ON THE REDUCTION TO A COMPLETE CLASS IN MULTIPLE DECISION PROBLEMS

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Summary

This paper is concerned with the reduction to a complete class of decision rule in case where actions, observations and states are finite in number. Discussions are made, first, from the viewpoint of the distribution of random observation and, secondly, from the viewpoint of loss function. With respect to loss function, "regret-relief ratio" criterion and "incremental loss-gain ratio" criterion are introduced and these usefulness are shown in 2-state of nature case.

1. Introduction

Let $L(\theta, a)$ be a loss function which is caused by taking action a when the state of nature is θ. The state of nature θ is unknown to the decision-maker, but he can get some information $x \in X$ about θ to guess the true state of nature. For each θ, there is a corresponding cumulative distribution function $F_X(x|\theta)$, which represents the distribution of X when the true state is θ. In this paper, we have assumed only the case Θ, A and X are finite, that is, $\Theta=\{\theta_1, \cdots, \theta_k\}$, $A=\{a_1, \cdots, a_n\}$, $X=\{x_1, \cdots, x_m\}$. We define a non-randomized decision rule (a non-randomized decision function) $d \in D$ and a randomized decision rule $\delta \in D^*$ as

$$a=d(x)\ , \qquad \text{and} \qquad \delta=\sum_{i=1}^{t} \pi_i d_i$$

where $\pi_i \geqq 0$ for all i and $\sum_{i=1}^{t} \pi_i=1$. The goodness of d and δ would be measured by the magnitude of risk defined by

$$(1.1) \qquad R(\theta, d)=\mathrm{E}_\theta\, L(\theta, d(X))=\int L(\theta, d(x))dF(x|\theta)$$
$$=\sum_{i=1}^{m} L(\theta, d(x_i))f(x_i|\theta)\ ,$$

and

$$R(\theta, \delta)=\sum_{i=1}^{t} \pi_i R(\theta, d_i) . \tag{1.2}$$

The minimax risk criterion for selecting the best decision rule requires to minimize the expected risk defined by

$$r(\delta)=\mathrm{E}_w R(\theta, \delta)=\sum_{s=1}^{k} w_s R(\theta_s, \delta) \tag{1.3}$$

where $W=(w_1, \cdots, w_k)$ is a weight function on Θ, usually called "prior distribution" or "degree of belief."

If the prior distribution W is not known, a reasonable way in choosing an action is to use the complete class, from which we pick up, in some way or other, a particular decision rule. It is the reasonable set of decision rules in the meaning of natural ordering.

DEFINITION 1 (Natural Ordering). A decision rule δ_1, is said to be as good as a rule δ_2, if $R(\theta, \delta_1) \leqq R(\theta, \delta_2)$ for all $\theta \in \Theta$. A rule δ_1 is said to be better than a rule δ_2 if $R(\theta, \delta_1) \leqq R(\theta, \delta_2)$ for all $\theta \in \Theta$ and $R(\theta, \delta_1) < R(\theta, \delta_2)$ for at least one $\theta \in \Theta$.

DEFINITION 2 (Complete Class). A class C of decision rules, $C \subset D^*$, is said to be complete, if given any rule $\delta \in D^*$ not in C, there exists a rule $\delta_0 \in C$ that is better than δ.

2. Reduction to a complete class using probability distribution

Let us first define the monotone decision problem in case where actions, observations and states are finite in number.

DEFINITION 3 (Monotone Multiple Decision Problem). A multiple decision problem (i.e., involving more than three actions) is said to be monotone if

$$L(\theta_s, a_j)-L(\theta_s, a_{j+1}) \leqq 0 \quad (s \leqq j)$$

$$L(\theta_s, a_j)-L(\theta_s, a_{j+1}) \geqq 0 \quad (s > j) \qquad \text{for } j=1, 2, \cdots, n-1 .$$

DEFINITION 4 (Non-randomized Monotone Decision Rule). A non-randomized decision rule d is said to be monotone if

$$d(x_i)=a_\alpha , \quad d(x_{i'})=a_{\alpha'} \qquad (i < i')$$

then $\alpha \leqq \alpha'$.

Karlin and Rubin [1] show that if the distribution of random observation has monotone likelihood ratio, defined below, the class of non-randomized monotone decision rules is essentially complete for a monotone multiple decision problem.

DEFINITION 5 (Monotone Likelihood Ratio). The family of distribution with density $f(x|\theta)$ is said to have a monotone likelihood ratio if there is a function $t(x)$ such that the likelihood ratio

$$\frac{f(x|\theta_i)}{f(x|\theta_j)} \qquad (\theta_i<\theta_j)$$

is monotone function of $t(x)$.

Remark. Definitions 4 and 5 are seemingly different from than those originally given by Karlin and Rubin themselves (see [1]). If, however, actions, observations and states are all real numbers, which they assume, rearrangement according to their magnitudes makes the definition here coincide with the original definition. Thus our definition here is more general than theirs in that it can admit actions, states and observations which are qualitative rather than quantitative.

To emphasize the usefulness of Karlin and Rubin's theorem, the following theorem may be helpful.

THEOREM 1. *In case that* $X=\{x_1,\cdots,x_m\}$ *and* $A=\{a_1,\cdots,a_n\}$, *the number of non-randomized monotone decision rules,* $K(m,n)$, *is*

$$K(m,n)=\binom{n+m-1}{m}. \tag{2.1}$$

PROOF. One non-randomized decision rule d assigns an action to each x. Since $X=\{x_1,\cdots,x_m\}$ and $A=\{a_1,\cdots,a_n\}$, the number of ways to assign m actions among A to possible m observations is n^m. This is the number of all possible non-randomized decision rules and also the number of permutations of m actions taken out of all possible n actions permitting repetitions. Now let ${}_n\mathrm{H}_m$ be the number of combinations of m actions taken out of n permitting repetitions. Then, we have

$${}_n\mathrm{H}_m={}_{n+m-1}\mathrm{C}_m=\binom{n+m-1}{m}. \tag{2.2}$$

For each combination, only one permutation has a monotone ordering with respect to the suffix of actions. Hence the number of non-randomized monotone decision rules is equal to the number of combinations of m actions taken out of n permitting repetitions. Therefore, we obtain

$$K(m,n)={}_n\mathrm{H}_m={}_{n+m-1}\mathrm{C}_m=\binom{n+m-1}{m}.$$

Now let $\tau(m, n)$ be the reduction ratio defined as follows.

$$\tau(m, n)=\frac{\text{Number of non-randomized monotone decision rules}}{\text{Number of non-randomized decision rules}} \tag{2.3}$$

$$=\frac{\binom{n+m-1}{m}}{n^m} .$$

We show $\tau(m, n)$ for some m and n in Fig. 1.

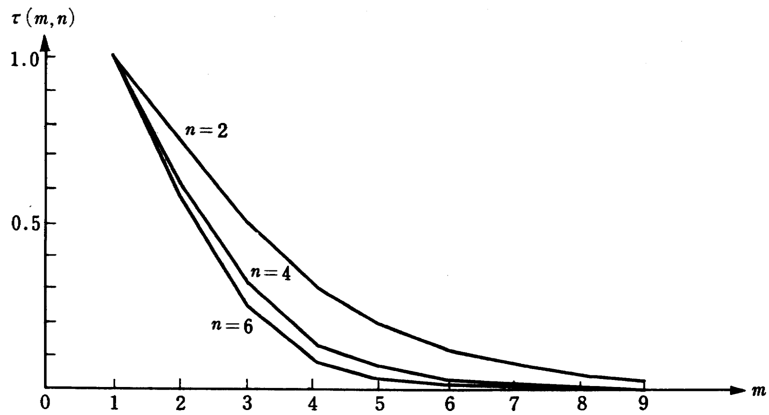

Fig. 1 $\tau(m, n)$

This figure shows Karlin and Rubin's theorem becomes very effective as m becomes larger.

3. Reduction to a complete class using loss function

In this section, as an opposite concept to "regret" of action a, "relief" of action a is defined and "regret-relief ratio" criterion and "incremental loss-gain ratio" criterion are introduced. Then using these criterions, the problem of reduction to the complete class in 2-state case is discussed with some examples.

3-1. *Regret-relief ratio criterion*

Let us define regret $\underline{l}(\theta, a)$, relief $\bar{l}(\theta, a)$ and regret-relief ratio $\gamma(\theta, a)$ of action a as follows.

DEFINITION 6 (Regret). The magnitude of regret caused by failing to take the best action is called regret $\underline{l}(\theta, a_k)$ of action a_k and defined by

$$\underline{l}(\theta, a_k)=L(\theta, a_k)-\min_{a \in A} L(\theta, a) . \tag{3.1}$$

DEFINITION 7 (Rerief). The magnitude of relief given by avoiding the worst action is called relief $\bar{l}(\theta, a_k)$ of action a_k and defined by

$$\bar{l}(\theta, a_k) = \max_{a \in A} L(\theta, a) - L(\theta, a_k) \ . \tag{3.2}$$

DEFINITION 8 (Regret-Relief Ratio). The regret-relief ratio $\gamma(\theta, a_k)$ of action a_k is defined by

$$\gamma(\theta, a_k) = \frac{\underline{l}(\theta, a_k)}{\bar{l}(\theta, a_k)} \ . \tag{3.3}$$

Remark. This might be infinity.

We shall now discuss the usefulness of regret-relief ratio $\gamma(\theta, a)$ in decision problems. The following consideration may help understand the concept of $\gamma(\theta, a)$. Let us suppose that the decision-maker is in doubt whether he should take action a_k or not. If $\bar{l}(\theta, a_k) < \underline{l}(\theta, a_k)$, then he may not take the decision rule which calls for a_k, and his behavior seems reasonable, for if he takes a_k then the magnitude of regret is greater than the magnitude of relief. In this case, regret-relief ratio is greater than 1,

$$\gamma(\theta, a_k) = \frac{\underline{l}(\theta, a_k)}{\bar{l}(\theta, a_k)} > 1 \ .$$

Therefore, from the preceding consideration, we can guess that regret-relief ratio may serve as a criterion for choosing a decision rule.

Now we examine the usefulness of regret-relief ratio in 2-state case. Let us suppose the case,

$$L(\theta_1, a_1) < L(\theta_1, a_2) < \cdots < L(\theta_1, a_n) \ ,$$

$$L(\theta_2, a_1) > L(\theta_2, a_2) > \cdots > L(\theta_2, a_n) \ ,$$

which reflects the preference orderings,

if the true state is θ_1, $a_1 \succ a_2 \succ \cdots \succ a_n$, and

if the true state is θ_2, $a_1 \prec a_2 \prec \cdots \prec a_n$,

where $a_i \succ a_j$ means that a_i is preferred to a_j. In other words, if the true state is θ_1, then a_1 is the best action and a_n the worst. On the contrary, if the true state is θ_2, then a_1 is the worst action and a_n the best. The regret and relief of action a_k are

$$\underline{l}(\theta_1, a_k) = L(\theta_1, a_k) - L(\theta_1, a_1) \ ,$$

$$\underline{l}(\theta_2, a_k) = L(\theta_2, a_k) - L(\theta_2, a_n) \ ,$$

$$\bar{l}(\theta_1, a_k) = L(\theta_1, a_n) - L(\theta_1, a_k) \ ,$$

$$\bar{l}(\theta_2, a_k)=L(\theta_2, a_1)-L(\theta_2, a_k)$$

and regret-relief ratios of action a_k are

$$\gamma(\theta_1, a_k)=\frac{l(\theta_1, a_k)}{\bar{l}(\theta_1, a_k)} \tag{3.4}$$

$$\gamma(\theta_2, a_k)=\frac{l(\theta_2, a_k)}{\bar{l}(\theta_2, a_k)} . \tag{3.5}$$

Using regret-relief ratio, we have obtained the following theorem and corollary.

THEOREM 2. *Assume that* $\Theta=\{\theta_1, \theta_2\}$, $X=\{x_1,\cdots, x_m\}$, $A=\{a_1,\cdots, a_n\}$ $w_s>0$ $(s=1, 2)$, $f(x_i|\theta)>0$ $(i=1, 2,\cdots, m;\ \theta\in\Theta)$ *and*

$$\begin{aligned} &L(\theta_1, a_1)<L(\theta_1, a_2)<\cdots<L(\theta_1, a_n) , \qquad \textit{and} \\ &L(\theta_2, a_1)>L(\theta_2, a_2)>\cdots>L(\theta_2, a_n) . \end{aligned} \tag{3.6}$$

For each $k=2,\cdots, n-1$, *the following holds. If the condition*

$$\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1 \tag{3.7}$$

is satisfied, then any non-randomized decision rule which takes action a_k *is dominated by some randomized decision rule.*

PROOF. Since any non-randomized decision rule $d\in D$ is defined by assigning $d(x_i)=a^i$ $(\in A)$ $(i=1, 2,\cdots, m)$, we write a non-randomized decision rule d in the form of an ordered m-tuple

$$d\equiv(a^1,\cdots, a^m) .$$

For $d=(a^1,\cdots, a^m)$, i $(i=1,\cdots, m)$ and $a_k\in A$, we define a new non-randomized decision rule

$$d^{(i)}*a_k=(a^1,\cdots, a^{i-1}, \underset{\substack{\uparrow \\ i\text{th place}}}{a_k}, a^{i+1},\cdots, a^m) .$$

That is, a non-randomized decision rule with a_k in ith place and remaining $a^1,\cdots, a^{i-1}, a^{i+1},\cdots, a^m$ unchanged from d. Then the non-randomized decision rules which take a_1 and a_n respectively in the case x_i is observed are written as follows;

$$d^{(i)}*a_1=(a^1,\cdots, a^{i-1}, a_1, a^{i+1},\cdots, a^m)$$

$$d^{(i)}*a_n=(a^1,\cdots, a^{i-1}, a_n, a^{i+1},\cdots, a^m) .$$

Note that a_1 is the best action in θ_1 and a_n the best in θ_2.

Now, we have to show, for each i $(i=1,\cdots, m)$ that some randomized decision rule $\delta^{(i)}$ which is the mixture of $d^{(i)} * a_1$ and $d^{(i)} * a_n$ dominates $d^{(i)} * a_k$. First, there exists a number q $(0<q<1)$ which satisfies the following equation;

$$R(\theta_1, d^{(i)} * a_k)=(1-q)R(\theta_1, d^{(i)} * a_1)+qR(\theta_1, d^{(i)} * a_n) \tag{3.8}$$

For let q be

$$\begin{aligned} q&=\frac{R(\theta_1, d^{(i)} * a_k)-R(\theta_1, d^{(i)} * a_1)}{R(\theta_1, d^{(i)} * a_n)-R(\theta_1, d^{(i)} * a_1)} \\ &=\frac{w_1 f(x_i|\theta_1)\{L(\theta_1, a_k)-L(\theta_1, a_1)\}}{w_1 f(x_i|\theta_1)\{L(\theta_1, a_n)-L(\theta_1, a_1)\}} \\ &=\frac{L(\theta_1, a_k)-L(\theta_1, a_1)}{L(\theta_1, a_n)-L(\theta_1, a_1)} . \end{aligned} \tag{3.9}$$

Then q satisfies (3.8) and $0<q<1$. Suppose $\delta^{(i)}$ is the randomized decision rule of $d^{(i)} * a_1$ and $d^{(i)} * a_n$ in the ratio of $1-q:q$. Then

$$\begin{aligned} R(\theta_1, \delta^{(i)})&=(1-q)R(\theta_1, d^{(i)} * a_1)+qR(\theta_1, d^{(i)} * a_n) \\ &=R(\theta_1, d^{(i)} * a_k) , \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} R(\theta_2, \delta^{(i)})&=(1-q)R(\theta_2, d^{(i)} * a_1)+qR(\theta_2, d^{(i)} * a_n) \\ &=\frac{1}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_n)-L(\theta_1, a_k)\}R(\theta_2, d^{(i)} * a_1) \\ &\quad +\{L(\theta_1, a_k)-L(\theta_1, a_1)\}R(\theta_2, d^{(i)} * a_n)] . \end{aligned} \tag{3.11}$$

Hence

$$\begin{aligned} &R(\theta_2, d^{(i)} * a_k)-R(\theta_2, \delta^{(i)}) \\ &=\frac{1}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_k)-L(\theta_1, a_1)\} \\ &\quad \cdot \{R(\theta_2, d^{(i)} * a_k)-R(\theta_2, d^{(i)} * a_n)\} \\ &\quad -\{L(\theta_1, a_n)-L(\theta_1, a_k)\}\{R(\theta_2, d^{(i)} * a_1)-R(\theta_2, d^{(i)} * a_k)\}] \\ &=\frac{w_2 f(x_i|\theta_2)}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_k)-L(\theta_1, a_1)\} \\ &\quad \cdot \{L(\theta_2, a_k)-L(\theta_2, a_n)\}-\{L(\theta_1, a_n)-L(\theta_1, a_k)\} \\ &\quad \cdot \{L(\theta_2, a_1)-L(\theta_2, a_k)\}] \\ &=\frac{w_2 f(x_i|\theta_2)}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_n)-L(\theta_1, a_k)\} \\ &\quad \cdot \{L(\theta_2, a_1)-L(\theta_2, a_k)\}] \\ &\quad \cdot \left\{\frac{L(\theta_1, a_k)-L(\theta_1, a_1)}{L(\theta_1, a_n)-L(\theta_1, a_k)}\,\frac{L(\theta_2, a_k)-L(\theta_2, a_n)}{L(\theta_2, a_1)-L(\theta_2, a_k)}-1\right\} \end{aligned} \tag{3.12}$$

$$=\frac{w_2 f(x_i|\theta_2)}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_n)-L(\theta_1, a_k)\}$$

$$\cdot\{L(\theta_2, a_1)-L(\theta_2, a_k)\}]\left\{\frac{l(\theta_1, a_k)}{\bar{l}(\theta_1, a_k)}\frac{l(\theta_2, a_k)}{\bar{l}(\theta_2, a_k)}-1\right\}$$

$$=\frac{w_2 f(x_i|\theta_2)}{L(\theta_1, a_n)-L(\theta_1, a_1)}[\{L(\theta_1, a_n)-L(\theta_1, a_k)\}$$

$$\cdot\{L(\theta_2, a_1)-L(\theta_2, a_k)\}]\{\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)-1\}\ .$$

Since $w_2>0$, $f(x_i|\theta_2)>0$, $L(\theta_1, a_n)-L(\theta_1, a_1)>0$, $L(\theta_1, a_n)-L(\theta_1, a_k)>0$, $L(\theta_2, a_1)-L(\theta_2, a_k)>0$ and $\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1$, we get

$$R(\theta_2, d^{(i)} * a_k)>R(\theta_2, \delta^{(i)}) \tag{3.13}$$

which shows, together with (3.10), that $\delta^{(i)}$ dominates $d^{(i)} * a_k$.

COROLLARY 1. *Assume the condition* (3.6) *and* (3.7) *of Theorem* 2. *If a randomized decision rule* δ *adopts with a positive probability such a non-randomized decision rule* d *that takes action* a_k, *then it is dominated by some randomized decision rule.*

PROOF. By Theorem 2, we can find a randomized decision rule δ^* that dominates d, i.e.,

$$R(\theta, \delta^*)\leqq R(\theta, d) \qquad \text{for all } \theta \tag{3.14}$$

and

$$R(\theta_0, \delta^*)<R(\theta_0, d) \qquad \text{for some } \theta_0\in\Theta\ . \tag{3.15}$$

We represent

$$\delta^*=\rho d+\sum_{d_i\neq d}\rho_i d_i$$

with

$$\rho+\sum_{d_i\neq d}\rho_i=1\ .$$

Then the risk of δ^* is

$$R(\theta, \delta^*)=\rho R(\theta, d)+\sum_{d_i\neq d}\rho_i R(\theta, d_i)\ . \tag{3.16}$$

Let δ adopt d with probability π and other d_i's ($\neq d$) with probability π_i where $\pi+\sum_{d_i\neq d}\pi_i=1$, i.e., symbolically,

$$\delta=\pi d+\sum_{d_i\neq d}\pi_i d_i\ .$$

Then the risk of δ is

$$R(\theta, \delta) = \pi R(\theta, d) + \sum_{d_i \neq d} \pi_i R(\theta, d_i) \ . \tag{3.17}$$

Now define a randomized decision rule δ^{**} such that;

i) if $d_i \neq d$, d_i is adopted with probability $\pi_i + \pi\rho_i$,

ii) if $d_i = d$, d_i is adopted with probability $\pi\rho$

i.e., symbolically,

$$\delta^{**} = (\pi\rho)d + \sum_{d_i \neq d} (\pi_i + \pi\rho_i)d_i \ .$$

Then

$$\begin{aligned} R(\theta, \delta^{**}) &= \pi\rho R(\theta, d) + \sum_{d_i \neq d} (\pi_i + \pi\rho_i) R(\theta, d_i) \\ &= \pi\{\rho R(\theta, d) + \sum_{d_i \neq d} \rho_i R(\theta, d_i)\} + \sum_{d_i \neq d} \pi_i R(\theta, d_i) \\ &= \pi R(\theta, \delta^*) + \sum_{d_i \neq d} \pi_i R(\theta, d_i) \\ &\leqq \pi R(\theta, d) + \sum_{d_i \neq d} \pi_i R(\theta, d_i) \\ &= R(\theta, \delta) \ . \end{aligned} \tag{3.18}$$

Using (3.14) and (3.15),

$$R(\theta, \delta^{**}) \leqq R(\theta, \delta) \qquad \text{for all } \theta \ ,$$

with strict inequality when $\theta = \theta_0$, i.e.,

$$R(\theta_0, \delta^{**}) < R(\theta_0, \delta) \qquad \text{for some } \theta_0 \in \Theta \ .$$

3-2. *Geometric interpretation*

Now we give a geometric interpretation of Theorem 2. Suppose risks of $d^{(i)} * a_1$, $d^{(i)} * a_k$, $d^{(i)} * a_n$ are shown by Fig. 2, and let Δ_1 denote the slope of line segment between $d^{(i)} * a_1$ and $d^{(i)} * a_k$ and Δ_2 between $d^{(i)} * a_k$ and $d^{(i)} * a_n$, that is,

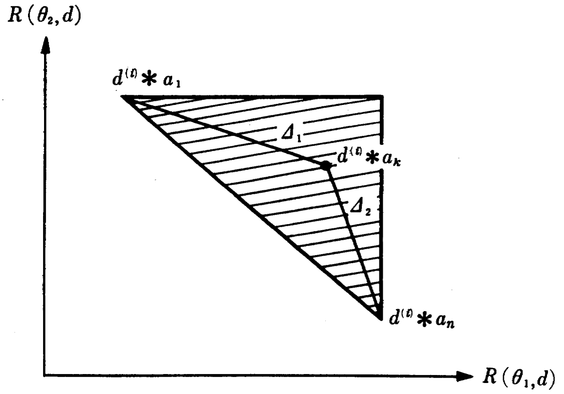

Fig. 2

$$\Delta_1=\frac{w_2 f(x_i|\theta_2)\{L(\theta_2, a_1)-L(\theta_2, a_k)\}}{w_1 f(x_i|\theta_1)\{L(\theta_1, a_1)-L(\theta_1, a_k)\}} \tag{3.19}$$

$$=-\frac{w_2 f(x_i|\theta_2)\bar{l}(\theta_2, a_k)}{w_1 f(x_i|\theta_1)\underline{l}(\theta_1, a_k)} \qquad (<0)$$

and

$$\Delta_2=\frac{w_2 f(x_i|\theta_2)\{L(\theta_2, a_k)-L(\theta_2, a_n)\}}{w_1 f(x_i|\theta_1)\{L(\theta_1, a_k)-L(\theta_1, a_n)\}} \tag{3.20}$$

$$=-\frac{w_2 f(x_i|\theta_2)\underline{l}(\theta_2, a_k)}{w_1 f(x_i|\theta_1)\bar{l}(\theta_1, a_k)} \qquad (<0)\,.$$

The ratio of slopes is

$$\frac{\Delta_2}{\Delta_1}=\frac{\underline{l}(\theta_1, a_k)\underline{l}(\theta_2, a_k)}{\bar{l}(\theta_1, a_k)\bar{l}(\theta_2, a_k)}=\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)\ , \tag{3.21}$$

and exactly equal to the product of regret-relief ratios in θ_1 and θ_2. This shows if $\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1$, then $\Delta_1>\Delta_2$ is satisfied, which implies that $d^{(i)} * a_k$ is contained in the shaded region and therefore that $d^{(i)} * a_k$ is dominated by some randomized decision rule of $d^{(i)} * a_1$ and $d^{(i)} * a_n$.

3-3. *Some examples of Theorem* 2 *and Corollary* 1

Example 1. Consider a decision problem of 2-state, 3-observation and 3-action with a loss function and a distribution of observation given by Tables 1 and 2, respectively. Prior probabilities assigned to θ_1, θ_2 are $w_1>0$, $w_2>0$, respectively.

Table 1. $L(\theta, a)$

θ \ a	a_1	a_2	a_3
θ_1	0	4	6
θ_2	5	2	0

Table 2. $f(x|\theta)$

θ \ x	x_1	x_2	x_3
θ_1	0.6	0.3	0.1
θ_2	0.2	0.3	0.5

The possible non-randomized decision rules are as follows;

$$\begin{array}{lll}
d_1=(a_1, a_1, a_1) & d_{10}=(a_2, a_1, a_1) & d_{19}=(a_3, a_1, a_1)\\
d_2=(a_1, a_1, a_2) & d_{11}=(a_2, a_1, a_2) & d_{20}=(a_3, a_1, a_2)\\
d_3=(a_1, a_1, a_3) & d_{12}=(a_2, a_1, a_3) & d_{21}=(a_3, a_1, a_3)\\
d_4=(a_1, a_2, a_1) & d_{13}=(a_2, a_2, a_1) & d_{22}=(a_3, a_2, a_1)\\
d_5=(a_1, a_2, a_2) & d_{14}=(a_2, a_2, a_2) & d_{23}=(a_3, a_2, a_2)\\
d_6=(a_1, a_2, a_3) & d_{15}=(a_2, a_2, a_3) & d_{24}=(a_3, a_2, a_3)\\
d_7=(a_1, a_3, a_1) & d_{16}=(a_2, a_3, a_1) & d_{25}=(a_3, a_3, a_1)\\
d_8=(a_1, a_3, a_2) & d_{17}=(a_2, a_3, a_2) & d_{26}=(a_3, a_3, a_2)\\
d_9=(a_1, a_3, a_3) & d_{18}=(a_2, a_3, a_3) & d_{27}=(a_3, a_3, a_3)
\end{array}$$

(Note; These are put in lexicographic orders with respect to the suffix of actions.) The product of regret-relief ratios of a_2 is

$$\gamma(\theta_1, a_2)\cdot\gamma(\theta_2, a_2)=\frac{4}{2}\,\frac{2}{3}=\frac{4}{3}>1 .$$

By Theorem 2 and Corollary 1, we know that a complete class is spanned by 8 decision rules d_1, d_3, d_7, d_9, d_{19}, d_{21}, d_{25}, d_{27}. Fig. 3 shows "risk points" $(R(\theta_1, d), R(\theta_2, d))$ of all 27 non-randomized decision rules and also shows that minimal complete class is spanned by 4 decision rules d_1, d_3, d_9 and d_{27}.

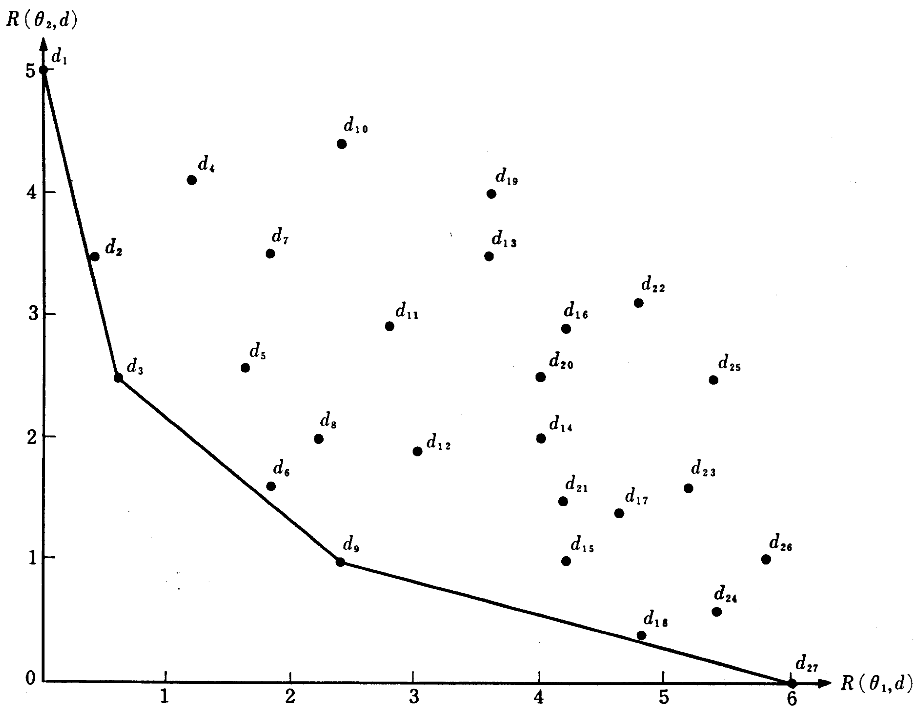

Fig. 3

Example 2. As an example which does not satisfy the condition in Theorem 2, let us suppose that only loss function changes into Table 3. Note that the condition,

$$\text{in case of } \theta_1,\ a_1 \succ a_2 \succ \cdots \succ a_n$$

$$\text{in case of } \theta_2,\ a_1 \prec a_2 \prec \cdots \prec a_n$$

still holds.

Table 3. $L(\theta, a)$

	a_1	a_2	a_3
θ_1	0	3	6
θ_2	5	2	0

Then the product of regret-relief ratios of action a_2 is

$$\gamma(\theta_1, a_2)\cdot\gamma(\theta_2, a_2)=1\cdot\frac{2}{3}<1 .$$

Risk points $(R(\theta_1, d), R(\theta_2, d))$ of all 27 non-randomized decision rules are shown by Fig. 4.

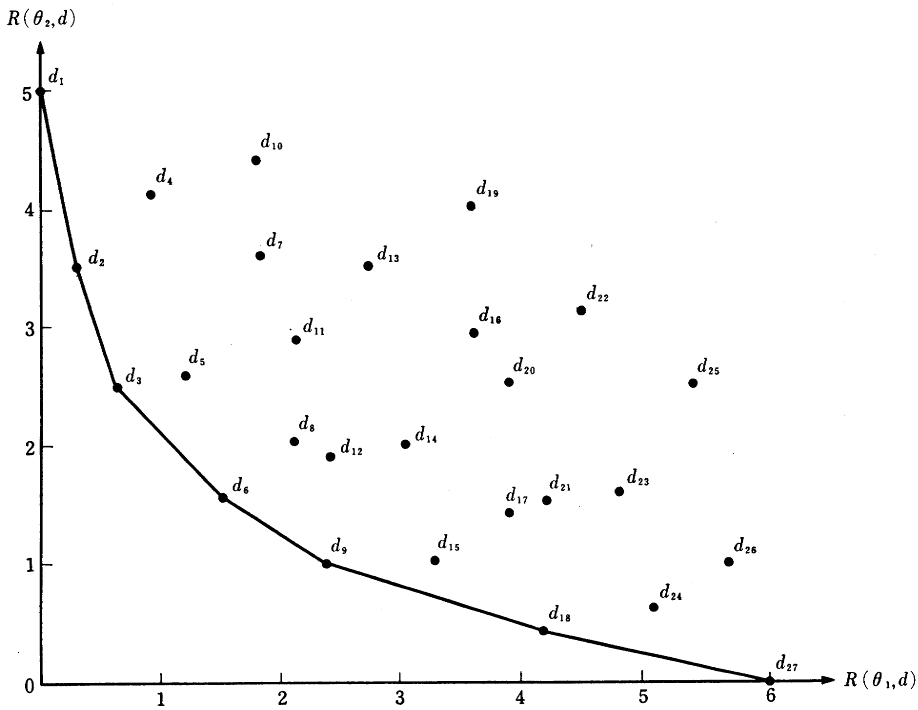

Fig. 4

As shown by this figure, the minimal complete class is spanned by 7 non-randomized decision rules d_1, d_2, d_3, d_6, d_9, d_{18}, d_{27} and decision rules which call for a_2 are contained. Thus the intermediate action, a_2, though never employed as the best one in any state of nature, becomes actually in use, when $\gamma(\theta_1, a_2)\cdot\gamma(\theta_2, a_2)<1$.

3-4. *The reduction in 2-state, 2-observation and n-action case*

The following theorem may be useful for the reduction to a complete class in 2-state, 2-observation and n-action case.

THEOREM 3. *Suppose that* $\Theta=\{\theta_1, \theta_2\}$, $X=\{x_1, x_2\}$, $A=\{a_1,\cdots,a_n\}$, $w_s>0$ $(s=1,2)$, $f(x_i|\theta)>0$ $(i=1,2;\ \theta\in\Theta)$ *and*

$$\begin{aligned}&L(\theta_1, a_1)<L(\theta_1, a_2)<\cdots<L(\theta_1, a_n) \qquad \textit{and}\\ &L(\theta_2, a_1)>L(\theta_2, a_2)>\cdots>L(\theta_2, a_n)\,.\end{aligned} \tag{3.22}$$

If the condition

$$\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1 \qquad \textit{for } k=2,\cdots,n-1 \tag{3.23}$$

is satisfied, then the minimal complete class is

i) *the set* C_3^b *of all randomized decision rules of the form either*

$$\lambda d_a+(1-\lambda)d_b \qquad (0\leqq\lambda\leqq1)$$

or

$$\lambda' d_b+(1-\lambda')d_d \qquad (0\leqq\lambda'\leqq1)$$

provided $f(x_2|\theta_1)<f(x_2|\theta_2)$, *and*

ii) *the set* C_3^c *of all randomized decision rules of the form either*

$$\lambda d_a+(1-\lambda)d_c \qquad (0\leqq\lambda\leqq1)$$

or

$$\lambda' d_c+(1-\lambda')d_d \qquad (0\leqq\lambda'\leqq1)$$

provided $f(x_2|\theta_1)>f(x_2|\theta_2)$, *where*

$$d_a=(a_1, a_1)\,, \quad d_b=(a_1, a_n)\,, \quad d_c=(a_n, a_1)\,, \quad d_d=(a_n, a_n)\,.$$

PROOF. By applying Theorem 2 and Corollary 1 to every a_k, $k=2,\cdots,n-1$ we conclude that a complete class is spanned by 4 non-randomized decision rules d_a, d_b, d_c, d_d, which call for only best actions. We express such complete class by

$$\begin{aligned}C_1=\{\rho_a d_a+\rho_b d_b+\rho_c d_c+\rho_d d_d;\ &\rho_a+\rho_b+\rho_c+\rho_d=1,\\ &\rho_a\geqq0,\ \rho_b\geqq0,\ \rho_c\geqq0,\ \rho_d\geqq0\}\,.\end{aligned}$$

The basic idea of further reduction from C_1 to C_3^b (or C_3^c) is as follows:

Step 1. Since d_a and d_d are not dominated and either d_b or d_c is dominated (this needs proof, which we omit here), we get a new complete class

$$C_2^b=\{\rho_a' d_a+\rho_b' d_b+\rho_d' d_d;\ \rho_a'+\rho_b'+\rho_d'=1,\ \rho_a'\geqq0,\ \rho_b'\geqq0,\ \rho_d'\geqq0\}$$
$$\text{if } f(x_2|\theta_1)<f(x_2|\theta_2)\,, \qquad \text{and}$$

$$C_2^c=\{\rho_a' d_a+\rho_c' d_c+\rho_d' d_d;\ \rho_a'+\rho_c'+\rho_d'=1,\ \rho_a'\geqq0,\ \rho_c'\geqq0,\ \rho_d'\geqq0\}$$
$$\text{if } f(x_2|\theta_1)>f(x_2|\theta_2)\,.$$

Step 2. C_2^b (or C_2^c) can be further reduced to a smaller complete class

$$C_3^b=\{\lambda d_a+(1-\lambda)d_b,\ \lambda' d_b+(1-\lambda')d_d;\ 0\leqq\lambda\leqq 1,\ 0\leqq\lambda'\leqq 1\}$$
$$\text{if } f(x_2|\theta_1)<f(x_2|\theta_2)\,, \qquad \text{and}$$

$$C_3^c=\{\lambda d_a+(1-\lambda)d_c,\ \lambda' d_c+(1-\lambda')d_d;\ 0\leqq\lambda\leqq 1,\ 0\leqq\lambda'\leqq 1\}$$
$$\text{if } f(x_2|\theta_1)>f(x_2|\theta_2)\,.$$

Step 3. C_3^b (or C_3^c) is minimal and terminates the reduction process.

If we prove C_3^b (or C_3^c) is complete, then the proof (*Step* 1) of completeness of C_2^b (or C_2^c) is bypassed since $C_3^b\subset C_2^b$ and $C_3^c\subset C_2^c$. Therefore we will prove that

i) C_3^b is complete, if $f(x_2|\theta_1)<f(x_2|\theta_2)$, and

ii) C_3^c is complete, if $f(x_2|\theta_1)>f(x_2|\theta_2)$.

i) According to Definition 2, it is enough to prove that given any randomized decision rule δ' not in C_3^b there exists a decision rule $\hat{\delta}\in C_3^b$ that is better than δ'. But when δ' ($\notin C_3^b$) is not in C_1, there exists

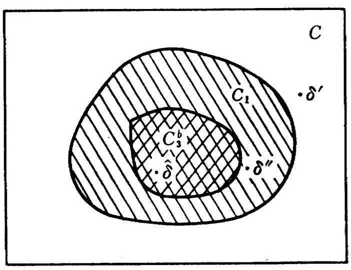

Fig. 5

δ'' in C_1 that is better than δ'. If $\delta''\in C_3^b$ then the proof is over. If on the other hand $\delta''\notin C_3^b$, it is enough to find $\delta'''\in C_3^b$ that is better than δ''. This condition enables us to assume without loss of generality that δ' is in C_1. Let us denote δ' as

$$\delta'=\rho_a d_a+\rho_b d_b+\rho_c d_c+\rho_d d_d\ ,$$

and introduce randomized decision rules

$$\hat{\delta}=\lambda d_a+(1-\lambda)d_b\ , \qquad \hat{\hat{\delta}}=\lambda' d_b+(1-\lambda')d_d\ ,$$

where

$$\lambda=\frac{\rho_a R(\theta_1, d_a)+\rho_b R(\theta_1, d_b)+\rho_c R(\theta_1, d_c)+\rho_d R(\theta_1, d_d)-R(\theta_1, d_b)}{R(\theta_1, d_a)-R(\theta_1, d_b)}$$
$$=\frac{f(x_2|\theta_1)(\rho_a+\rho_c)-f(x_1|\theta_1)(\rho_c+\rho_d)}{f(x_2|\theta_1)}\ ,$$

$$\lambda' = \frac{\rho_a R(\theta_1, d_a) + \rho_b R(\theta_1, d_b) + \rho_c R(\theta_1, d_c) + \rho_d R(\theta_1, d_d) - R(\theta_1, d_d)}{R(\theta_1, d_b) - R(\theta_1, d_d)}$$
$$= \frac{f(x_2 | \theta_1)(\rho_a + \rho_c) + f(x_1 | \theta_1)(\rho_a + \rho_b)}{f(x_1 | \theta_1)}$$

and $\delta' \neq \hat{\delta}$, $\delta' \neq \hat{\hat{\delta}}$. Then

(3.24) $$R(\theta_1, \delta') - R(\theta_1, \hat{\delta})$$
$$= \rho_a R(\theta_1, d_a) + \rho_b R(\theta_1, d_b) + \rho_c R(\theta_1, d_c) + \rho_d R(\theta_1, d_d)$$
$$- \lambda R(\theta_1, d_a) - (1 - \lambda) R(\theta_1, d_b)$$
$$= 0$$

$$R(\theta_2, \delta') - R(\theta_2, \hat{\delta})$$
$$= \rho_a R(\theta_2, d_a) + \rho_b R(\theta_2, d_b) + \rho_c R(\theta_2, d_c) + \rho_d R(\theta_2, d_d)$$
$$- \lambda R(\theta_2, d_a) - (1 - \lambda) R(\theta_2, d_b)$$
$$= \{L(\theta_2, a_1) - L(\theta_2, a_n)\} \{f(x_2 | \theta_2)(\rho_a + \rho_c)$$
$$- f(x_1 | \theta_2)(\rho_c + \rho_d) - \lambda f(x_2 | \theta_2)\}$$
$$= \frac{f(x_2 | \theta_2) - f(x_2 | \theta_1)}{f(x_2 | \theta_1)} \{L(\theta_2, a_1) - L(\theta_2, a_n)\} (\rho_c + \rho_d) ,$$

and

(3.25) $$R(\theta_1, \delta') - R(\theta_1, \hat{\hat{\delta}})$$
$$= \rho_a R(\theta_1, d_a) + \rho_b R(\theta_1, d_b) + \rho_c R(\theta_1, d_c) + \rho_d R(\theta_1, d_d)$$
$$- \lambda' R(\theta_1, d_b) - (1 - \lambda') R(\theta_1, d_d)$$
$$= 0$$

$$R(\theta_2, \delta') - R(\theta_2, \hat{\hat{\delta}})$$
$$= \rho_a R(\theta_2, d_a) + \rho_b R(\theta_2, d_b) + \rho_c R(\theta_2, d_c) + \rho_d R(\theta_2, d_d)$$
$$- \lambda' R(\theta_2, d_b) - (1 - \lambda') R(\theta_2, d_d)$$
$$= \frac{f(x_2 | \theta_2) - f(x_2 | \theta_1)}{f(x_1 | \theta_1)} \{L(\theta_2, a_1) - L(\theta_2, a_n)\} (\rho_a + \rho_c) .$$

Since $f(x_2 | \theta_2) - f(x_2 | \theta_1) > 0$, $f(x_1 | \theta_1) > 0$, $f(x_2 | \theta_1) > 0$, $L(\theta_2, a_1) - L(\theta_2, a_n) > 0$, $\rho_a + \rho_c > 0$ and $\rho_c + \rho_d > 0$, we get

(3.26) $$R(\theta_2, \delta') > R(\theta_2, \hat{\delta})$$

and

(3.27) $$R(\theta_2, \delta') > R(\theta_2, \hat{\hat{\delta}}) .$$

Taken together (3.24), (3.25), (3.26) and (3.27), it is concluded that C_3^b is complete.

ii) Similarly as in i), it is concluded that C_3^c is complete.

Next we prove that C_3^b (or C_3^c) is minimal. Let us suppose that C_3^b is not minimal and a minimal complete class does not contain a randomized decision rule δ^*, for example, such that $\delta^*=\rho^* d_a+(1-\rho^*)d_b$ $(0<\rho^*<1)$. Then in the minimal complete class, there exists δ^{**} such that $\delta^{**}\succ\delta^*$, where $\delta^{**}=\rho^{**}d_a+(1-\rho^{**})d_b$ or $\delta^{**}=\rho^{**}d_b+(1-\rho^{**})d_d$ and $\rho^{**}\neq\rho^*$.

1) In case $\delta^{**}=\rho^{**}d_a+(1-\rho^{**})d_b$ and $\delta^*=\rho^* d_a+(1-\rho^*)d_b$
For all θ, the following inequality must hold

$$\rho^{**}R(\theta, d_a)+(1-\rho^{**})R(\theta, d_b)\leqq\rho^* R(\theta, d_a)+(1-\rho^*)R(\theta, d_b)\ .$$

This implies that

$$\text{if } \rho^{**}-\rho^*>0 \text{ then } R(\theta, d_a)\leqq R(\theta, d_b) \text{ for all } \theta \tag{3.28}$$

or

$$\text{if } \rho^{**}-\rho^*<0 \text{ then } R(\theta, d_a)\geqq R(\theta, d_b) \text{ for all } \theta\ . \tag{3.29}$$

But both (3.28) and (3.29) contradict our conditions $R(\theta_1, d_a)<R(\theta_1, d_b)$ and $R(\theta_2, d_a)>R(\theta_2, d_b)$.

2) In case $\delta^{**}=\rho^{**}d_a+(1-\rho^{**})d_b$ and $\delta^*=\rho^* d_b+(1-\rho^*)d_d$
Then the following inequality must hold

$$\rho^{**}R(\theta_2, d_a)+(1-\rho^{**})R(\theta_2, d_b)\leqq\rho^* R(\theta_2, d_b)+(1-\rho^*)R(\theta_2, d_d)\ .$$

But since $R(\theta_2, d_a)>R(\theta_2, d_b)>R(\theta_2, d_d)$, we get the contradiction

$$\rho^{**}\leqq\frac{(1-\rho^*)\{R(\theta_2, d_d)-R(\theta_2, d_b)\}}{\{R(\theta_2, d_a)-R(\theta_2, d_b)\}}<0\ .$$

In other cases such that

$$\delta^{**}=\rho^{**}d_b+(1-\rho^{**})d_d \quad\text{and}\quad \delta^*=\rho^* d_a+(1-\rho^*)d_d$$

and

$$\delta^{**}=\rho^{**}d_b+(1-\rho^{**})d_d \quad\text{and}\quad \delta^*=\rho^* d_b+(1-\rho^*)d_d\ ,$$

similarly as in 1) and 2), we get a contradiction.

The proof of minimality of C_3^c can be carried out by similar way which we used in the proof of minimality of C_3^b. Hence we conclude that C_3^b (or C_3^c) is the minimal complete class and terminate the proof.

3-5. *Some example of Theorem* 3

Example 3. Consider a decision problem of 2-state, 2-observation and 5-action. A loss function and a distribution of observation are given by Tables 4 and 5 respectively and $w_s>0$ $(s=1, 2)$.

Table 4. $L(\theta, a)$

	a_1	a_2	a_3	a_4	a_5
θ_1	0	5	9	12	13
θ_2	15	10	6	2	0

Table 5. $f(x \mid \theta)$

	x_1	x_2
θ_1	0.8	0.2
θ_2	0.3	0.7

The possible non-randomized decision rules are

$$
\begin{array}{lllll}
d_1=(a_1, a_1) & d_6=(a_2, a_1) & a_{11}=(a_3, a_1) & d_{16}=(a_4, a_1) & d_{21}=(a_5, a_1) \\
d_2=(a_1, a_2) & d_7=(a_2, a_2) & d_{12}=(a_3, a_2) & d_{17}=(a_4, a_2) & d_{22}=(a_5, a_2) \\
d_3=(a_1, a_3) & d_8=(a_2, a_3) & d_{13}=(a_3, a_3) & d_{18}=(a_4, a_3) & d_{23}=(a_5, a_3) \\
d_4=(a_1, a_4) & d_9=(a_2, a_4) & d_{14}=(a_3, a_4) & d_{19}=(a_4, a_4) & d_{24}=(a_5, a_4) \\
d_5=(a_1, a_5) & d_{10}=(a_2, a_5) & d_{15}=(a_3, a_5) & d_{20}=(a_4, a_5) & d_{25}=(a_5, a_5) .
\end{array}
$$

Products of regret-relief ratios of action a_2, a_3 and a_4 are

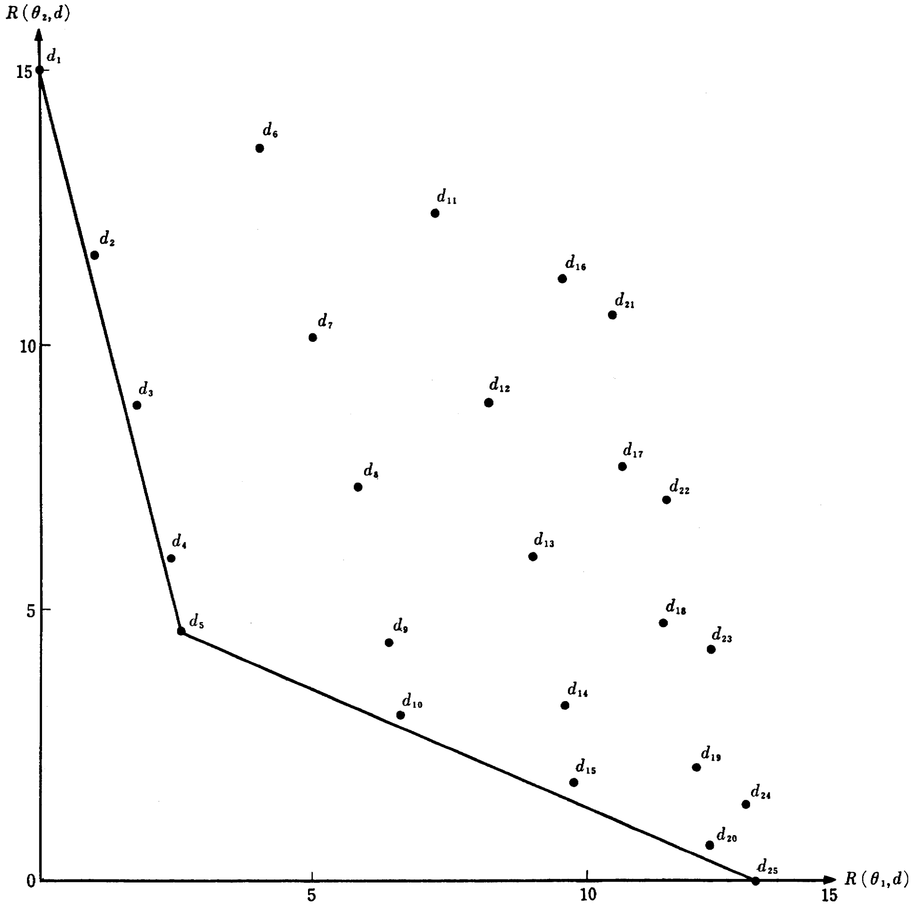

Fig. 6

$$\gamma(\theta_1, a_2)\cdot\gamma(\theta_2, a_2)=\frac{5}{8}\frac{2}{1}=\frac{5}{4}>1\ ,$$

$$\gamma(\theta_1, a_3)\cdot\gamma(\theta_2, a_3)=\frac{9}{4}\frac{2}{3}=\frac{3}{2}>1\ ,$$

$$\gamma(\theta_1, a_4)\cdot\gamma(\theta_2, a_4)=\frac{12}{1}\frac{2}{13}=\frac{24}{13}>1$$

and $f(x_2|\theta_1)<f(x_2|\theta_2)$. Therefore, by Theorem 3, we conclude that the complete class is the set of all randomized decision rules of the form either

$$\lambda d_1+(1-\lambda)d_5 \qquad (0\leqq\lambda\leqq1)$$

or

$$\lambda' d_5+(1-\lambda')d_{25} \qquad (0\leqq\lambda'\leqq1)\ .$$

Fig. 6 shows "risk points"$(R(\theta_1, d), R(\theta_2, d))$ of 25 decision rules of this problem.

3-6. *Incremental loss-gain ratio criterion*

For the reduction to a complete class in a multiple decision problem, "incremental loss-gain ratio" criterion, similar to regret-relief ratio criterion, will be introduced.

Let us define incremental loss $\underline{e}(\theta, a)$, incremental gain $\bar{e}(\theta, a)$ and incremental loss-gain ratio $\eta(\theta, a)$ of action a as follows.

DEFINITION 9 (Incremental Loss). If

$$\text{(3.30)} \quad \begin{array}{ll} L(\theta_1, a_1)<L(\theta_1, a_2)<\cdots<L(\theta_1, a_n)\ , & \text{and} \\ L(\theta_2, a_1)>L(\theta_2, a_2)>\cdots>L(\theta_2, a_n)\ , & \end{array}$$

then we define incremental loss $\underline{e}(\theta, a_k)$ of action a_k to be the increment of loss caused by taking an action a_k instead of more preferred action a_{k-1} (when $\theta=\theta_1$) or a_{k+1} (when $\theta=\theta_2$). More precisely,

$$\underline{e}(\theta_1, a_k)=L(\theta_1, a_k)-L(\theta_1, a_{k-1})$$

or

$$\underline{e}(\theta_2, a_k)=L(\theta_2, a_k)-L(\theta_2, a_{k+1})\ .$$

DEFINITION 10 (Incremental Gain). Under the condition (3.30) of Definition 9, we define incremental gain $\bar{e}(\theta, a_k)$ of action a_k to be the increment of gain given by taking an action a_k instead of less preferred action a_{k+1} (when $\theta=\theta_1$) or a_{k-1} (when $\theta=\theta_2$). More precisely,

$$\bar{e}(\theta_1, a_k) = L(\theta_1, a_{k+1}) - L(\theta_1, a_k)$$

or

$$\bar{e}(\theta_2, a_k) = L(\theta_2, a_{k-1}) - L(\theta_2, a_k) \ .$$

DEFINITION 11 (Incremental Loss-gain Ratio). The incremental loss-gain ratio $\eta(\theta, a_k)$ of action a_k is defined by

$$\eta(\theta, a_k) = \frac{\underline{e}(\theta, a_k)}{\bar{e}(\theta, a_k)} \ .$$

Based on the fact that "incremental loss-gain ratio" is very similar to "regret-relief ratio," we can imagine that the incremental loss-gain ratio will be also useful as a criterion for choosing a decision rule. In fact, the following theorem and its corollary may help demonstrate the above consideration.

THEOREM 4. *Suppose that* $\Theta = \{\theta_1, \theta_2\}$, $X = \{x_1, \cdots, x_m\}$, $A = \{a_1, \cdots, a_n\}$ $w_s > 0$ $(s = 1, 2)$, $f(x_i | \theta) > 0$ $(i = 1, \cdots, m;\ \theta \in \Theta)$ *and*

$$\begin{aligned} &L(\theta_1, a_1) < L(\theta_1, a_2) < \cdots < L(\theta_1, a_n) \ , \qquad \textit{and} \\ &L(\theta_2, a_1) > L(\theta_2, a_2) > \cdots > L(\theta_2, a_n) \ . \end{aligned} \tag{3.31}$$

For each $k = 2, \cdots, n-1$, *the following holds. If the condition*

$$\eta(\theta_1, a_k) \cdot \eta(\theta_2, a_k) > 1$$

is satisfied, then any non-randomized decision rule which takes action a_k *is dominated by some randomized decision rule.*

PROOF. Using the inequality that if

$$x_1 < x_2 < x_3 \ , \qquad y_1 > y_2 > y_3$$

and

$$\frac{y_1 - y_2}{x_2 - x_1} < \frac{y_2 - y_3}{x_3 - x_2}$$

then

$$\frac{y_1 - y_2}{x_2 - x_1} < \frac{y_1 - y_3}{x_3 - x_1} < \frac{y_2 - y_3}{x_3 - x_2} \ ,$$

we have

$$\begin{aligned} \frac{L(\theta_2, a_1) - L(\theta_2, a_k)}{L(\theta_1, a_k) - L(\theta_1, a_1)} &< \frac{L(\theta_2, a_{k-1}) - L(\theta_2, a_k)}{L(\theta_1, a_k) - L(\theta_1, a_{k-1})} \\ &< \frac{L(\theta_2, a_k) - L(\theta_2, a_{k+1})}{L(\theta_1, a_{k+1}) - L(\theta_1, a_k)} < \frac{L(\theta_2, a_k) - L(\theta_2, a_n)}{L(\theta_1, a_n) - L(\theta_1, a_k)} \ , \end{aligned}$$

where the second inequality is by the assumption. Consequently, we get

$$\frac{L(\theta_2, a_1)-L(\theta_2, a_k)}{L(\theta_1, a_k)-L(\theta_1, a_1)} < \frac{L(\theta_2, a_k)-L(\theta_2, a_n)}{L(\theta_1, a_n)-L(\theta_1, a_k)} ,$$

that is,

$$\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1 .$$

Thus the assumption of Theorem 2 is satisfied, which completes the proof.

Remark. Thus, one should notice that incremental loss-gain ratio criterion implies regret-relief ratio criterion.

COROLLARY 2. *Assume the condition* (3.31) *of Theorem* 4. *If the condition*

$$\eta(\theta_1, a_k)\cdot\eta(\theta_2, a_k)>1 \qquad \textit{for } k=2,\cdots, n-1$$

is satisfied, then the minimal complete class is

i) *the set of all randomized decision rules of the form*; *either*

$$\lambda d_a+(1-\lambda)d_b \qquad (0\leqq\lambda\leqq1)$$

or

$$\lambda' d_b+(1-\lambda')d_d \qquad (0\leqq\lambda'\leqq1)$$

provided $f(x_2|\theta_1)<f(x_2|\theta_2)$,

ii) *the set of all randomized decision rules of the form*; *either*

$$\lambda d_a+(1-\lambda)d_c \qquad (0\leqq\lambda\leqq1)$$

or

$$\lambda' d_c+(1-\lambda')d_d \qquad (0\leqq\lambda'\leqq1) ,$$

provided $f(x_2|\theta_1)>f(x_2|\theta_2)$, *where*

$$d_a=(a_1, a_1) , \quad d_b=(a_1, a_n) , \quad d_c=(a_n, a_1) \quad \textit{and} \quad d_d=(a_n, a_n) .$$

PROOF. Similarly as in the proof of Theorem 4, for each $k=2,\cdots, n-1$ we get

$$\gamma(\theta_1, a_k)\cdot\gamma(\theta_2, a_k)>1 .$$

Thus the assumptions of Theorem 3 are satisfied, which completes the proof.

3-7. *Comparison of the results of reduction*

We now compare the results of reduction when using monotone likelihood ratio criterion with when using regret-relief ratio criterion in the case of Example 3.

1) The case using monotone likelihood ratio criterion

By Theorem 1, we know that a complete class is spanned by 15 monotone decision rules d_1, d_2, d_3, d_4, d_5, d_7, d_8, d_9, d_{10}, d_{13}, d_{14}, d_{15}, d_{19}, d_{20}, d_{25}.

$$K(5, 2)=\binom{n+m-1}{m}=\binom{5+2-1}{2}=\binom{6}{2}=15 .$$

2) The case using regret-relief ratio criterion

In this case, by Theorem 2 and Corollary 1, we know that a complete class is spanned by only 4 decision rules d_1, d_5, d_{21}, d_{25}. (If we use a relation among $f(x|\theta)$, then we get a minimal complete class by Theorem 3.)

For the problem of Example 3, regret-relief ratio criterion is very effective in the reduction to a complete class.

Acknowledgement

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REFERENCES

[1] Karlin, S. and Rubin, H. (1956). The theory of decision procedures for distributions with monotone likelihood ratio, *Ann. Math. Statist.*, **27**, 272-299.

[2] Chernoff, H. and Moses, L. E. (1959). *Elementary Decision Theory*, John Wiley & Sons, New York.

[3] Ferguson, T. S. (1967). *Mathematical Statistics: A Decision Theoretic Approach*, Academic Press, New York.

Ann. Inst. Statist. Math.
28 (1976), Part A, 508

CORRECTION TO "ON THE REDUCTION TO A COMPLETE CLASS IN MULTIPLE DECISION PROBLEMS"

Masakatsu Murakami

In the above titled paper (this Annals 28(1976), pp. 145–165), the following correction should be made:

On page 146, line 3:

The minimax risk criterion $\longrightarrow$ The minimum risk criterion.