\cdots(s+i+1)}\\&\qquad-\frac{s}{6}\sum_{i=1}^{\infty}\frac{1}{(i+1)(s+i+1)}\cdot\frac{\Gamma(i+1)}{(s+1)\cdots(s+i)},\end{aligned} \tag{4.21}$$

by applying Lemma 3.3, for $s>1$

$$\ge\frac{1}{6(s+1)}-\frac{s}{12(s+2)}\left(\frac{s}{s-1}-1\right)$$

$$=\frac{s^2+s-4}{12(s-1)(s+1)(s+2)}$$
$$=\frac{1}{12(s-1)}-\frac{1}{6(s-1)(s+1)}-\frac{1}{6(s-1)(s+1)(s+2)}=\underline{T}(s)$$
$$\geq\frac{1}{72(s-1)}=\bar{T}(s)/6\,.$$

Thus, we have completed the proof (iv) of the theorem.

4.4. *The proof of the inequality* (2.25)

Since

$$\frac{1}{a^2}=\int_0^\infty te^{-at}dt \qquad (a>0), \tag{4.22}$$

we can represent

$$\sum_{i=1}^{p}\frac{1}{(x+i)^2}=\int_0^\infty te^{-(x+1)t}\frac{1-e^{-pt}}{1-e^{-t}}dt \tag{4.23}$$
$$=\int_0^\infty te^{-(x+1)t}(1-e^{-t})^{-1}dt-\int_0^\infty te^{-(x+p+1)t}(1-e^{-t})^{-1}dt$$
$$=\psi'(x+1)-\psi'(x+p+1)\,,$$

where for $s>0$

$$\psi'(s)=\frac{d^2}{ds^2}\log\Gamma(s)=\int_0^\infty te^{-st}(1-e^{-t})^{-1}dt \tag{4.24}$$

(cf. [5], p. 261). Making use of the transformation (3.5) and the Maclaulin expansion,

$$\psi'(s)=\int_0^1 y^{-1}(1-y)^{s-1}\{-\ln(1-y)\}dy \tag{4.25}$$
$$=\sum_{i=1}^{\infty}\frac{d_i}{s(s+1)\cdots(s+i-1)}=U(s)$$

where d_i is the same coefficient defined in (2.27). Thus, from (4.23) and (4.25), we have the formula (2.24).

Now we shall prove the inequality (2.25). By Lemma 3.3 we can evaluate $U(s)$ as

$$U(s)=\frac{1}{s}+\frac{1}{2s(s+1)}+\sum_{j=2}^{\infty}\frac{\Gamma(j+1)}{(j+1)s(s+1)\cdots(s+j)} \tag{4.26}$$
$$\geq\frac{1}{s}+\frac{1}{2s(s+1)}+\frac{2}{3}\sum_{j=2}^{\infty}\frac{\Gamma(j)}{s(s+1)\cdots(s+j)}$$
$$=\frac{1}{s}+\frac{1}{2s(s+1)}+\frac{2}{3s^2(s+1)}=\underline{U}(s)\,.$$

On the other hand,

$$
\begin{aligned}
(4.27) \qquad U(s) &< \frac{1}{s} + \frac{1}{2s(s+1)} + \frac{1}{3} \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{s(s+1)\cdots(s+j)} \\
&= \frac{1}{s} + \frac{1}{2s(s+1)} + \frac{\Gamma(s)}{3} \sum_{j=2}^{\infty} \frac{\Gamma(j+1)}{\Gamma(s+j+1)} \\
&= \frac{1}{s} + \frac{1}{2s(s+1)} + \frac{2}{3s(s-1)(s+1)} = \bar{U}(s) .
\end{aligned}
$$

Therefore, we have completed the proof of the theorem.

Finally, we remark that our inequalities can be improved more accurately by considering some additional terms in evaluations of the underlying inverse factorial series.

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