# FURTHER RESULTS ON SIMULTANEOUS CONFIDENCE INTERVALS FOR THE NORMAL DISTRIBUTION

R. SRINIVASAN AND R. M. WHARTON

(Received Oct. 30, 1972; revised Apr. 27, 1974)


## Summary

Based on a random sample from the normal cumulative distribution function $\Phi(x; \mu, \sigma)$ with unknown parameters $\mu$ and $\sigma$, one-sided confidence contours for $\Phi(x; \mu, \sigma)$, $-\infty<x<\infty$, and simultaneous confidence intervals for $\Phi(y; \mu, \sigma)-\Phi(x; \mu, \sigma)$, $-\infty<x<y<\infty$, are constructed using the method outlined in [3]. Small sample and asymptotic distributions of the relevant statistics are provided so that the construction could be completely carried out in any practical situation.


## 1. Introduction

Let $x_1, x_2, \cdots, x_n$ be a random sample of size $n$ from a normal cumulative distribution function (c.d.f.) $\Phi(x; \mu, \sigma)$ with mean $\mu$ $(-\infty<\mu<\infty)$ and variance $\sigma^2$ both of which are unknown. Let $\bar{x}$ and $s^2$ denote respectively the sample mean and the sample variance with divisor $n-1$.

For a given $\alpha \in (0, 1)$, in Section 2 of this paper, we construct an upper confidence contour for $\Phi(x; \mu, \sigma)$ with confidence level $\alpha$ based on our random sample. More specifically, a random function $U(x; \bar{x}, s)$, $-\infty<x<\infty$, is given with the property that

$$\Pr\{\Phi(x; \mu, \sigma)\leqq U(x; \bar{x}, s),\ -\infty<x<\infty\}=\alpha . \tag{1.1}$$

Small sample as well as asymptotic cases are considered. The methods employed in our construction have already been outlined in [3] and [8] where a two-sided confidence band for $\Phi(x; \mu, \sigma)$ is developed. Reference may also be made to [7] where similar results are given for the exponential c.d.f., and [6] which is concerned with the Weibull c.d.f. Section 2 also contains an analogous lower confidence contour for $\Phi(x; \mu, \sigma)$, that is, a random function $L(x; \bar{x}, s)$, with the property that

$$\Pr\{\Phi(x; \mu, \sigma)\geqq L(x; \bar{x}, s),\ -\infty<x<\infty\}=\alpha . \tag{1.2}$$

In Section 3 a set of simultaneous confidence intervals for the interval probabilities of $\Phi(x;\mu,\sigma)$ with confidence level $\alpha$ is constructed. Using an analogue of Kuiper's statistic [4], two random interval functions $K_l[(x,y);\bar{x},s]$ and $K_u[(x,y);\bar{x},s]$ are given such that

$$(1.3)\qquad \Pr\{K_l[(x,y);\bar{x},s]\leqq\Phi(y;\mu,\sigma)-\Phi(x;\mu,\sigma)\leqq K_u[(x,y);\bar{x},s],\ \infty<x\leqq y<\infty\}=\alpha\ .$$

Both small sample and asymptotic results are provided as in Section 2.

## 2. One-sided confidence contours for $\Phi(x;\mu,\sigma)$

Basic to our construction of an upper confidence contour for $\Phi(x;\mu,\sigma)$ is the statistic

$$(2.1)\qquad L_n^+=\sup_{-\infty<x<\infty}[\Phi(x;\mu,\sigma)-\Phi(x;\bar{x},s)]\ .$$

By transforming the original observations $x_i$'s to $z_i$'s by means of the standardizing transformation, $x=\sigma z+\mu$, it is readily seen that

$$(2.2)\qquad L_n^+=\sup_{-\infty<z<\infty}[\Phi(z;0,1)-\Phi(z;\bar{z},s_z)]\ ,$$

where $\bar{z}$ and $s_z$ are respectively the sample mean and standard deviation of the $z_i$'s. It follows that the distribution of $L_n^+$ does not depend on the unknown parameters $\mu$ and $\sigma$. In deriving the distribution of $L_n^+$ we can therefore assume that we are sampling from a standard normal population, that is, $\mu=0$ and $\sigma=1$. Thus $L_n^+$ would stand for its standardized form (2.2) throughout the rest of this section.

For $\alpha\in(0,1)$, if $l_\alpha^+$ is the $\alpha$-quantile of $L_n^+$,

$$(2.3)\qquad \Pr\{L_n^+\leqq l_\alpha^+\}=\alpha\ ,$$

an upper confidence contour for $\Phi(x;\mu,\sigma)$ with confidence level $\alpha$ is immediately provided by taking, in (1.1).

$$(2.4)\qquad U(x;\bar{x},s)=\min\{\Phi(x;\bar{x},s)+l_\alpha^+,1\}\ .$$

The derivation of the distribution of $L_n^+$ is now in order. We shall first do this, and then show that the resulting confidence region is 'full' in the sense to be explicitly defined later.

Using certain results of [3] it is possible to obtain a simple expression for $L_n^+$ which is much more manageable than (2.2). For this purpose consider the function

$$(2.5)\qquad w(x;\mu_1,\mu_2,\sigma_1,\sigma_2)=\Phi(x;\mu_1,\sigma_1)-\Phi(x;\mu_2,\sigma_2)\ ,$$

where $\mu_1,\mu_2,\sigma_1>0$ and $\sigma_2>0$ are fixed. It is proved in [3] that, whenever $\sigma_1 < \sigma_2$, we have

$$\max_x w = w(x_m; \mu_1, \mu_2, \sigma_1, \sigma_2) , \tag{2.6}$$

and

$$\min_x w = w(x_p; \mu_1, \mu_2, \sigma_1, \sigma_2) , \tag{2.7}$$

where

$$x_m, x_p = (\sigma_1^2 - \sigma_2^2)^{-1}[\sigma_1^2\mu_2 - \sigma_2^2\mu_1 \mp \sigma_1\sigma_2\sqrt{(\mu_1-\mu_2)^2 + (\sigma_1^2-\sigma_2^2)\ln\{(\sigma_1/\sigma_2)^2\}}] . \tag{2.8}$$

Applying this result to (2.2) with (0, 1) and $(\bar{z}, s_z)$ playing the roles of $(\mu_1, \sigma_1)$ and $(\mu_2, \sigma_2)$ interchangeably, and keeping in mind that $\Pr\{s_z = 1\} = 0$, we get that

$$\Pr\{L_n^+ = w(z_u; 0, \bar{z}, 1, s_z)\} = 1 , \tag{2.9}$$

where

$$z_u = (1 - s_z^2)^{-1}[\bar{z} - s_z\sqrt{\bar{z}^2 - 2(1-s_z^2)\ln s_z}] . \tag{2.10}$$

The distribution of $L_n^+$ can now be obtained from (2.9) and (2.10) via Monte Carlo methods using repeated samples of size $n$ of standard

Table 1 Quantiles, $l_\alpha^+$ of $L_n^+ = \sup_x [\Phi(x; \mu, \sigma) - \Phi(x; \bar{x}, s)]$

| Sample size $n$ | Confidence level $\alpha = \Pr\{L_n^+ \leq l_\alpha^+\}$ | | | | |
|---|---|---|---|---|---|
| | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
| 5 | 0.20 | 0.24 | 0.29 | 0.35 | 0.48 |
| 6 | 0.18 | 0.21 | 0.25 | 0.31 | 0.43 |
| 7 | 0.17 | 0.20 | 0.23 | 0.29 | 0.39 |
| 8 | 0.16 | 0.18 | 0.21 | 0.27 | 0.37 |
| 9 | 0.14 | 0.17 | 0.20 | 0.25 | 0.34 |
| 10 | 0.14 | 0.16 | 0.19 | 0.24 | 0.33 |
| 11 | 0.13 | 0.15 | 0.18 | 0.22 | 0.31 |
| 12 | 0.12 | 0.14 | 0.17 | 0.21 | 0.30 |
| 13 | 0.12 | 0.14 | 0.16 | 0.20 | 0.29 |
| 14 | 0.11 | 0.13 | 0.16 | 0.20 | 0.28 |
| 15 | 0.11 | 0.13 | 0.15 | 0.19 | 0.26 |
| 16 | 0.11 | 0.12 | 0.15 | 0.18 | 0.25 |
| 17 | 0.10 | 0.12 | 0.14 | 0.18 | 0.24 |
| 18 | 0.10 | 0.12 | 0.14 | 0.17 | 0.24 |
| 19 | 0.10 | 0.11 | 0.13 | 0.17 | 0.23 |
| 20 | 0.09 | 0.11 | 0.13 | 0.16 | 0.23 |
| 21 | 0.09 | 0.11 | 0.13 | 0.16 | 0.22 |
| 22 | 0.09 | 0.10 | 0.12 | 0.15 | 0.21 |
| 23 | 0.09 | 0.10 | 0.12 | 0.15 | 0.21 |
| 24 | 0.09 | 0.10 | 0.12 | 0.15 | 0.20 |
| 25 | 0.08 | 0.10 | 0.12 | 0.15 | 0.20 |
| 26 | 0.08 | 0.10 | 0.11 | 0.14 | 0.20 |
| 27 | 0.08 | 0.09 | 0.11 | 0.14 | 0.19 |
| 28 | 0.08 | 0.09 | 0.11 | 0.14 | 0.19 |
| 29 | 0.08 | 0.09 | 0.11 | 0.13 | 0.19 |
| 30 | 0.08 | 0.09 | 0.11 | 0.13 | 0.18 |

normal deviates. This has been done with a Monte Carlo sample size of 10,000 in each case for $n=5(1)30$. The resulting $\alpha$-quantiles $l_\alpha^+$ of $L_n^+$, defined in (2.3), are presented in Table 1 for several commonly used values of $\alpha$; the entries are believed to be accurate to two decimal places.

We shall now derive the asymptotic distribution of $\sqrt{n}\,L_n^+$, and show that this can be used to obtain quite accurate estimates of the quantiles of $L_n^+$ for $n>30$. Our derivation depends heavily on the results of [8] to which the reader is referred for the details. It is shown there that the stochastic process

$$\sqrt{n}\,w(u;\bar{z},0,s_z,1)\,,\qquad -\infty<u<\infty\,, \tag{2.11}$$

converges weakly, as $n\to\infty$, to the Gaussian process $X_u$, $-\infty<u<\infty$, specified by

$$\mathrm{E}\,\{X_u\}=0\,,\qquad -\infty<u<\infty \tag{2.12}$$

and

$$\mathrm{E}\,\{X_uX_v\}=\phi(u)\phi(v)\Big(1+\frac{uv}{2}\Big)\,,\qquad -\infty<u,\ v<\infty\,, \tag{2.13}$$

where $\phi(u)=(1/\sqrt{2\pi})e^{-u^2/2}$ is the standard normal density. Applying the continuous mapping theorem ([1], Theorem 5.1) to this result, we see that the limit distribution of $\sqrt{n}\,L_n^+$ is identical with that of the random variable

$$X_0=-\mathrm{Inf}\,\{X_u,\ -\infty<u<\infty\}\,. \tag{2.14}$$

In order to get the distribution of $X_0$ we note that a representation for the Gaussian process is given by

$$X_u=\phi(u)\Big(Z_1+\frac{uZ_2}{\sqrt{2}}\Big)\,,\qquad -\infty<u<\infty\,, \tag{2.15}$$

where $Z_1$ and $Z_2$ are independent standard normal random variables. Thus, for $\lambda\in[0,\infty)$, we have

$$\Pr\,\{X_0\leqq\lambda\}=\Pr\Big\{Z_1+\frac{uZ_2}{\sqrt{2}}\geqq-\frac{\lambda}{\phi(u)},\ -\infty<u<\infty\Big\}\,. \tag{2.16}$$

The right-hand side of (2.16) is simply the planar measure of all straight lines which lie entirely above the convex curve $y=-\lambda/\phi(u)$, where the measure referred to is the one corresponding to the joint c.d.f. of $Z_1$ and $Z_2/\sqrt{2}$. Exploiting the obvious symmetry involved in the problem, that is, considering only lines with positive slope, and employing the transformation $Z_2=\sqrt{2}\cdot\lambda W/\phi(W)$, that is, sweeping through parallel lines, we get

$$(2.17)\quad \lim_{n\to\infty} \Pr\{\sqrt{n}\, L_n^+ \leqq \lambda\} = 2\sqrt{2}\cdot\lambda \int_0^\infty (1+w^2)\Phi(\lambda\sqrt{2\pi}(1-w^2)e^{w^2/2}; 0, 1) \cdot \exp\left\{-\frac{w^2}{2}(4\lambda^2\pi e^{w^2}-1)\right\} dw .$$

Table 2 gives the $\lambda_\alpha$'s corresponding to several values of $\alpha$ in the equation

$$(2.18)\quad \lim_{n\to\infty} \Pr\{\sqrt{n}\, L_n^+ \leqq \lambda_\alpha\} = \alpha .$$

Table 2. Quantiles of the limit distribution of $\sqrt{n}\, L_n^+$

| $\alpha$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| $\lambda_\alpha$ | 0.427 | 0.496 | 0.584 | 0.718 | 0.967 |

In order to get a rough idea of the speed of convergence of $\sqrt{n}\, L_n^+$ to its limit distribution we have computed the empirical $\alpha$-quantiles of $L_n^+$ for $n=40(20)100$ for various values of $\alpha$ based on a Monte Carlo sample size of 10,000 in each case. These are presented in Table 3 along with the corresponding values obtained from (2.17) through Table 2. The results clearly indicate that for $n>30$, we are quite safe in approximating the exact quantiles of $L_n^+$ from its limit distribution.

Let us now turn our attention to an important question concerning the upper confidence contour (2.4) derived through $L_n^+$. Denote the planar region determined by $y=U(x;\bar{x}, s)$ and $y=0$ by $C_U$ so that

$$(2.19)\quad C_U = \{(x, y):\ -\infty < x < \infty,\ 0 \leqq y \leqq U(x;\bar{x}, s)\} .$$

Let $\Phi(\mu, \sigma)$ denote the graph of the c.d.f. $\Phi(x;\mu, \sigma)$:

$$(2.20)\quad \Phi(\mu, \sigma) = \{(x, y):\ -\infty < x < \infty,\ y = \Phi(x;\mu, \sigma)\} .$$

With this notation we can rewrite (1.1) as

$$(2.21)\quad \Pr\{\Phi(\mu, \sigma) \subset C_U\} = \alpha .$$

Does $C_U$ contain any superfluous areas that can be removed from it without affecting the confidence statement associated with it? We shall answer this question in the following paragraph in the negative by showing that $C_U$ is 'full' in the sense that, given any $(x_0, y_0) \in C_U$, there exists a normal c.d.f. $\Phi(x;\mu_0, \sigma_0)$ passing through $(x_0, y_0)$ such that $\Phi(\mu_0, \sigma_0) \subset C_U$. We mention in passing that $C_U$ differs markedly in this respect from the two-sided band $B_L$ developed in [3] which was not full and was considerably whittled down.

In our proof that $C_U$ is full we assume, without loss, that $\bar{x}=0$

Table 3 Comparison of the 'exact' and asymptotic quantiles of $L_n^+$

| $n \diagdown \alpha$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| 40 | 0.067<br>0.068 | 0.080<br>0.079 | 0.093<br>0.092 | 0.115<br>0.114 | 0.156<br>0.153 |
| 60 | 0.054<br>0.055 | 0.065<br>0.064 | 0.076<br>0.075 | 0.093<br>0.093 | 0.124<br>0.125 |
| 80 | 0.047<br>0.048 | 0.055<br>0.055 | 0.066<br>0.065 | 0.082<br>0.080 | 0.109<br>0.108 |
| 100 | 0.043<br>0.043 | 0.050<br>0.050 | 0.059<br>0.058 | 0.072<br>0.072 | 0.096<br>0.097 |

Note: For each $n$, the top row gives the Monte Carlo quantiles, and the bottom the values obtained using Table 2.

and $s=1$; only slight modifications are needed in the general case. Consider the function $f_\alpha(x)$ defined on the interval $(-\infty, q_{1-l_\alpha^+}]$ by the equation

$$\Phi(f_\alpha(x); 0, 1)-\Phi(x; 0, 1)=l_\alpha^+ , \tag{2.22}$$

where $q_\beta$ denotes the $\beta$-quantile of $\Phi(x; 0, 1)$. It is not difficult to see that $\Phi(\mu, \sigma)\subset C_U$ if and only if $(x-\mu)/\sigma\leqq f_\alpha(x)$ for $x \in (-\infty, q_{1-l_\alpha^+})$. The problem of proving that there exists a c.d.f. $\Phi(x; \mu_0, \sigma_0)$ through $(x_0, y_0)$ satisfying $\Phi(\mu_0, \sigma_0)\subset C_U$ thus reduces to that of showing that there exists a line $y=(x-\mu_0)/\sigma_0$ through the point $(x_0, y_0)$ and lying entirely below the curve $y=f_\alpha(x)$. Now it can be shown that the planar region $F$ defined by

$$F=\{(x, y): -\infty<x\leqq q_{1-l_\alpha}, \ y\geqq f_\alpha(x)\} \tag{2.23}$$

is closed and convex, and therefore, given any point $(x^*, f_\alpha(x^*))$ on its boundary, there exists, by Minkowski's theorem [9], a line through this point so that $F$ lies entirely in one of the half-planes determined by this line. It follows that the line through $(x_0, y_0)$ parallel to the Minkowski line through $(x_0, f_\alpha(x))$ satisfies our requirements, thus proving that $C_U$ is full.

Our construction of a lower confidence contour for $\Phi(x; \mu, \sigma)$ parallels that of upper contour. Analogous to (2.1), define the statistic

$$L_n^- = \sup_{-\infty<x<\infty} [\Phi(x; \bar{x}, s)-\Phi(x; \mu, \sigma)] . \tag{2.24}$$

The distribution of $L_n^-$ is also independent of $\mu$ and $\sigma$ and a standardized version of $L_n^-$ analogous to (2.2) can be seen to hold. If $l_\alpha^-$ is the $\alpha$-quantile of $L_n^-$, then our level-$\alpha$ lower contour is provided by

$$L(x; \bar{x}, s)=\max\{\Phi(x; \bar{x}, s)-l_\alpha^-, 0\} \tag{2.25}$$

satisfying (1.2). The resulting region is full, and this can be established quite easily. Finally, concerning the distribution of $L_n^-$, we shall now show that it is identical with that of $L_n^+$. To see this, simply apply (2.6) and (2.7) to the standardized $L_n^-$ to get, similar to (2.9),

$$\Pr\{L_n^- = w(-z_u;\, -\bar{z}, 0, s_z, 1)\} = 1 , \tag{2.26}$$

where $z_u$ is given by (2.10). In deriving (2.26) we have made use of the fact that $(\bar{z}, s_z)$ and $(-\bar{z}, s_z)$ are identically jointly distributed. Since $\phi(x)$ is symmetric about 0, the desired result follows from (2.9) and (2.26).

## 3. Simultaneous confidence intervals for the interval probabilities of $\Phi(x;\mu,\sigma)$

Let $\mathcal{G}$ denote the class of all intervals $(x, y)$, $-\infty < x < y < \infty$. Let $P$ denote the probability measure on the Borel subsets of the real line corresponding to the c.d.f. $\Phi(x;\mu,\sigma)$, and $\tilde{P}$ the measure corresponding to $\Phi(x;\bar{x}, s)$. Consider the statistic $V_n$ defined by

$$V_n = \sup_{I \in \mathcal{G}} |P(I) - \tilde{P}(I)| . \tag{3.1}$$

Since the class $\mathcal{G}$ remains invariant under the linear transformation $x = \sigma z + \mu$, it is seen that (3.1) can be rewritten in the following standardized form

$$V_n = \sup_{I \in \mathcal{G}} |P_0(I) - \tilde{P}_0(I)| , \tag{3.2}$$

where $P_0$ and $\tilde{P}_0$ correspond respectively to $\Phi(x;0,1)$ and $\Phi(x;\bar{z}, s_z)$ with $\bar{z}$ and $s_z$ defined as in Section 2. The distribution of $V_n$ is thus independent of $\mu$ and $\sigma$. If $v_\alpha$ is the $\alpha$-quantile of $V_n$, then it is clear from (3.1) that a set of simultaneous confidence intervals at level $\alpha$ for the interval probabilities of $\Phi(x;\mu,\sigma)$ is given by

$$K_l[(x, y);\bar{x}, s] = \Phi(y;\bar{x}, s) - \Phi(x;\bar{x}, s) - v_\alpha , \tag{3.3}$$

and

$$K_u[(x, y);\bar{x}, s] = \Phi(y;\bar{x}, s) - \Phi(x;\bar{x}, s) + v_\alpha , \tag{3.4}$$

so that (1.3) is satisfied.

The distribution of $V_n$ is easily derived once we note that it can be equivalently expressed as

$$V_n = \sup_{-\infty<z<\infty} [\Phi(z;0,1) - \Phi(z;\bar{z}, s_z)] - \inf_{-\infty<z<\infty} [\Phi(z;0,1) - \Phi(z;\bar{z}, s_z)] . \tag{3.5}$$

(See [2] and [15] where a distribution-free version of $V_n$ is considered),

Table 4 Quantiles, $v_\alpha$ of $V_n = \sup_{I \in \mathcal{G}} |P(I) - \tilde{P}(I)|$

| Sample size $n$ | Confidence level $\alpha = \Pr\{V_n \leqq v_\alpha\}$ | | | | |
|---|---|---|---|---|---|
| | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
| 5 | 0.32 | 0.36 | 0.40 | 0.47 | 0.61 |
| 6 | 0.28 | 0.31 | 0.35 | 0.42 | 0.55 |
| 7 | 0.26 | 0.28 | 0.32 | 0.38 | 0.50 |
| 8 | 0.24 | 0.26 | 0.30 | 0.35 | 0.45 |
| 9 | 0.22 | 0.25 | 0.28 | 0.32 | 0.41 |
| 10 | 0.21 | 0.23 | 0.26 | 0.30 | 0.40 |
| 11 | 0.20 | 0.22 | 0.25 | 0.29 | 0.37 |
| 12 | 0.19 | 0.21 | 0.23 | 0.27 | 0.36 |
| 13 | 0.18 | 0.20 | 0.22 | 0.26 | 0.35 |
| 14 | 0.18 | 0.20 | 0.22 | 0.25 | 0.33 |
| 15 | 0.17 | 0.19 | 0.21 | 0.24 | 0.31 |
| 16 | 0.17 | 0.18 | 0.20 | 0.23 | 0.29 |
| 17 | 0.16 | 0.18 | 0.19 | 0.22 | 0.28 |
| 18 | 0.16 | 0.17 | 0.19 | 0.22 | 0.28 |
| 19 | 0.15 | 0.16 | 0.18 | 0.21 | 0.27 |
| 20 | 0.15 | 0.16 | 0.18 | 0.21 | 0.27 |
| 21 | 0.14 | 0.16 | 0.17 | 0.20 | 0.26 |
| 22 | 0.14 | 0.15 | 0.17 | 0.19 | 0.25 |
| 23 | 0.14 | 0.15 | 0.16 | 0.19 | 0.24 |
| 24 | 0.13 | 0.15 | 0.16 | 0.19 | 0.24 |
| 25 | 0.13 | 0.14 | 0.16 | 0.18 | 0.23 |
| 26 | 0.13 | 0.14 | 0.16 | 0.18 | 0.23 |
| 27 | 0.13 | 0.14 | 0.15 | 0.18 | 0.23 |
| 28 | 0.12 | 0.14 | 0.15 | 0.17 | 0.22 |
| 29 | 0.12 | 0.13 | 0.15 | 0.17 | 0.21 |
| 30 | 0.12 | 0.13 | 0.14 | 0.17 | 0.21 |

that is, $V_n = L_n^+ + L_n^-$. Table 4 presents the Monte Carlo quantiles of $V_n$ for $n=5(1)30$. To get the limit distribution of $\sqrt{n}\, V_n$ we see, as in the case of $L_n^+$, from the weak convergence of the process $\sqrt{n}\, w(u; \bar{z}, 0, s_z, 1)$ to the Gaussian process $X_u$, and the continuous mapping theorem, that it is identical with the distribution of the random variable

$$W = \phi(W_1)\left|Z_1 + \frac{W_1 Z_2}{\sqrt{2}}\right| + \phi(W_2)\left|Z_1 + \frac{W_2 Z_2}{\sqrt{2}}\right|, \tag{3.6}$$

where

$$W_1, W_2 = \frac{-Z_1 \pm \sqrt{Z_1^2 + 2Z_2^2}}{\sqrt{2}\, Z_2}, \tag{3.7}$$

and $Z_1$ and $Z_2$ are independent standard normal random variables. The quantiles of $W$ are given in Table 5.

As in the previous section, we have compared the Monte Carlo quantiles of $V_n$ with those obtained using Table 5 for various values

Table 5 Quantiles of the limit distribution of $\sqrt{n}\, V_n$

| $\alpha$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| $w_\alpha$ | 0.651 | 0.706 | 0.776 | 0.888 | 1.095 |

Table 6 Comparison of the 'exact' and asymptotic quantiles of $V_n$

| $n$ \ $\alpha$ | 0.80 | 0.85 | 0.90 | 0.95 | 0.99 |
|---|---|---|---|---|---|
| 40 | 0.102 | 0.113 | 0.125 | 0.142 | 0.178 |
| | 0.102 | 0.112 | 0.123 | 0.140 | 0.173 |
| 60 | 0.084 | 0.091 | 0.101 | 0.116 | 0.144 |
| | 0.084 | 0.091 | 0.100 | 0.115 | 0.141 |
| 80 | 0.073 | 0.080 | 0.088 | 0.101 | 0.124 |
| | 0.073 | 0.080 | 0.087 | 0.099 | 0.122 |
| 100 | 0.065 | 0.071 | 0.078 | 0.089 | 0.113 |
| | 0.065 | 0.071 | 0.078 | 0.089 | 0.109 |

Note: For each $n$, the top row gives the Monte Carlo quantiles, and the bottom the values obtained using Table 5.

of $n$. The results, given in Table 6, suggest that Table 5 provides sufficiently accurate approximations for $n>30$.

The problem of determining whether or not the region in 3-space generated by $V_n$ is full remains unsolved.

All the Monte Carlo results reported in this paper were obtained in the CDC 6400 computer at Temple University using the methods given in Knuth, D. E., *Semi Numerical Algorithms*, Reading, Mass: Addison-Wesley, 1969. The results are believed to be accurate to at least two decimal places.

TEMPLE UNIVERSITY
TRENTON STATE COLLEGE

## REFERENCES

[1] Billingsley, P. (1968). *Convergence of Probability Measures*, John Wiley, New York.
[2] Brunk, H. D. (1962). On the range of the difference between hypothetical distribution function and Pyke's modified empirical distribution function, *Ann. Math. Statist.*, **33**, 525-532.
[3] Kanofsky, P. and Srinivasan, R. (1972). An approach to the construction of parametric confidence bands on cumulative distribution functions, *Biometrika*, **59**, 623-631.
[4] Kuiper, N. H. (1960). Tests concerning random points on a circle, *Proc. Kon. Ned. Akad. Wetensch.*, A **63**, *Indag. Math.*, **22**, 38-47.
[5] Rosenblatt, J. (1963). Test and confidence intervals based on the metric $d_2$, *Ann. Math. Statist.*, **34**, 618-623.
[6] Saunders, S. C. (1970). On maximum likelihood estimators of shape and scale parameters and their application in constructing confidence contours, *Mathematical and Information Sciences Report* No. 5, Boeing Scientific Research Laboratories.
[7] Srinivasan, R., Kanofsky, P. and Wharton, R. M. (1976). Some simultaneous confidence intervals for the exponential distribution, *Sankhya*, B, **35**, to appear.
[8] Srinivasan, R. and Wharton, R. M. (1973). The limit distribution of a random variable used in the construction of confidence bands, *Biometrika*, **60**, 431-432.
[9] Valentine, F. A. (1964). *Convex Sets*, McGraw-Hill, New York.