SPECTRAL ANALYSIS FOR A RANDOM PROCESS ON THE SPHERE

ROCH ROY*

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1. Introduction

In the present paper, a random process on the sphere is defined as a family of random variables indexed by the position $P \in S_2$, the unit sphere of the three-dimensional space R^3, and by the time $t \in Z$, the set of integers, i.e. a family of the form $\{X(P, t): (P, t) \in S_2 \times Z\}$. This paper develops a spectral analysis for a process $X(P, t)$ which is homogeneous (with respect to P) and stationary (with respect to t).

The problems of filtering and sampling for a process on the sphere have been studied in Hannan [4], [5]. For a process which is time-independent rather than stationary (i.e. for which the realizations at different times are mutually independent), a spectral analysis has been developed by Jones [8]; the estimation of the covariance function has been made by Roy [11]. Applications of random processes on the sphere to meteorology can be found in Jones [7], Cohen and Jones [3].

In Section 2 of this paper, we consider the spherical harmonic series expansion of $X(P, t)$; the coefficients are stochastic processes indexed only by time. For a homogeneous and stationary process $\{X(P, t): (P, t) \in S_2 \times Z\}$, a spectral representation is given. This representation allows us to develop a spectral analysis for a process on the sphere analogous to the one described in [10] for a process on the circle. Given a positive integer T, we suppose that complete realizations of the process have been observed at times $t=0, 1, \cdots, T-1$. From these data, we can compute the values of the coefficients in the spherical harmonic series expansion at $t=0, 1, \cdots, T-1$. In Section 3, conditions are given which allow us to apply results of Brillinger [1], [2], for vector-valued time series to the coefficients of $X(P, t)$. First, a theorem gives the asymptotic distribution of the finite Fourier transform of the coefficients. From this theorem, the asymptotic distribution of the family of periodograms is deduced. In Section 4, a class of consistent estimates of the spectral densities is studied. Finally in Section 5, a sample covariance function for the process $X(P, t)$ is proposed and its asymptotic distribution derived.

* Now at Département d'Informatique, Université de Montréal.

2. Spectral representation

In this section, we consider a real-valued process $\{X(P,t): (P,t) \in S_2 \times Z\}$ with finite second-order moments such that for each t, the process $\{X(P,t): P \in S_2\}$ is continuous in quadratic mean (q.m.). Thus, for each t, one can expand $X(P,t)$ in a spherical harmonic series which is convergent in q.m. If for each $n \geqq 0$, $\{Y_{nk}(P): -n \leqq k \leqq n\}$ denotes an orthonormal basis for the real spherical harmonics of order n (see Sansone [12], p. 262), we can write:

$$X(P,t) = \sum_{n=0}^{\infty} \sum_{k=-n}^{n} Z_{nk}(t) Y_{nk}(P) , \qquad P \in S_2 \tag{2.1}$$

where

$$Z_{nk}(t) = \int_{S_2} X(P,t) Y_{nk}(P) \sigma(dP) \tag{2.2}$$

σ being the Lebesgue measure on S_2. The integral in (2.2) is the integral in the q.m. sense and the series (2.1) converges in q.m.

In the following, we will say that the process $X(P,t)$ is *second-order homogeneous* (with respect to P) and *stationary* (with respect to t) if

$$\mathrm{E}[X(P_1,t)] \equiv \mu , \qquad \mathrm{E}[X(P_1,t+s)X(P_2,s)] = R(\theta,t) , \tag{2.3}$$

for every $P_1, P_2 \in S_2$ and for every $s, t \in Z$; θ being the angular distance between P_1 and P_2.

Without loss of generality, we can suppose that $\mu = 0$, which implies that $\mathrm{E}[Z_{nk}(t)] \equiv 0$ for every k and n. The following theorem gives a spectral representation of the process $X(P,t)$.

THEOREM 2.1. *If the process on the sphere $\{X(P,t): (P,t) \in S_2 \times Z\}$ is second-order homogeneous and stationary, then*

$$\mathrm{E}[Z_{nk}(t+s) Z_{mh}(s)] = \delta_{nm} \delta_{kh} \int_{-\pi}^{\pi} e^{i\lambda t} dF_n(\lambda) \tag{2.4}$$

for $-n \leqq k \leqq n$, $-m \leqq h \leqq m$ and $n, m \geqq 0$. Also,

$$\begin{aligned} &\mathrm{E}[X(P_1,t+s)X(P_2,s)] \\ &\quad = R(\theta,t) = (4\pi)^{-1} \sum_{n=0}^{\infty} (2n+1) P_n(\cos\theta) \int_{-\pi}^{\pi} e^{i\lambda t} dF_n(\lambda) \end{aligned} \tag{2.5}$$

where $\{F_n(\lambda)\}_{n=0}^{\infty}$ is a sequence of real non-decreasing functions which are unique to an additive constant such that

$$(4\pi)^{-1} \sum_{n=0}^{\infty} (2n+1) \int_{-\pi}^{\pi} dF_n(\lambda) < \infty , \tag{2.6}$$

and $P_n(\cdot)$ *denotes the Legendre polynomial of degree* n.

The representation given here could be deduced from the general theory of second-order stationary processes on a homogeneous space (see Hannan [6], p. 106). A detailed proof is given in Roy [9], Chapter 1.

3. Finite Fourier transforms and periodograms

The basic tool for the construction of spectral estimates will be the finite Fourier transform of the coefficients. In order to study its asymptotic behavior, we must make some stronger assumptions about the process $X(P, t)$. In the following, we will say that the process $X(P, t)$ is *strictly homogeneous and stationary* if for any finite collection of points $(P_1, t_1), \cdots, (P_k, t_k)$, the joint distribution of $X(g(P_1), t_1+t), \cdots, X(g(P_k), t_k+t)$ is independent of g and t for all $g \in G$ the group of rotations of the sphere, for all $t \in Z$ and $k=1, 2, \cdots$. Thus, a strictly homogeneous and stationary process with finite second-order moments is second-order homogeneous and stationary and Theorem 2.1 is valid for such a process.

ASSUMPTION I. For each t, $\{X(P, t): P \in S_2\}$ is continuous in q.m. Also, $X(P, t)$ is strictly homogeneous and stationary all of whose moments exist and are bounded with respect to P i.e.

$$|\mathrm{E}[X(P_1, t_1)\cdots X(P_k, t_k)]| \leqq M_k(t_1, \cdots, t_k)$$

uniformly in $P_1, \cdots, P_k$ for all $t_1, \cdots, t_k$ and $k=1, 2, \cdots$.

In this paper, $\mathrm{cum}\{X_1, \cdots, X_j\}$ will denote the joint cumulant of order j of the random variables $X_1, \cdots, X_j$. By an argument analogous to the one used in Roy [10], under Assumption I, it follows that cum $\{Z_{n_1k_1}(u_1+t), \cdots, Z_{n_{j-1}k_{j-1}}(u_{j-1}+t), Z_{n_jk_j}(t)\}$ is independent of t and from now, it will be denoted by $c_{n_1k_1,\cdots,n_jk_j}(u_1, \cdots, u_{j-1})$.

The main assumption about the process $X(P, t)$ will be the following:

ASSUMPTION II(l). For a given $l \geqq 0$,

$$\sum_{u_1,\cdots,u_{j-1}=-\infty}^{\infty} \{1+|u_i|^l\}|\mathrm{cum}\{X(P_1, u_1+t), \cdots, X(P_{j-1}, u_{j-1}+t), X(P_j, t)\}| \leqq C_j < \infty$$

uniformly in $P_1, \cdots, P_j$, for $i=1, \cdots, j-1$ and $j=2, 3, \cdots$.

Under Assumption II(0), the jth order cumulant spectrum $f_{n_1k_1,\cdots,n_jk_j}(\lambda_1, \cdots, \lambda_{j-1})$ of the coefficients $Z_{n_1k_1}(t), \cdots, Z_{n_jk_j}(t)$ is defined by

(3.1) $f_{n_1k_1,\cdots,n_jk_j}(\lambda_1, \cdots, \lambda_{j-1})$

$$=(2\pi)^{-j+1}\sum_{u_1,\cdots,u_{j-1}=-\infty}^{\infty} c_{n_1k_1,\cdots,n_jk_j}(u_1,\cdots,u_{j-1})\exp\left\{-i\sum_{h=1}^{j-1}u_h\lambda_h\right\}$$

for $-\infty<\lambda_h<\infty$, $h=1,\cdots,j-1$ and $j=2,3,\cdots$.

For the second-order spectra, by Theorem 2.1, we have that

$$f_{nk,mh}(\lambda)=\delta_{nm}\delta_{kh}f_n(\lambda) \tag{3.2}$$

for $-n\leqq k\leqq n$, $-m\leqq h\leqq m$ and $m,n\geqq 0$. Assumption II(0) garantees us that the functions $F_n(\lambda)$ of Theorem 2.1 are differentiable and $f_n(\lambda)=F_n'(\lambda)$.

Now, the estimation of the spectral densities $f_n(\lambda)$, $n\geqq 0$, and of the covariance function $R(\theta,t)$ can be done in a similar way to the estimation of the corresponding parameters for a process on the circle. The main difference being that for a process on the sphere, $f_n(\lambda)$ is the spectral density of $2n+1$ coefficients rather than 2 in the case of a process on the circle.

Given the values $X(P,t)$, $P\in S_2$, $t=0,1,\cdots,T-1$, the finite Fourier transform of the coefficient $Z_{nk}(t)$ is defined by

$$d_{nk}^{(T)}(\lambda)=\sum_{t=0}^{T-1}Z_{nk}(t)e^{-i\lambda t}\,,\qquad \lambda\in R\,. \tag{3.3}$$

For an arbitrary subset of subscripts $S=\{n_1k_1,\cdots,n_sk_s\}$, $\boldsymbol{d}_S^{(T)}(\lambda)$ denotes the column vector $(d_{n_1k_1}^{(T)}(\lambda),\cdots,d_{n_sk_s}^{(T)}(\lambda))$. Similarly, $\boldsymbol{Z}_S(t)$ represents the column vector $(Z_{n_1k_1}(t),\cdots,Z_{n_sk_s}(t))$ and $\boldsymbol{f}_S(\lambda)$ the matrix of second-order spectra of the process $\boldsymbol{Z}_S(t)$. Also, $N_k(\mu,\Sigma)$ represents a real k vector-valued normal variable and $N_k^c(\mu,\Sigma)$ a complex k vector-valued normal variable. The next theorem which gives the asymptotic distribution of $\boldsymbol{d}_S^{(T)}(\lambda)$ as $T\to\infty$, follows from Theorem 4.4.2 of Brillinger [2].

THEOREM 3.1. *Let $X(P,t)$ satisfy Assumptions* I, II(1) *and have mean zero. Let $d_{nk}^{(T)}(\lambda)$ be defined by* (3.3) *and suppose that $2\lambda_j$, $\lambda_j\pm\lambda_k\not\equiv 0$ (mod 2π) for $1\leqq j<k\leqq J$. Then, $\boldsymbol{d}_S^{(T)}(\lambda_j)$, $j=1,\cdots,J$ are asymptotically independent $N_s^c(0,2\pi T\boldsymbol{f}_S(\lambda_j))$. Also if $\lambda\equiv 0$ (mod π), $\boldsymbol{d}_S^{(T)}(\lambda)$ is asymptotically $N_s(0,2\pi T\boldsymbol{f}_S(\lambda))$ independently of the previous variables.*

From (3.2) and Theorem 3.1, we see that

$$\lim_{T\to\infty}\frac{1}{T}\,\mathrm{E}\,[|d_{nk}^{(T)}(\lambda)|^2]=2\pi f_n(\lambda)\,,\qquad -n\leqq k\leqq n$$

and as a preliminary estimate of $f_n(\lambda)$, we can consider

$$I_n^{(T)}(\lambda)=1/\{2\pi(2n+1)T\}\sum_{k=-n}^{n}|d_{nk}^{(T)}(\lambda)|^2\,,\qquad n\geqq 0\,. \tag{3.4}$$

The statistic $I_n^{(T)}(\lambda)$ will be called the *periodogram of order* n corresponding to $X(P, t)$. From Theorem 3.1, we deduce easily the asymptotic distribution of the family of periodograms $I_n^{(T)}(\lambda)$, $n \geqq 0$. In the following theorem, χ_n^2 denotes a chi-square random variable with n degrees of freedom.

THEOREM 3.2. *Suppose the conditions of Theorem* 3.1 *are satisfied and let* $I_n^{(T)}(\lambda)$ *be given by* (3.4). *Then for* $n \geqq 0$, $I_n^{(T)}(\lambda_j)$, $j=1, \cdots, J$ *are asymptotically independent* $f_n(\lambda_j)\chi^2_{2(2n+1)}/2(2n+1)$ *respectively. For* $\lambda \equiv 0$ $(\text{mod } \pi)$, $I_n^{(T)}(\lambda)$ *is* $f_n(\lambda)\chi^2_{2n+1}/2n+1$ *independently of the previous variables. Also for* $n \neq m$ *the processes* $I_n^{(T)}(\lambda)$ *and* $I_m^{(T)}(\lambda)$ *are asymptotically independent.*

4. A class of consistent spectral densities estimates

Taking $I_n^{(T)}(\lambda)$ as the basic statistic, a consistent estimate of the spectral density $f_n(\lambda)$ can be obtained by the usual technique of estimation of the spectral density of a stationary time series. We choose first a weight function $H(\alpha)$ $(-\pi < \alpha \leqq \pi)$ which is bounded, symmetric about zero, has a bounded first derivative and is such that $\int_{-\pi}^{\pi} H(\alpha)d\alpha = 1$. Given $B_T > 0$, we define $H^{(T)}(\alpha) = B_T^{-1}H(B_T^{-1}\alpha)$ and as an estimate of $f_n(\lambda)$, we consider for $n \geqq 0$

$$f_n^{(T)}(\lambda) = \int_{-\pi}^{\pi} H^{(T)}(\alpha) I_n^{(T)}(\lambda - \alpha) d\alpha \tag{4.1}$$

where $I_n^{(T)}(\lambda)$ is defined by (3.4). We see that

$$f_n^{(T)}(\lambda) = \frac{1}{2n+1} \sum_{k=-n}^{n} f_{nk}^{(T)}(\lambda) \tag{4.2}$$

with

$$f_{nk}^{(T)}(\lambda) = \int_{-\pi}^{\pi} H^{(T)}(\alpha) \{(2\pi T)^{-1} |d_{nk}^{(T)}(\lambda - \alpha)|^2\} d\alpha ,$$

$-n \leqq k \leqq n$. The statistic $f_{nk}^{(T)}(\lambda)$ is the usual spectral density estimate corresponding to the series $Z_{nk}(t)$, $t = 0, 1, \cdots, T-1$.

In the following, the function $\eta(\lambda)$ is defined by

$$\eta(\lambda) = \begin{cases} 1 & \text{if } \lambda \equiv 0 \pmod{2\pi} \\ 0 & \text{otherwise}. \end{cases}$$

By taking advantage of (4.2), the next theorem can be deduced from Theorems 6.1 and 6.2 of Brillinger [1].

THEOREM 4.1. *Let* $X(P, t)$ *satisfy Assumptions* I, II(1) *and have* *mean zero. Let* $f_n^{(T)}(\lambda)$, $n \geqq 0$, *be given by* (4.1). *If* $B_T \to 0$, $TB_T \to \infty$ *as* $T \to \infty$, *then*

$$\lim_{T\to\infty} \mathrm{E}\,[f_n^{(T)}(\lambda)] = f_n(\lambda)$$

and

$$(4.3)\quad \lim_{T\to\infty} TB_T \operatorname{cov}[f_n^{(T)}(\lambda), f_m^{(T)}(\mu)] = \frac{2\pi\delta_{nm}}{2n+1}[\eta(\lambda-\mu)+\eta(\lambda+\mu)]f_n^2(\lambda)\int_{-\infty}^{\infty} H^2(\alpha)d\alpha \qquad \text{for } n, m \geqq 0\,.$$

Also, the random variables $(B_T T)^{1/2}\{f_n^{(T)}(\lambda_1) - \mathrm{E}\,[f_n^{(T)}(\lambda_1)]\}, \cdots, (B_T T)^{1/2}\cdot \{f_{n_J}^{(T)}(\lambda_J) - \mathrm{E}\,[f_{n_J}^{(T)}(\lambda_J)]\}$ *are asymptotically jointly normal with mean* 0 *and covariance structure given by* (4.3), $n_j \geqq 0$, $j=1,\cdots,J$ *and* $J=1,2,\cdots$.

5. Estimation of the covariance function

Let $m_n(u)$, $u=0, \pm 1, \cdots$, denote the covariance function of the coefficients $Z_{nk}(t)$, $-n \leqq k \leqq n$. The function $m_n(u)$ can be estimated by $m_n^{(T)}(u) = \int_{-\pi}^{\pi} e^{i\lambda u} I_n^{(T)}(\lambda) d\lambda$, $n \geqq 0$, $u=0, \pm 1, \cdots$, where $I_n^{(T)}(\lambda)$ is given by (3.4). From the definition of $I_n^{(T)}(\lambda)$, we see that

$$(5.1)\qquad m_n^{(T)}(u) = \frac{1}{2n+1}\sum_{k=-n}^{n} m_{nk}^{(T)}(u)$$

where $m_{nk}^{(T)}(u)$ is the sample covariance function corresponding to the series $Z_{nk}(t)$, $t=0,1,\cdots,T-1$.

For N fixed, as an estimate of $R(\theta, u)$, we consider

$$(5.2)\qquad R_N^{(T)}(\theta, u) = \sum_{n=0}^{N}\left(\frac{2n+1}{4\pi}\right) P_n(\cos\theta) m_n^{(T)}(u)\,.$$

In order to study the asymptotic distribution of $R_N^{(T)}(\theta, u)$, we must introduce the following parameter:

$$(5.3)\qquad g_{n_1 n_2}(\lambda_1, \lambda_2, \lambda_3) = \sum_{k_1=-n_1}^{n_1}\sum_{k_2=-n_2}^{n_2} f_{n_1k_1, n_1k_1, n_2k_2, n_2k_2}(\lambda_1, \lambda_2, \lambda_3)$$

where $f_{n_1k_1, n_1k_1, n_2k_2, n_2k_2}(\lambda_1, \lambda_2, \lambda_3)$ the cumulant spectrum of order four of the coefficients $Z_{n_1k_1}(t)$, $Z_{n_1k_1}(t)$, $Z_{n_2k_2}(t)$, $Z_{n_2k_2}(t)$ is defined by (3.1).

Using (5.1), the next theorem follows from Theorems 5.1 and 5.2 of Brillinger [1].

THEOREM 5.1. *Let* $X(P, t)$ *satisfy Assumptions* I, II(1) *and have mean zero. Let* $R_N^{(T)}(\theta, u)$ *be given by* (5.2) *with* N *fixed. Then,*

$$\text{(5.4)} \qquad \mathrm{E}\,[R_N^{(T)}(\theta, u)] = \sum_{n=0}^{N} \left(\frac{2n+1}{4\pi}\right) P_n(\cos\theta) m_n(u) + O_N(T^{-1})$$

with the error term (a function of N) is uniform in θ, u.

$$\text{(5.5)} \qquad \lim_{T\to\infty} T \operatorname{cov}\{R_N^{(T)}(\theta_1, u_1), R_N^{(T)}(\theta_2, u_2)\}$$

$$= (8\pi)^{-1} \sum_{n=0}^{N} (2n+1) P_n(\cos\theta_1) P_n(\cos\theta_2) \int_0^{2\pi} [\exp\{-i\alpha(u_1+u_2)\}$$

$$+ \exp\{i\alpha(u_1-u_2)\}] f_n^2(\alpha) d\alpha + (8\pi)^{-1} \sum_{n_1, n_2=0}^{N} P_{n_1}(\cos\theta_1) P_{n_2}(\cos\theta_2)$$

$$\times \int_0^{2\pi} \int_0^{2\pi} \exp\{i(\alpha_1 u_1 - \alpha_2 u_2)\} g_{n_1 n_2}(\alpha_1, \alpha_2, -\alpha_1) d\alpha_1 d\alpha_2$$

and $R_N^{(T)}(\theta_i, u_i)$, $i=1, \cdots, I$ are asymptotically jointly normal with the above first and second-order moment structure; $0 \leqq \theta_i \leqq \pi$, $u_i = 0, \pm 1, \cdots$, $i = 1, \cdots, I$ *and* $I = 1, 2, \cdots$.

From the previous theorem, we obtain an asymptotically unbiased estimate of $R(\theta, u)$ by letting $N \to \infty$.

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