# PARTIAL DIFFERENTIAL EQUATIONS FOR HYPERGEOMETRIC FUNCTIONS OF COMPLEX ARGUMENT MATRICES AND THEIR APPLICATIONS

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## 1. Introduction and summary

Statistical analysis based on the complex multivariate normal distribution has been developed as a counterpart of classical statistical analysis based on the real multivariate normal distribution, and especially in connection with the study of the spectral density matrix of a multiple stationary Gaussian time series. The reader may be referred to Goodman [5], Goodman and Dubman [6], James [10] and Khatri [14], [15] for the general theory, and Brillinger [1], Fujikoshi [4], Hayakawa [7], [8], Khatri [13], [16], Sugiyama [24] and Priestley, Subba Rao and Tong [22] for the further distributional results and the applications in time series analysis.

In multivariate analysis based on the complex multivariate normal distribution, many of the distributions of the matrix variates and of the latent roots can be expressed in terms of hypergeometric functions ${}_p\tilde{F}_q$ of one and two complex argument matrices respectively. These results may be compared with the corresponding results in real multivariate analysis, discussed in terms of hypergeometric functions ${}_pF_q$ of one and two real argument matrices. The hypergeometric functions ${}_p\tilde{F}_q$ are defined as the power series representation (James [10])

$$(1.1)\qquad {}_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(A)}{k!}$$

and

$$(1.2)\qquad {}_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; A, B)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{[a_1]_\kappa\cdots[a_p]_\kappa}{[b_1]_\kappa\cdots[b_q]_\kappa}\frac{\tilde{C}_\kappa(A)\tilde{C}_\kappa(B)}{\tilde{C}_\kappa(I_m)k!}\ ,$$

where $a_1, \cdots, a_p$, $b_1, \cdots, b_q$ are real or complex constants, $[a]_\kappa$ is the complex multivariate hypergeometric coefficient defined by

$$[a]_\kappa=\prod_{i=1}^{m}(a-i+1)_{k_i}\,,\qquad (a)_n=a(a+1)\cdots(a+n-1)\,.$$

$\tilde{C}_\kappa(A)$ is the zonal polynomial of the $m\times m$ Hermitian matrix $A$ corresponding to the partition $\kappa=(k_1, k_2, \cdots, k_m)$, $k_1\geqq k_2\geqq\cdots\geqq k_m\geqq 0$, of the integer $k$ into not more than $m$ parts. It is defined by

$$\tilde{C}_\kappa(A)=\chi_{[\kappa]}(1)\chi_{[\kappa]}(A)\,, \tag{1.3}$$

where $\chi_{[\kappa]}(1)$ is the dimension of the representation $[\kappa]$ of the symmetric group and is given by

$$\chi_{[\kappa]}(1)=k!\prod_{i<j}^{m}(k_i-k_j-i+j)\Big/\prod_{i=1}^{m}(k_i+m-1)!\,,$$

and $\chi_{[\kappa]}(A)$ is the character of the representation $\{\kappa\}$ of the linear group and is given as a symmetric function of the latent roots of $A$.

The following simple recursive formula of $\tilde{C}_\kappa(A)$ is given in Sugiyama [24]

$$\begin{aligned}\tilde{C}_{(k_1,\cdots,k_m)}(A)=a_{\hat{k}_1}\tilde{C}_{(\hat{k}_2,\cdots,\hat{k}_m)}(A)-a_{\hat{k}_2-1}\tilde{C}_{(\hat{k}_1+1,\hat{k}_3,\cdots,\hat{k}_m)}(A)\\ +a_{\hat{k}_3-2}\tilde{C}_{(\hat{k}_1+1,\hat{k}_2+1,\hat{k}_4,\cdots,\hat{k}_m)}(A)-\cdots\,,\end{aligned}$$

where $a_i$ is the $i$th elementary symmetric function and $(\hat{k}_1, \hat{k}_2, \cdots, \hat{k}_m)$ is the conjugate partition to $\kappa=(k_1, k_2, \cdots, k_m)$. From this recursive formula, starting with $\tilde{C}_{(k)}(A)=a_k$, the zonal polynomial of a Hermitian matrix can be calculated much more easily than for the real case. However, the determination of the series (1.1) and (1.2) requires an enourmous amount of calculation, and so, from a practical point of view, it is useful to derive asymptotic expansions for the distributions involved. Such examples in the complex case are in Fujikoshi [4] and Hayakawa [8]. It is known that partial differential equations (p.d.e.'s) are useful tools in obtaining asymptotic expansions in the real case (see e.g. Muirhead [18], [19], [20], Muirhead and Chikuse [21] and Chikuse [2]), and it appears that obtaining p.d.e.'s for the complex case could be certainly worth while. Sugiura [23] obtained derivatives of any latent root of a symmetric matrix, and applied them to deriving p.d.e.'s for zonal polynomials and to giving an asymptotic expansion for the distribution of any latent root of a Wishart matrix. Similar results were also given for the complex case.

In this paper, it is shown that the function ${}_2\tilde{F}_1(a, b; c; A)$ satisfies the system of p.d.e.'s

$$A_i(1-A_i)\frac{\partial^2\tilde{F}}{\partial A_i^2}+\left[c-m+1-(a+b-m+a)A_i+\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{A_i(1-A_i)}{A_i-A_j}\right]\frac{\partial\tilde{F}}{\partial A_i} \tag{1.4}$$

$$-\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{A_j(1-A_j)}{A_i-A_j}\frac{\partial\tilde{F}}{\partial A_j}=ab\tilde{F}\qquad (i=1,2,\cdots,m)$$

and that the function ${}_2\tilde{F}_1(a, b; c; A, B)$ satisfies the p.d.e.

$$(1.5)\quad \sum_{i=1}^{m}A_i\frac{\partial^2\tilde{F}}{\partial A_i^2}+2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{A_i}{A_i-A_j}\frac{\partial\tilde{F}}{\partial A_i}+(c-m+1)\sum_{i=1}^{m}\frac{\partial\tilde{F}}{\partial A_i}$$
$$-(a+b-2m+3)\sum_{i=1}^{m}B_i^2\frac{\partial\tilde{F}}{\partial B_i}-\sum_{i=1}^{m}B_i^3\frac{\partial^2\tilde{F}}{\partial B_i^2}-2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{B_i^3}{B_i-B_j}\frac{\partial\tilde{F}}{\partial B_i}$$
$$=ab[\mathrm{tr}\,(B)]\tilde{F},$$

where $A_1, A_2, \cdots, A_m$ and $B_1, B_2, \cdots, B_m$ are the latent roots of the $m\times m$ Hermitian matrices $A$ and $B$ respectively. We note here that the latent roots of a Hermitian matrix are real numbers.

The methods adopted are essentially extensions of those, due to Muirhead [18] and Constantine and Muirhead [3], used for deriving the p.d.e.'s satisfied by the hypergeometric functions ${}_2F_1$ in the real case, to the complex case. Using (1.4) and (1.5) we can obtain the systems of p.d.e's and the p.d.e.'s for the functions ${}_1\tilde{F}_1$, ${}_0\tilde{F}_1$, ${}_1\tilde{F}_0$ and ${}_0\tilde{F}_0$. The results are shown in Sections 2 and 3 for the cases of one argument matrix and of two argument matrices respectively.

The paper concludes with presenting useful applications of the p.d.e.'s to obtaining asymptotic expansions for distributions in complex multivariate analysis.

## 2. A system of partial differential equations

In this section it is shown that the function ${}_2\tilde{F}_1(a, b; c; A)$ satisfies the system of p.d.e.'s (1.4). Muirhead [18] derived a system of p.d.e.'s satisfied by the ${}_2F_1$ function of a real argument matrix. His method is extended to our complex case.

Denote by $\binom{\kappa}{\sigma}$ the coefficient of $\tilde{C}_\sigma(A)/\tilde{C}_\sigma(I)$ in the "complex binomial" expansion

$$(2.1)\qquad \tilde{C}_\kappa(I+A)/\tilde{C}_\kappa(I)=\sum_{s=0}^{k}\sum_{\sigma}\binom{\kappa}{\sigma}\tilde{C}_\sigma(A)\Big/\tilde{C}_\sigma(I)\,.$$

We use the following differential operators

$$(2.2)\qquad E=\sum_{i=1}^{m}A_i\partial/\partial A_i\,,$$

$$(2.3)\qquad \varepsilon=\sum_{i=1}^{m}\partial/\partial A_i\,,$$

$$\tilde{D}^* = \sum_{i=1}^{m} A_i^2 \partial^2/\partial A_i^2 + 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} [A_i^2/(A_i - A_j)]\partial/\partial A_i \tag{2.4}$$

and

$$\tilde{\delta}^* = \sum_{i=1}^{m} A_i \partial^2/\partial A_i^2 + 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} [A_i/(A_i - A_j)]\partial/\partial A_i . \tag{2.5}$$

Here we notice the slight difference between the coefficients of the terms in $\tilde{D}^*$ and $\tilde{\delta}^*$ and those in $D^*$ and $\delta^*$ needed for the real case in Muirhead [18]. Corresponding to the partition $\kappa$, let

$$\kappa_i = (k_1, k_2, \cdots, k_i + 1, \cdots, k_m) \quad \text{and} \quad \kappa^{(i)} = (k_1, k_2, \cdots, k_i - 1, \cdots, k_m) \tag{2.6}$$

wherever they are admissible, i.e. so long as the parts are in non-increasing order.

We obtain the effect of the operators $E$, $\varepsilon$, $\tilde{D}^*$ and $\tilde{\delta}^*$ in

LEMMA 2.1.

$$E\tilde{C}_\kappa(A) = k\tilde{C}_\kappa(A) , \tag{2.7}$$

$$\varepsilon\tilde{C}_\kappa(A)/\tilde{C}_\kappa(I) = \sum_{i=1}^{m} \binom{\kappa}{\kappa^{(i)}} \tilde{C}_{\kappa^{(i)}}(A) \Big/ \tilde{C}_{\kappa^{(i)}}(I) , \tag{2.8}$$

$$\tilde{D}^*\tilde{C}_\kappa(A) = \left[\sum_{i=1}^{m} k_i(k_i - 2i) + k(2m-1)\right]\tilde{C}_\kappa(A) \tag{2.9}$$

*and*

$$\tilde{\delta}^*\tilde{C}_\kappa(A)/\tilde{C}_\kappa(I) = \sum_{i=1}^{m} \binom{\kappa}{\kappa^{(i)}} (k_i - i + m - 1)\tilde{C}_{\kappa^{(i)}}(A) \Big/ \tilde{C}_{\kappa^{(i)}}(I) . \tag{2.10}$$

PROOF. We can prove them in the same manner as James [11] and Muirhead [18] did for the real case. Therefore, it suffices to note that $\tilde{C}_\kappa(A)$ is a homogeneons polynomial of degree $k$, that $\tilde{C}_\kappa(A) = c_\kappa A_1^{k_1} A_2^{k_2} \cdots A_m^{k_m}$ + terms of lower weight, which is shown by (1.3) and James [10], eq. (113), and that $\tilde{\delta}^* = (\varepsilon\tilde{D}^* - \tilde{D}^*\varepsilon)/2$, in order to prove (2.7), (2.9) and (2.10) respectively.

We apply Muirhead's approach [18] to our complex case, using the results obtained in Lemma 2.1. The details are omitted here and only the final result is summarized in the following

THEOREM 2.1. *The function* ${}_2\tilde{F}_1(a, b; c; A)$ *is the unique solution of each of the differential equations*

$$A_i(1 - A_i)\frac{\partial^2 \tilde{F}}{\partial A_i^2} + \left[c - m + 1 - (a + b - m + 2)A_i + \sum_{\substack{j=1\\ j\neq i}}^{m} \frac{A_i(1 - A_i)}{A_i - A_j}\right]\frac{\partial \tilde{F}}{\partial A_i} \tag{2.11}$$

$$-\sum_{\substack{j=1\\j\neq i}}^{m}\frac{A_j(1-A_j)}{A_i-A_j}\frac{\partial\tilde{F}}{\partial A_j}=ab\tilde{F}\qquad(i=1,2,\cdots,m),$$

*subject to the conditions*

(a) $\tilde{F}$ *is a symmetric function of* $A_1, A_2, \cdots, A_m$, *and*

(b) $\tilde{F}$ *is analytic about* $A=0$, *and* $\tilde{F}(0)=1$.

From Theorem 2.1 and the confluences

$$\lim_{b\to\infty}{}_2\tilde{F}_1(a, b; c; b^{-1}A)={}_1\tilde{F}_1(a; c; A)\quad\text{and}$$

$$\lim_{a\to\infty}{}_1\tilde{F}_1(a; c; a^{-1}A)={}_0\tilde{F}_1(c; A),$$

the systems of p.d.e.'s satisfied by the ${}_1\tilde{F}_1$ and ${}_0\tilde{F}_1$ functions are given in

COROLLARY 2.1. *The function* ${}_1\tilde{F}_1(a; c; A)$ *is the unique solution of the system of p.d.e.'s*

$$A_i\frac{\partial^2\tilde{F}}{\partial A_i^2}+\Big[c-m+1-A_i+\sum_{\substack{j=1\\j\neq i}}^{m}\frac{A_i}{A_i-A_j}\Big]\frac{\partial\tilde{F}}{\partial A_i}-\sum_{\substack{j=1\\j\neq i}}^{m}\frac{A_i}{A_i-A_j}\frac{\partial\tilde{F}}{\partial A_j}=a\tilde{F}\qquad(i=1,2,\cdots,m),\tag{2.12}$$

*and the function* ${}_0\tilde{F}_1(c; A)$ *is the unique solution of the system of p.d.e.'s*

$$A_i\frac{\partial^2\tilde{F}}{\partial A_i^2}+\Big[c-m+1+\sum_{\substack{j=1\\j\neq i}}^{m}\frac{A_i}{A_i-A_j}\Big]\frac{\partial\tilde{F}}{\partial A_i}-\sum_{\substack{j=1\\j\neq i}}^{m}\frac{A_j}{A_i-A_j}\frac{\partial\tilde{F}}{\partial A_j}=\tilde{F}\qquad(i=1,2,\cdots,m),\tag{2.13}$$

*subject to the same conditions as in Theorem* 2.1.

## 3. A partial differential equation

In this section, we shall show that the function ${}_2\tilde{F}_1(a, b; c; A, B)$ satisfies the p.d.e. (1.5). Constantine and Muirhead [3] established a p.d.e. satisfied by the ${}_2F_1$ function of two real argument matrices. Their method can be extended to our complex case.

Let us use the following differential operators

$$\tilde{D}_A^*=\sum_{i=1}^{m}A_i^2\partial^2/\partial A_i^2+2\sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m}[A_i^2/(A_i-A_j)]\partial/\partial A_i,\tag{3.1}$$

$$\tilde{\delta}_A^*=\sum_{i=1}^{m}A_i\partial^2/\partial A_i^2+2\sum_{i=1}^{m}\sum_{\substack{j=1\\j\neq i}}^{m}[A_i/(A_i-A_j)]\partial/\partial A_i,\tag{3.2}$$

$$\varepsilon_A=\sum_{i=1}^{m}\partial/\partial A_i,\tag{3.3}$$

$$\gamma_A = \sum_{i=1}^{m} A_i^2 \partial/\partial A_i \tag{3.4}$$

and

$$\tilde{\eta}_A = \sum_{i=1}^{m} A_i^3 \partial^2/\partial A_i^2 + 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} [A_i^3/(A_i - A_j)]\partial/\partial A_i + (2-m)\gamma_A\,. \tag{3.5}$$

Here we notice the difference between the coefficients of the terms in $\tilde{D}_A^*$, $\tilde{\delta}_A^*$ and $\tilde{\eta}_A$ and those in $D_A^*$, $\delta_A^*$ and $\eta_A$ used for the real case in Constantine and Muirhead [3]. We use the same notation as in Section 2.

From Lemma 2.1 we have

$$\tilde{D}_A^* \tilde{C}_\kappa(A) = \left[\sum_{i=1}^{m} k_i(k_i - 2i) + k(2m-1)\right]\tilde{C}_\kappa(A) \tag{3.6}$$

$$\tilde{\delta}_A^* \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I) = \sum_{i=1}^{m} \binom{\kappa}{\kappa^{(i)}}(k_i - i + m - 1)\tilde{C}_{\kappa^{(i)}}(A)\Big/\tilde{C}_{\kappa^{(i)}}(I) \tag{3.7}$$

and

$$\varepsilon_A \tilde{C}_\kappa(A)/\tilde{C}_\kappa(I) = \sum_{i=1}^{m} \binom{\kappa}{\kappa^{(i)}} \tilde{C}_{\kappa^{(i)}}(A)\Big/\tilde{C}_{\kappa^{(i)}}(I)\,. \tag{3.8}$$

Applying the operator $\varepsilon_B$ with (3.8) to both sides of the well-known result (James [10])

$$\int_{U(m)} \operatorname{etr}(AUB\bar{U}')dU = \sum_{k=0}^{\infty}\sum_{\kappa} \tilde{C}_\kappa(A)\tilde{C}_\kappa(B)/\tilde{C}_\kappa(I)k!$$

and comparing the coefficient of $\tilde{C}_\kappa(B)$ on both sides gives

$$\sum_{i=0}^{\infty} \binom{\kappa_i}{\kappa} \tilde{C}_{\kappa_i}(A) = (k+1)\operatorname{tr}(A)\tilde{C}_\kappa(A)\,. \tag{3.9}$$

The effect of the operator $\gamma_A$ and $\tilde{\eta}_A$ are given in

LEMMA 3.1.

$$\gamma_A \tilde{C}_\kappa(A) = \frac{1}{k+1}\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(k_i - i + 1)\tilde{C}_{\kappa_i}(A) \tag{3.10}$$

*and*

$$\tilde{\eta}_A \tilde{C}_\kappa(A) = \frac{1}{k+1}\sum_{i=1}^{m} \binom{\kappa_i}{\kappa}(k_i - i + 1)(k_i - i + m)\tilde{C}_{\kappa_i}(A)\,. \tag{3.11}$$

PROOF. The proof is similar to that for the real case; so that it suffices to note the fact $\tilde{\eta}_A = (\tilde{D}_A^* \gamma_A - \gamma_A \tilde{D}_A^*)/2$.

Applying Constantine and Muirhead's approach [3] to our complex case, with the results obtained above, gives the final result in

THEOREM 3.1. *The function* ${}_2\tilde{F}_1(a, b; c; A, B)$ *is the unique solution of the p.d.e.*

$$\tilde{\delta}_A^* \tilde{F} + (c-m+1)\varepsilon_A \tilde{F} - (a+b-m+1)\gamma_B \tilde{F} - \tilde{\eta}_B \tilde{F} = ab[\mathrm{tr}\,(B)]\tilde{F} \tag{3.12}$$

*i.e.*

$$\sum_{i=1}^m A_i \frac{\partial^2 \tilde{F}}{\partial A_i^2} + 2\sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m \frac{A_i}{A_i - A_j} \frac{\partial \tilde{F}}{\partial A_i} + (c-m+1)\sum_{i=1}^m \frac{\partial \tilde{F}}{\partial A_i} - (a+b-2m+3)\sum_{i=1}^m B_i^2 \frac{\partial \tilde{F}}{\partial B_i} - \sum_{i=1}^m B_i^3 \frac{\partial^2 \tilde{F}}{\partial B_i^2} - 2\sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m \frac{B_i^3}{B_i - B_j} \frac{\partial \tilde{F}}{\partial B_i} = ab[\mathrm{tr}\,(B)]\tilde{F}\,, \tag{3.13}$$

*subject to the condition that* $\tilde{F}$ *may be expressed in the series form*

$$\tilde{F}(A, B) = \sum_{k=0}^m \sum_{\kappa} \alpha_\kappa \tilde{C}_\kappa(A)\tilde{C}_\kappa(B)/\tilde{C}_\kappa(I)\,,$$

*where* $\tilde{F}(0, 0) = 1$, *i.e.* $\alpha_{(0)} = 1$.

From Theorem 3.1 and the confluences

$$\lim_{b\to\infty} {}_2\tilde{F}_1(a, b; c; A, b^{-1}B) = {}_1\tilde{F}_1(a; c; A, B) \quad \text{and}$$

$$\lim_{a\to\infty} {}_1\tilde{F}_1(a; c; A, a^{-1}B) = {}_0\tilde{F}_1(c; A, B)\,,$$

the p.d.e.'s for the ${}_1\tilde{F}_1$ and ${}_0\tilde{F}_1$ functions are established in

COROLLARY 3.1. *The function* ${}_1\tilde{F}_1(a; c; A, B)$ *satisfies the p.d.e.*

$$\sum_{i=1}^m A_i \frac{\partial^2 \tilde{F}}{\partial A_i^2} + 2\sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m \frac{A_i}{A_i - A_j} \frac{\partial \tilde{F}}{\partial A_i} + (c-m+1)\sum_{i=1}^m \frac{\partial \tilde{F}}{\partial A_i} - \sum_{i=1}^m B_i^2 \frac{\partial \tilde{F}}{\partial B_i} = a[\mathrm{tr}\,(B)]\tilde{F} \tag{3.14}$$

*and the function* ${}_0\tilde{F}_1(c; A, B)$ *satisfies the p.d.e.*

$$\sum_{i=1}^m A_i \frac{\partial^2 \tilde{F}}{\partial A_i^2} + 2\sum_{i=1}^m \sum_{\substack{j=1 \\ j\neq i}}^m \frac{A_i}{A_i - A_j} \frac{\partial \tilde{F}}{\partial A_i} + (c-m+1)\sum_{i=1}^m \frac{\partial \tilde{F}}{\partial A_i} = \mathrm{tr}\,(B)\tilde{F}\,, \tag{3.15}$$

*subject to the condition in Theorem* 3.1.

Putting $b=c=m-1$ in (3.13) and $a=c=m-1$ in (3.14) establishes

COROLLARY 3.2. *The function* ${}_1\tilde{F}_0(a; A, B)$ *satisfies the p.d.e.*

$$(3.16)\quad \sum_{i=1}^{m} A_i \frac{\partial^2 \tilde{F}}{\partial A_i^2} + 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{A_i}{A_i - A_j}\frac{\partial \tilde{F}}{\partial A_i} - (a-m+2)\sum_{i=1}^{m} B_i^2 \frac{\partial \tilde{F}}{\partial B_i} - \sum_{i=1}^{m} B_i^3 \frac{\partial^2 \tilde{F}}{\partial B_i^2}$$
$$-2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{B_i^3}{B_i - B_j}\frac{\partial \tilde{F}}{\partial B_i} = (m-1)a[\operatorname{tr}(B)]\tilde{F}$$

*and the function* ${}_0\tilde{F}_0(A, B)$ *satisfies the p.d.e.*

$$(3.17)\quad \sum_{i=1}^{m} A_i \frac{\partial^2 \tilde{F}}{\partial A_i^2} + 2\sum_{i=1}^{m}\sum_{\substack{j=1\\ j\neq i}}^{m} \frac{A_i}{A_i - A_j}\frac{\partial \tilde{F}}{\partial A_i} - \sum_{i=1}^{m} B_i^2 \frac{\partial \tilde{F}}{\partial B_i} = (m-1)\operatorname{tr}(B)\tilde{F},$$

*subject to the condition in Theorem* 3.1.

## 4. Applications

We conclude the paper with presenting applications of the p.d.e.'s for the hypergeometric functions, developed in the previous sections. Let $A$ be the covariance matrix formed from a sample of size $n+1$ drawn from a complex $m$-variate normal distribution with population covariance matrix $\Sigma$ (assumed to be positive definite); then $nA$ has the complex Wishart distribution $W_m^c(n, \Sigma)$ (see e.g. James [10], p. 489). Let $l_1 > l_2 > \cdots > l_m > 0$ and $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_m > 0$ denote the latent roots of $A$ and $\Sigma$ respectively. In this section we shall derive asymptotic expansions for the distributions of the extreme roots $l_1$ and $l_m$ of $A$ by a partial differential equation method, when the population roots $\lambda_1, \lambda_2, \cdots, \lambda_m$ are simple.

The sample spectral density matrix of a multiple stationary Gaussian time series is distributed as complex Wishart. In this connection, it is of great use to work with the distribution of the latent roots $l_1, l_2, \cdots, l_m$ in our notation. Priestley, Subba Rao and Tong [22] obtained asymptotic distributions of the likelihood ratio criteria for testing hypotheses concerning $\lambda_i$.

The problem of deriving asymptotic expansions for the distributions of the latent roots of the sample covariance matrix formed from a real multivariate normal distribution has been considered by Muirhead and Chikuse [21].

Use will be made of the following results.

LEMMA 4.1. *The complex generalization of Laplace transform is given as*

$$(4.1)\quad \frac{1}{\tilde{\Gamma}_m(a)}\int_{0<\bar{T}'=T} \operatorname{etr}(-T)\det T^{a-m}\, {}_p\tilde{F}_q(a_1, \cdots, a_p; b_1, \cdots, b_q; TA)dT$$
$$= {}_{p+1}\tilde{F}_q(a_1, \cdots, a_p, a; b_1, \cdots, b_q; A).$$

*The "complex" confluent hypergeometric function has the integral representation*

(4.2) $${}_1\tilde{F}_1(a\,;c\,;A) = \frac{\tilde{\Gamma}_m(c)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(c-a)}\int_{0<\bar{T}'=T<I} \operatorname{etr}(TA)\det T^{a-m}\det(I-T)^{c-a-m}dT\,,$$

*holding for all* $A$, $\operatorname{Re} a>m-1$ *and* $\operatorname{Re}(c-a)>m-1$. *The "complex" Gaussian hypergeometric function has the integral representation*

(4.3) $${}_2\tilde{F}_1(a, b\,;c\,;A) = \frac{\tilde{\Gamma}_m(c)}{\tilde{\Gamma}_m(a)\tilde{\Gamma}_m(c-a)} \cdot\int_{0<\bar{T}'=T<I} \det T^{a-m}\det(I-T)^{c-a-m}\det(I-AT)^{-b}dT\,,$$

*valid for* $\operatorname{Re} A<I$, $\operatorname{Re} a>m-1$, $\operatorname{Re}(c-a)>m-1$ *and all* $b$. *Here we have*

$$\tilde{\Gamma}_m(a)=\pi^{m(m-1)/2}\prod_{i=1}^{m}(a-i+1)\,.$$

Proof. (4.1) and (4.2) are verified by Fujikoshi [4]. (4.3) follows from using (4.1) and (4.2) and interchanging the order of integration (see Herz [9] for the proof in the real case).

We consider first the distribution function of the largest root $l_1$. Since $nA=S$ is distributed as Wishart $W_m^c(n, \Sigma)$, we have, with (4.2)

(4.4) $$\begin{aligned} \mathrm{P}(l_1<y) &= \mathrm{P}(A<yI)=\mathrm{P}(S<nyI) \\ &= \int_{0<\bar{S}=S<nyI} \frac{1}{\tilde{\Gamma}_m(n)\det\Sigma^n}\operatorname{etr}(-\Sigma^{-1}S)\det S^{n-m}dS \\ &= \frac{(ny)^{nm}}{\tilde{\Gamma}_m(n)\det\Sigma^n}\int_{0<\bar{T}'=T<I}\operatorname{etr}(-ny\Sigma^{-1}T)\det T^{n-m}dT \\ &= \frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(n+m)}\det(ny\Sigma^{-1})^n{}_1\tilde{F}_1(n\,;n+m\,;-ny\Sigma^{-1})\,. \end{aligned}$$

Sugiyama [24] has obtained an approximation to $\mathrm{P}(l_1<y)$ in terms of a product of $\chi^2$ probabilities. Since (4.4) depends on $\Sigma$ only via its latent roots, we can regard $\Sigma$ as being diagonal i.e. $\Sigma=\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. From (4.4) the distribution function of $x_1=n^{1/2}(l_1/\lambda_1-1)$ can be written as

(4.5) $$\mathrm{P}(x_1<x)=\frac{\tilde{\Gamma}_m(m)}{\tilde{\Gamma}_m(n+m)}\det R^n{}_1\tilde{F}_1(n\,;n+m\,;-R)\,,$$

where $R=\operatorname{diag}(r_1, r_2, \cdots, r_m)$ with $r_i=(n+n^{1/2}x)z_i$, $z_i=\lambda_1/\lambda_i$ $(i=1, 2, \cdots,$

$m$). Starting with the system of p.d.e.'s (2.12) satisfied by the ${}_1\tilde{F}_1$ function it can be readily verified that $P \equiv \mathrm{P}(x_1 < x)$ satisfies each of the $m$ p.d.e.'s

$$\begin{aligned}(4.6)\quad & \left(1+\frac{2x}{\sqrt{n}}+\frac{x^2}{n}\right)\frac{\partial^2 P}{\partial x^2}+\left[x+\frac{1}{\sqrt{n}}(1+x^2-A_1)+\frac{x}{n}(1-A_1)\right]\frac{\partial P}{\partial x} \\ & -\sum_{k=2}^{m}\left[\frac{x}{\sqrt{n}}-\frac{1}{n}\left(1+A_1-\frac{1}{1-z_k}\right)\right]z_k\frac{\partial P}{\partial z_k} \\ & -2\sum_{k=2}^{m}\left(\frac{1}{\sqrt{n}}+\frac{x}{n}\right)z_k\frac{\partial^2 P}{\partial x\partial z_k}+\frac{1}{n}\sum_{k=2}^{m}\sum_{j=2}^{m}z_k z_j\frac{\partial^2 P}{\partial z_k\partial z_j}=0\end{aligned}$$

and

$$\begin{aligned}(4.7)\quad & \frac{z_i}{n}\frac{\partial^2 P}{\partial z_i^2}+\frac{1}{1-z_i}\left(\frac{1}{\sqrt{n}}+\frac{x}{n}\right)\frac{\partial P}{\partial x} \\ & +\left[z_i-1+\frac{x}{\sqrt{n}}z_i+\frac{1}{n}\left(1+\sum_{\substack{j=1\\ j\neq i}}^{m}\frac{z_i}{z_i-z_j}\right)\right]\frac{\partial P}{\partial z_i} \\ & -\frac{1}{n}\frac{1}{1-z_i}\sum_{k=2}^{m}z_k\frac{\partial P}{\partial z_k}-\frac{1}{n}\sum_{\substack{j=2\\ j\neq i}}^{m}\frac{z_j}{z_i-z_j}\frac{\partial P}{\partial z_j}=0 \qquad (i=2,\cdots,m),\end{aligned}$$

with $A_1=\sum_{j=2}^{m}1/(z_j-1)$.

We now look for a solution of these $m$ p.d.e.'s (4.6) and (4.7) of the form

$$P=\Phi(x)+n^{-k/2}Q_k\,, \tag{4.8}$$

where $\Phi(\cdot)$ denotes the standard normal distribution function and the $Q_k$ are functions of $x, z_2, \cdots, z_m$.

Now that $P$ possesses such an expansion as (4.8) follows from the asymptotic expansion for the joint density function of $l_1, l_2, \cdots, l_m$, given by Li, Pillai and Chang [17], eq. (5.29). Adopting the same manner as in Muirhead and Chikuse [21] used for the real case, we can show that if $\lambda_i$ is a simple root then, for large $n$, $l_i$ is asymptotically independent of the other sample roots and the limiting distribution of $n^{1/2}(l_i/\lambda_i-1)$ is standard normal $N(0,1)$.

Substituting (4.8) into (4.6) and (4.7) and solving the resulting equations gives the expansion in

THEOREM 4.1. *Let $l_1$ and $\lambda_1$ be the largest roots of $A$ and $\Sigma$ respectively, where $nA$ is distributed as complex Wishart $W_m^c(n,\Sigma)$ and the roots of $\Sigma$ are simple. Then the distribution function of $x_1=n^{1/2}(l_1/\lambda_1-1)$ can be expanded for large $n$ as*

$$\mathrm{P}(x_1<x)=\Phi(x)+n^{-1/2}Q_1+n^{-1}Q_2+O(n^{-3/2})\,, \tag{4.9}$$

*where $\Phi(\cdot)$ denotes the standard normal distribution function,*

$$(4.10)\qquad Q_1=-\frac{1}{3}\phi(x)[H_2(x)+3A_1H_0(x)]$$

*and*

$$(4.11)\qquad Q_2=-\frac{1}{36}\phi(x)[2H_5(x)+3(3+4A_1)H_3(x)+18(A_1^2-B_1)H_1(x)]\,,$$

*with*

$$A_1=\sum_{j=2}^{m}1/(z_j-1)\,,\quad B_1=\sum_{j=2}^{m}1/(z_j-1)^2\,,\quad z_j=\lambda_1/\lambda_j\qquad (j=1,2,\cdots,m),$$

*and* $H_j(x)$ *the Hermite polynomial of degree* $j$ *tabulated to* $j=10$ *in Kendall and Stuart* [12], p. 155.

We now consider the distribution of the smallest root $l_m$.

$$\begin{aligned}(4.12)\quad & \mathrm{P}\,(l_m>y)\\ &=\mathrm{P}\,(S>nyI)\\ &=\int_{nyI<\bar S'=S}\frac{1}{\tilde\Gamma_m(n)\det\Sigma^n}\operatorname{etr}(-\Sigma^{-1}S)\det S^{n-m}dS\\ &=\frac{(ny)^{nm}\operatorname{etr}(-ny\Sigma^{-1})}{\tilde\Gamma_m(n)\det\Sigma^n}\int_{0<\bar T'=T}\operatorname{etr}(-ny\Sigma^{-1}T)\det(I+T)^{n-m}dT\\ &\hspace{20em}(\because\quad T=(ny)^{-1}S-I)\\ &=\frac{\tilde\Gamma_m(m)}{\tilde\Gamma_m(n)}\det(ny\Sigma^{-1})^n\operatorname{etr}(-ny\Sigma^{-1})\tilde\Psi(m,n+m;ny\Sigma^{-1})\,.\end{aligned}$$

Here we define another confluent hypergeometric function of a Hermitian argument matrix by

DEFINITION.

$$(4.13)\quad \tilde\Psi(a,c;A)=\frac{1}{\tilde\Gamma_m(a)}\int_{0<\bar T'=T}\operatorname{etr}(-AT)\det T^{a-m}\det(I+T)^{c-a-m}dT\,,$$

holding for $\operatorname{Re}A>0$ and $\operatorname{Re}a>m-1$.

We need the following

LEMMA 4.2. *The functions* ${}_1\tilde F_1(a;c;A)$ *and* $\tilde\Psi(a,c;A)$ *both satisfy the same system of p.d.e.'s* (2.12).

PROOF. This is proved by the same argument as for the real case in Muirhead [19]. We can easily show, using Lemma 4.1, that

$$\lim_{c\to\infty}{}_2\tilde F_1(a,b;c;I-cA^{-1})=\det A^b\tilde\Psi(b,b-a+m;A)\,,$$

and then the required result follows from the system of p.d.e.'s for the ${}_2\tilde{F}_1$ function given by (2.11).

Put $x_m = n^{1/2}(l_m/\lambda_m - 1)$. Then using the system of p.d.e.'s (2.12) satisfied by the $\tilde{\Psi}$ function, with (4.12), we can readily obtain the expansion for the distribution function of $x_m$. Hence

THEOREM 4.2. *Let $l_m$ and $\lambda_m$ be the smallest roots of $A$ and $\Sigma$ respectively, where $nA$ is distributed as complex Wishart $W_m^c(n, \Sigma)$ and the roots of $\Sigma$ are simple. Then the distribution function of $x_m = n^{1/2}(l_m/\lambda_m - 1)$ can be expanded for large $n$ as*

$$\mathrm{P}\,(x_m < x) = \Phi(x) + n^{-1/2}Q_1 + n^{-1}Q_2 + O(n^{-3/2}) \,, \tag{4.14}$$

*where $z_i = \lambda_m/\lambda_{m-i+1}$ in $Q_1$ and $Q_2$ given by* (4.10) *and* (4.11) *respectively.*

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