APPROXIMATELY MINIMAX TESTS FOR TESTING HYPOTHESES ABOUT A RANDOM PARAMETER WITH UNKNOWN DISTRIBUTION

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1. Introduction

Let X be a real-valued random variable with a family of possible distributions indexed by $\lambda \in \Omega$, λ is the realization of a random variable Λ taking values in the space Ω. For each λ, let f_λ denote the conditional density of X given $\Lambda=\lambda$ with respect to some σ-finite measure μ. Let $\mathcal{G}$ be a family of possible a priori distributions G for Λ.

Consider the problem, after observing X, of testing $H: \lambda \in \omega$ against $K: \lambda \in \omega^c$ where ω is a subset of Ω, ω^c its complement and H, K are both composite hypotheses. Recently Meeden ([2], [3]) formulated this problem in a way which bears an obvious analogy with the corresponding problem in classical analysis, involving, as usual, the two kinds of errors proper under such set up. Under certain assumptions about $\mathcal{G}$, $\{f_\lambda(x)\}$ and ω, Meeden suggested best tests for the above problem, minimizing the second kind of error and dominating the first kind of error by a pre-assigned quantity, uniformly for $G \in \mathcal{G}$. These have been derived primarily on the basis of their counterparts in the classical Neyman-Pearson theory and seem to be the best one can obtain under the assumptions made. Therefore, whenever these assumptions cease to hold, it is difficult to obtain best tests in the above sense.

The purpose of this note is to suggest, under certain assumptions, some 'approximately minimax' test procedures for the above problem. Specifically, the object here is to derive upper bounds of the two kinds of errors for variations in $G \in \mathcal{G}$ and then to suggest test procedures proper for controlling the upper bound of the first kind of error and minimizing that of the second kind of error.

2. Formulation of the problem and derivation of main results

Throughout the following, we adhere to the notations used by Meeden. Quite naturally, the occurrences of the two types of errors

are as follows:
type (i) error: $\Lambda \in \omega^c$ decided and $\Lambda \in \omega$ occurs,
type (ii) error: $\Lambda \in \omega$ decided and $\Lambda \in \omega^c$ occurs;
and the probabilities of these errors, assuming that G is the true distribution of Λ, are

$$\text{(2.1)} \qquad \mathrm{P}_G\,(\text{type (i) error}) = \int_{\omega} \left\{ \int_{\mathcal{X}} \delta(x) f_\lambda(x) d\mu(x) \right\} dG(\lambda)$$

$$\text{(2.2)} \qquad \mathrm{P}_G\,(\text{type (ii) error}) = \int_{\omega^c} \left\{ \int_{\mathcal{X}} (1-\delta(x)) f_\lambda(x) d\mu(x) \right\} dG(\lambda)$$

where, as always, $\delta(x)$ is the (randomized) test function defined on the range of X which takes on values in the interval $[0, 1]$ and if $X=x$ is observed, K is decided to be true with probability $\delta(x)$ and H with probability $(1-\delta(x))$.

Analogous to the problem of finding uniformly most powerful (UMP) level α tests is the problem:

$$\text{(2.3)} \qquad \text{subject to:} \quad \mathrm{P}_G\,(\text{type (i) error}) \leqq \alpha \qquad \text{for all } G \in \mathcal{G}$$

$$\text{(2.4)} \qquad \text{minimize:} \quad \mathrm{P}_G\,(\text{type (ii) error}), \quad \text{uniformly for } G \in \mathcal{G}.$$

A test which achieves this is called UMP level α test relative to $\mathcal{G}$. Under certain assumptions about $\mathcal{G}$, $f_\lambda(x)$ and ω, Meeden provided solutions to this problem.

Our object here is to provide, under some different conditions, a kind of minimax solutions for testing H against K.

General procedure

Under the assumption that $\mathcal{G}$ is a parametric family of a priori distributions i.e., $\mathcal{G} = \{G(\lambda|\theta),\ \theta \in [\underline{\theta}, \bar{\theta}]$, the form of G known, $\underline{\theta}$, $\bar{\theta}$ known$\}$, the two kinds of errors have the expressions

$$\begin{aligned}\text{(2.5)} \qquad \mathrm{P}_\theta\,(\text{type (i) error}) &= \int_{\omega} \left\{ \int_{\mathcal{X}} \delta(x) f_\lambda(x) d\mu(x) \right\} dG(\lambda|\theta) \\ &= \int_{\mathcal{X}} \delta(x) \left\{ \int_{\omega} f_\lambda(x) dG(\lambda|\theta) \right\} d\mu(x) \\ &\qquad \text{(by applying Fubini's theorem)} \\ &= \int_{\mathcal{X}} \delta(x) f_{\omega}^{G(\theta)}(x) d\mu(x)\end{aligned}$$

and

$$\begin{aligned}\text{(2.6)} \qquad \mathrm{P}_\theta\,(\text{type (ii) error}) &= \int_{\omega^c} \left\{ \int_{\mathcal{X}} (1-\delta(x)) f_\lambda(x) d\mu(x) \right\} dG(\lambda|\theta) \\ &= \int_{\mathcal{X}} \left\{ \int_{\omega^c} f_\lambda(x) dG(\lambda|\theta) \right\} (1-\delta(x)) d\mu(x)\end{aligned}$$

$$=\int_{\mathcal{X}}(1-\delta(x))f_{\omega^c}^{G(\theta)}(x)d\mu(x)$$

where we write $f_{\omega}^{G(\theta)}(x)=\int_{\omega} f_{\lambda}(x)dG(\lambda|\theta)$ and $f_{\omega^c}^{G(\theta)}=\int_{\omega^c} f_{\lambda}(x)dG(\lambda|\theta)$. It is, of course, assumed that $\mathcal{X}$ does not depend on λ. The general problem is then to find a test function $\delta(x)$ such that subject to: $\int_{\mathcal{X}}\delta(x)\cdot f_{\omega}^{G(\theta)}(x)d\mu(x)\leqq\alpha$ for all $\theta\in[\underline{\theta},\bar{\theta}]$, the quantity $\int_{\mathcal{X}}(1-\delta(x))f_{\omega^c}^{G(\theta)}(x)d\mu(x)$ is a minimum, uniformly for $\theta\in[\underline{\theta},\bar{\theta}]$. We shall now find upper bounds to each of these errors and choose $\delta(x)$ such that the above conditions are satisfied wrt these "upper" functions. To this end, we have a number of procedures.

Procedure 1.

Since

$$\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\left\{\int_{\mathcal{X}}\delta(x)f_{\omega}^{G(\theta)}(x)d\mu(x)\right\}\leqq\int_{\mathcal{X}}\delta(x)\{\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\{f_{\omega}^{G(\theta)}(x)\}\}d\mu(x)$$

and

$$\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\left\{\int_{\mathcal{X}}(1-\delta(x))f_{\omega^c}^{G(\theta)}(x)d\mu(x)\right\}\leqq\int_{\mathcal{X}}(1-\delta(x))\{\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\{f_{\omega^c}^{G(\theta)}(x)\}\}d\mu(x)\,,$$

and further that, whenever $f_{\omega}^{G(\theta)}(x)$ and $f_{\omega^c}^{G(\theta)}(x)$ are continuous in θ, uniformly in x, these suprema are attained (the functions being defined over closed domain), it follows that if we choose $\delta(x)$ minimizing

$$\int_{\mathcal{X}}\{1-\delta(x)\}\{\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\{f_{\omega^c}^{G(\theta)}(x)\}\}d\mu(x)$$

subject to

$$\int_{\mathcal{X}}\delta(x)\{\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]}\{f_{\omega}^{G(\theta)}(x)\}\}d\mu(x)\leqq\alpha \tag{2.7}$$

then this $\delta(x)$ provides a 'kind of minimax solution.' Quite evidently, $\delta(x)$ is given by

$$\delta(x)=\begin{cases}1\,, & \text{for } \operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]} f_{\omega^c}^{G(\theta)}(x)>k\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]} f_{\omega}^{G(\theta)}(x)\,,\\ d\,, & \text{for } \operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]} f_{\omega^c}^{G(\theta)}(x)=k\operatorname*{Sup}_{\theta\in[\underline{\theta},\bar{\theta}]} f_{\omega}^{G(\theta)}(x)\,,\\ 0\,, & \text{otherwise}\end{cases}$$

where k and d are such that (2.7) is satisfied. Strictly speaking, this $\delta(x)$ satisfies (2.3) exactly and minimizes 'some sort' of maximum of (2.4).

In practice, however, this procedure is not very suitable since determination of the 'sup functions' is not at all easy. The following two examples will make this point clear.

Example 1. Let X, given $\Lambda=\lambda$, have the hypergeometric (n, λ, m) distribution where $\lambda \in (0, 1, \cdots, n)=\Omega$ and m and n are known positive integers such that $0<m<n$. $\mathcal{G}$ is the class of binomial (n, p) distributions for $p \in M$, a closed subset of $[0, 1]$. Suppose that our object is to test on the basis of an observed value x of X the hypothesis $H: \lambda \leqq \lambda_0$ against $K: \lambda>\lambda_0$ where λ_0 is a given integer $\in \Omega$. Obviously we must have $x \leqq \min(\lambda, m)$.

To apply Procedure 1, we require to compute $\operatorname*{Sup}_{p\in M} f_\omega^{G(p)}(x)$ and $\operatorname*{Sup}_{p\in M} f_{\omega^c}^{G(p)}(x)$ where, as can be easily verified,

$$f_\omega^{G(p)}(x)=\sum_{\lambda\leqq\lambda_0} f_\lambda(x)dG(\lambda\,|\,p)=\binom{m}{x}p^x q^{m-x}\frac{\int_0^q z^{n-m-(\lambda_0-x)-1}(1-z)^{\lambda_0-x}dz}{B(n-m-\lambda_0+x,\ \lambda_0-x+1)}$$

and

$$f_{\omega^c}^{G(p)}(x)=\sum_{\lambda>\lambda_0} f_\lambda(x)dG(\lambda\,|\,p)=\binom{m}{x}p^x q^{m-x}\frac{\int_q^1 z^{n-m-(\lambda_0-x)-1}(1-z)^{\lambda_0-x}dz}{B(n-m-\lambda_0+x,\ \lambda_0-x+1)}\ .$$

If then α $(\geqq 0)$ and β $(\leqq 1)$ are the infimum and supremum of M, we have

$$\operatorname*{Sup}_{p\in M} f_\omega^{G(p)}(x)=\begin{cases} f_\omega^{G(\alpha)}(x)\ , & \text{for } x\leqq m\alpha\ ,\\ f_\omega^{G(\beta)}(x)\ , & \text{for } x\geqq \max\left\{m\beta,\ \dfrac{\beta}{1-\beta}(n-\lambda_0)\right\}\ ,\\ \text{a function of } x,\ \lambda_0,\ n,\ m\ , & \text{otherwise}\end{cases}$$

and it is really the explicit determination of the last composite function which is difficult. Similar expressions for $\operatorname*{Sup}_{p\in M} f_{\omega^c}^{G(p)}(x)$ can also be obtained and the same difficulty is involved.

Example 2. Let the distribution of X given $\Lambda=\lambda$ be normal (λ, σ_1^2) and the distribution of Λ be normal (θ, σ_2^2) where σ_1^2 and σ_2^2 are known and $\theta \in M$, a closed set of real numbers, is unknown.

As before, if our object is to test $H_0: \lambda \leqq \lambda_0$ against $K: \lambda>\lambda_0$ (where λ_0 is a fixed number), we need to compute $\operatorname*{Sup}_{\theta\in M} f_\omega^{G(\theta)}(x)$ and $\operatorname*{Sup}_{\theta\in M} f_{\omega^c}^{G(\theta)}(x)$ where, quite evidently,

$$f_{\omega}^{G(\theta)}(x)=\Phi\left[\left(\lambda_0-\frac{x/\sigma_1^2+\theta/\sigma_2^2}{1/\sigma_1^2+1/\sigma_2^2}\right)\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right]f(x\,|\,\theta,\,\sigma_1^2+\sigma_2^2)$$

and

$$f_{\omega^c}^{G(\theta)}(x)=\left\{1-\Phi\left[\left(\lambda_0-\frac{x/\sigma_1^2+\theta/\sigma_2^2}{1/\sigma_1^2+1/\sigma_2^2}\right)\sqrt{\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}}\right]\right\}f(x\,|\,\theta,\,\sigma_1^2+\sigma_2^2)\ ,$$

$f(x\,|\,\mu,\sigma^2)$ denoting the normal density with mean μ, variance σ^2 and $\Phi(z)=\int_{-\infty}^{z}f(x\,|\,0,1)dx$. Here also determination of the 'sup functions' cannot be easily performed.

Procedure 2.

Here we assume that the family $\{G(\lambda\,|\,\theta)\}$ admits a density function $g(\lambda\,|\,\theta)$ wrt some σ-finite measure μ^* over Ω. Since then

$$\begin{aligned}\operatorname*{Sup}_{\theta\in M}\mathrm{P}_\theta\,(\text{type (i) error})&=\operatorname*{Sup}_{\theta\in M}\left\{\int_{\mathcal{X}}\int_{\omega}\delta(x)f_\lambda(x)g(\lambda\,|\,\theta)d\mu^*(\lambda)d\mu(x)\right\}\\&\leqq\int_{\mathcal{X}}\int_{\omega}\delta(x)f_\lambda(x)\{\operatorname*{Sup}_{\theta\in M}g(\lambda\,|\,\theta)\}d\mu^*(\lambda)d\mu(x)\end{aligned}$$

and

$$\begin{aligned}\operatorname*{Sup}_{\theta\in M}\mathrm{P}_\theta\,(\text{type (ii) error})\leqq\int_{\mathcal{X}}\int_{\omega^c}&(1-\delta(x))f_\lambda(x)\{\operatorname*{Sup}_{\theta\in M}g(\lambda\,|\,\theta)\}\\&\cdot d\mu^*(\lambda)d\mu(x)\ ,\end{aligned}$$

we can determine $\delta(x)$ such that subject to

(2.8)

$$\int_{\mathcal{X}}\delta(x)\bar{f}_{\omega}^{G}(x)d\mu(x)\leqq\alpha$$

$$\int_{\mathcal{X}}(1-\delta(x))\bar{f}_{\omega^c}^{G}(x)d\mu(x)$$

is a minimum where we write

$$\bar{f}_{\omega}^{G}(x)=\int_{\omega}f_\lambda(x)\{\operatorname*{Sup}_{\theta\in M}g(\lambda\,|\,\theta)\}d\mu^*(\lambda)$$

and

$$\bar{f}_{\omega^c}^{G}(x)=\int_{\omega^c}f_\lambda(x)\{\operatorname*{Sup}_{\theta\in M}g(\lambda\,|\,\theta)\}d\mu^*(\lambda)\,.$$

Such a test function $\delta(x)$ obviously satisfies (2.3) and minimizes 'some sort' of maximum of (2.4). Further, specification of $\delta(x)$ presents no difficulty since in most cases determination of $\bar{f}_{\omega}^{G}(x)$ and $\bar{f}_{\omega^c}^{G}(x)$ is rather easy. By way of illustration, we outline the procedure for determining

$\bar{f}_{\omega}^{G}(x)$ with reference to the Example 1 mentioned earlier.

From what has been discussed there, it follows that we have

$$\bar{f}_{\omega}^{G}(x)=\sum_{\lambda\leqq\lambda_0} f_{\lambda}(x)\left\{\operatorname*{Sup}_{p\in M}\binom{n}{\lambda}p^{\lambda}q^{n-\lambda}\right\}$$
$$=\sum_{\lambda\leqq\lambda_0}\varepsilon_1 f_{\lambda}(x)\binom{n}{\lambda}\alpha^{\lambda}(1-\alpha)^{n-\lambda}+\sum_{\lambda\leqq\lambda_0}\varepsilon_2(\lambda)f_{\lambda}(x)\binom{n}{\lambda}\hat{p}^{\lambda}\hat{q}^{n-\lambda}|_{\hat{p}=\lambda/n}$$
$$+\sum_{\lambda\leqq\lambda_0}\varepsilon_3(\lambda)f_{\lambda}(x)\binom{n}{\lambda}\beta^{\lambda}(1-\beta)^{n-\lambda}$$

where

$$\varepsilon_1(\lambda)=\begin{cases}1, & \text{for } \lambda\leqq\lambda_1 \text{ where } \lambda_1/n\leqq\alpha\\ 0, & \text{otherwise};\end{cases}$$

$$\varepsilon_2(\lambda)=\begin{cases}1, & \text{for } 1+\lambda_1\leqq\lambda\leqq\lambda_2 \text{ where } \alpha<(\lambda_1+1)/n<\lambda_2/n<\beta\\ 0, & \text{otherwise};\end{cases}$$

and

$$\varepsilon_3(\lambda)=\begin{cases}1, & \text{for } \lambda_0\geqq\lambda\geqq\lambda_2+1 \text{ where } (1+\lambda_2)/n\geqq\beta\\ 0, & \text{otherwise}.\end{cases}$$

We, of course, assume M, for the sake of simplicity, to be the closed interval $[\alpha, \beta]$. Therefore, $\bar{f}_{\omega}^{G}(x)$ is known for every x. Similarly $\bar{f}_{\omega^c}^{G}(x)$ can also be explicitly obtained, thereby leading to the specification of $\delta(x)$.

Procedure 3.

Under certain assumptions, we shall now give a procedure which will enable us to determine quite generally the explicit expression for the 'approximately minimax' test function.

ASSUMPTIONS. (i) $\mathcal{G}=\{G(\lambda|\theta),\ \theta\in[\underline{\theta},\bar{\theta}]\}$ is a family of distributions having monotone likelihood ratio in λ.
(ii) The density $f_{\lambda}(x)$ is a nondecreasing function of λ, for $\lambda\in\Omega$, uniformly in x.

We shall presently see how with the help of these assumptions we can obtain explicit expressions for the upper bounds of the probabilities of the two types of errors. Now

$$P_{\theta}\,(\text{type (i) error})=\int_{\mathcal{X}}\delta(x)\left\{\int_{\omega}f_{\lambda}(x)dG(\lambda|\theta)\right\}d\mu(x)\ .$$

We take $\omega=(0, \lambda_0]$ where λ_0, a fixed number, $\in\Omega$. Define

$$\Psi(\lambda, x)=\begin{cases} f_\lambda(x) & \text{for } \lambda \in \omega\,, \\ \eta & \text{for } \lambda \in \omega^c \text{ where } \eta \text{ (independent of } \lambda, \\ & \text{may or may not depend on } x) \geqq f_{\lambda_0}(x)\,. \end{cases}$$

Then, by assumption (ii) and a lemma of Lehmann ([1], p. 74), it follows that $E_\theta \Psi(\lambda, x)=\int_\Omega \Psi(\lambda, x)dG(\lambda|\theta)$ is nondecreasing in θ, uniformly in x. Making use of this fact, it is easy to verify that

$$\text{(2.9)}\qquad P_\theta\,(\text{type (i) error}) \leqq \int_{\mathcal{X}} \delta(x)\,[f^\omega(x\,|\,\bar{\theta})+\eta\{G^{\omega^c}(\bar{\theta})-G^{\omega^c}(\underline{\theta})\}]\,d\mu(x)$$

where

$$f^\omega(x\,|\,\bar{\theta})=\int_{\lambda\in\omega} f_\lambda(x)dG(\lambda\,|\,\bar{\theta})\,, \qquad G^{\omega^c}(\theta)=\int_{\omega^c} dG(\lambda\,|\,\theta)\,.$$

Again

$$P_\theta\,(\text{type (ii) error})=\int_{\mathcal{X}} (1-\delta(x))\left\{\int_{\omega^c} f_\lambda(x)dG(\lambda\,|\,\theta)\right\}d\mu(x)\,.$$

Define

$$\Psi^*(\lambda, x)=\begin{cases} \eta^* & \text{for } \lambda \in \omega\,, \\ f_\lambda(x) & \text{for } \lambda \in \omega^c \end{cases}$$

where η^* (non-negative, independent of λ, may or may not depend on $x) \leqq f_{\lambda_0}(x)$. Then $E_\theta \Psi^*(\lambda, x)$ is nondecreasing in θ, uniformly in x and we have quite readily

$$\text{(2.10)}\quad P_\theta\,(\text{type (ii) error}) \leqq \int_{\mathcal{X}} (1-\delta(x))\,[f^{\omega^c}(x\,|\,\bar{\theta})+\eta^*\{G^\omega(\bar{\theta})-G^\omega(\underline{\theta})\}]\,d\mu(x)$$

where, as before, $f^{\omega^c}(x\,|\,\bar{\theta})=\int_{\omega^c} f_\lambda(x)dG(\lambda\,|\,\bar{\theta})$ and $G^\omega(\theta)=\int_\omega dG(\lambda\,|\,\theta)$. The critical function $\delta(x)$ which satisfies the size condition and minimizes the r.h.s. of (2.10) is then explicitly given by

$$\text{(2.11)}\qquad \delta(x)=\begin{cases} 1 & \text{for } f^{\omega^c}(x\,|\,\bar{\theta})+\eta^*\{G^\omega(\bar{\theta})-G^\omega(\underline{\theta})\} \\ & \qquad > k[f^\omega(x\,|\,\bar{\theta})+\eta\{G^{\omega^c}(\bar{\theta})-G^{\omega^c}(\underline{\theta})\}]\,, \\ c & \text{for } f^{\omega^c}(x\,|\,\bar{\theta})+\eta^*\{G^\omega(\bar{\theta})-G^\omega(\underline{\theta})\} \\ & \qquad = k[f^\omega(x\,|\,\bar{\theta})+\eta\{G^{\omega^c}(\bar{\theta})-G^{\omega^c}(\underline{\theta})\}]\,, \\ 0\,, & \text{otherwise} \end{cases}$$

where c and k are determined such that

$$\text{(2.12)}\qquad \int_{\mathcal{X}} \delta(x)\,[f^\omega(x\,|\,\bar{\theta})+\eta\{G^{\omega^c}(\bar{\theta})-G^{\omega^c}(\underline{\theta})\}]\,d\mu(x) \leqq \alpha\,.$$

This function $\delta(x)$ obviously satisfies the basic condition (2.3) and moreover, minimizes 'some sort' of maximum of the second kind of error.

We now illustrate the application of the procedure with the help of the following

Example 3. Let X given $\Lambda=\lambda$ have the truncated exponential distribution, truncated at λ, with the density $f_\lambda(x)=e^{-(x-\lambda)}$, $x\geqq\lambda$ (wrt the Lebesgue measure) where $\lambda\in(0,\infty)=\Omega$. $\mathcal{G}$ is the class of exponential densities (wrt the Lebesgue measure) with a parameter θ i.e., $dG(\lambda|\theta)=1/\theta\cdot e^{-\lambda/\theta}d\lambda$, $\theta\in[\underline{\theta},\bar{\theta}]$, a closed subset of real numbers with $\underline{\theta}>0$. It may be seen that the assumptions (i) and (ii) are satisfied in this case. Suppose now that on the basis of an observed value x of X our object is to test $H:\lambda\leqq\lambda_0$ against $K:\lambda>\lambda_0$, λ_0 being a fixed number $\in\Omega$. It is to be noted, however, that the procedure outlined in (2.11) is not readily applicable in this case since $\mathcal{X}$, the domain of variation of X, depends on λ. The appropriate test function $\delta(x)$ can however be constructed without any difficulty following lines very similar to those stated earlier.

For this purpose, we begin from the definitions of the two kinds of errors. We remember that

$$
\begin{aligned}
(2.13)\quad \mathrm{P}_\theta(\text{type (i) error}) &=\int_{\lambda\in\omega}\left[\int_{\mathcal{X}}\delta(x)f_\lambda(x)d\mu(x)\right]dG(\lambda|\theta)\\
&=\int_{0\leqq\lambda\leqq\lambda_0}\left[\int_\lambda^\infty\delta(x)f_\lambda(x)d\mu(x)\right]dG(\lambda|\theta)\\
&=\int_0^\infty\delta(x)\left[\int_{0\leqq\lambda\leqq\min(\lambda_0,x)}f_\lambda(x)dG(\lambda|\theta)\right]d\mu(x)\\
&<\int_0^\infty\delta(x)\left\{\int_{0\leqq\lambda\leqq\lambda_0}f_\lambda(x)dG(\lambda|\theta)\right\}d\mu(x)\\
&\leqq\int_0^\infty\delta(x)[f^\omega(x|\bar{\theta})+\eta\{G^{\omega^c}(\bar{\theta})-G^{\omega^c}(\underline{\theta})\}]d\mu(x)\\
&\qquad\qquad\text{(using (2.9))}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
(2.14)\quad \mathrm{P}_\theta(\text{type (ii) error}) &=\int_{\lambda\in\omega^c}\left[\int_{\mathcal{X}}(1-\delta(x))f_\lambda(x)d\mu(x)\right]dG(\lambda|\theta)\\
&=\int_{\lambda_0}^\infty(1-\delta(x))\left[\int_{\lambda_0}^x f_\lambda(x)dG(\lambda|\theta)\right]d\mu(x)\\
&=\int_0^\infty(1-\delta(x))\left[\int_{\lambda_0}^x f_\lambda(x)dG(\lambda|\theta)\right]d\mu(x)\\
&\quad+\int_0^{\lambda_0}(1-\delta(x))\left[\int_x^{\lambda_0}f_\lambda(x)dG(\lambda|\theta)\right]d\mu(x)\,.
\end{aligned}
$$

Since for any critical function $\delta(x)$, we can define, *a forteori*, $\delta(x)=0$

for $x \leqq \lambda_0$, it is indeed with the first part of (2.14) that we are concerned. Defining, then,

$$\Psi^*(\lambda)=\begin{cases} 0 & \text{for } \lambda<\lambda_0\,, \\ f_\lambda(x) & \text{for } \lambda_0\leqq\lambda\leqq x\,, \\ \eta^{**} & \text{for } \lambda>x \text{ where } f_\lambda(x)\leqq\eta^{**} \text{ for all } \lambda\leqq x \end{cases}$$

it can be easily seen that

$$\text{(2.15)}\quad \text{first part of (2.14)} \leqq \int_0^\infty (1-\delta(x))\Big[\int_{\lambda_0}^x f_\lambda(x)dG(\lambda\,|\,\bar\theta) + \eta^{**}\Big\{\int_x^\infty dG(\lambda\,|\,\bar\theta)-\int_x^\infty dG(\lambda\,|\,\underline{\theta})\Big\}\Big]d\mu(x)\,.$$

Performing necessary integrations, we get the minimax critical function $\delta(x)$ proper for this problem, analogous to (2.11), in the following form

$$\delta(x)=\begin{cases} 1\,, & \text{for } e^{-x}\Big\{\dfrac{e^{-\lambda_0(1/\bar\theta-1)}-e^{-x(1/\bar\theta-1)}}{1-\bar\theta}\Big\}+\eta^{**}(e^{-x/\bar\theta}-e^{-x/\underline{\theta}}) \\ & \qquad \geqq k\Big[e^{-x}\Big\{\dfrac{1-e^{-\lambda_0(1/\bar\theta-1)}}{1-\bar\theta}\Big\}+\eta(e^{-\lambda_0/\bar\theta}-e^{-\lambda_0/\underline{\theta}})\Big]\,, \\ 0\,, & \text{otherwise, for } \bar\theta\neq 1 \end{cases}$$

and

$$\delta(x)=\begin{cases} 1\,, & \text{for } e^{-x}(x-\lambda_0)+\eta^{**}(e^{-x}-e^{-x/\underline{\theta}})\geqq k[e^{-x}\lambda_0+\eta(e^{-\lambda_0}-e^{-\lambda_0/\underline{\theta}})] \\ 0\,, & \text{otherwise, for } \bar\theta=1\,. \end{cases}$$

Taking, in particular, $\eta=f_{\lambda_0}(x)$ and $\eta^{**}=1$, it is readily found that the minimax critical function $\delta(x)$ is given by

$$\text{(2.16)}\qquad \delta(x)=\left.\begin{cases} 1 & \text{for } x\geqq c \\ 0\,, & \text{otherwise} \end{cases}\right\}\ \text{whatever } \underline{\theta},\ \bar\theta$$

where the constant c is such that

$$\int_c^\infty [f^\omega(x\,|\,\bar\theta)+f_{\lambda_0}(x)\{G^{\omega^c}(\bar\theta)-G^{\omega^c}(\underline{\theta})\}]dx\leqq\alpha$$

i.e.,

$$\text{(2.17)}\qquad e^{-c}[\xi(\lambda_0,\,\bar\theta,\,\underline{\theta})]\leqq\alpha$$

where ξ is a completely known function of λ_0, $\bar\theta$ and $\underline{\theta}$. It may be noted that (2.16) (with c to be appropriately chosen) represents the

UMP critical function for testing H againt K in the classical theory of hypothesis testing where λ is treated as a constant.

Remarks. Minimax test procedures are useful and meaningful only when the quantity viz., the maximum value of the second kind of error, which we minimize, is really assumed for some value of the relevant argument belonging to the appropriate domain. In our context, this means that the expression for the supremum of the second kind of error (considered as a function of θ), which we are really minimizing (by choosing $\delta(x)$ properly), should be the value of the second kind of error itself for some $\theta \in M=[\underline{\theta}, \bar{\theta}]$. It may however, be noted that this is not generally so in our problems unless the closed interval M is degenerate, a very trivial case. The results therefore hold only under the assumption that $\underline{\theta}$ and $\bar{\theta}$ are 'close' to each other. This is revealed by the term 'approximately minimax tests' that we have used implying thereby that in all cases $\delta(x)$ satisfies (2.3) exactly and minimizes 'some sort' of maximum of the second kind of error given by (2.4).

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REFERENCES

[1] Lehmann, E. L. (1966). *Testing Statistical Hypotheses*, John Wiley and Sons, New York.

[2] Meeden, G. (1970). Best tests for testing hypotheses about a random parameter with unknown distribution, *Ann. Math. Statist.*, **41**, 585-591.

[3] Meeden, G. (1971). Finding best tests approximately for testing hypotheses about a random parameter, *Ann. Math. Statist.*, **42**, 1452-1454.