TESTING THE EQUALITY OF COVARIANCE MATRICES UNDER INTRACLASS CORRELATION MODELS

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Summary

The modified likelihood ratio test for the equality of covariance matrices under intraclass correlation models is obtained and its asymptotic distributions are derived. This test is compared with the test derived by using Roy's union-intersection principle, by a Monte Carlo study. It is found that in general the modified likelihood ratio test has a larger power. When the covariance matrices are such that one has small eigenvalues, one has large eigenvalues and the eigenvalues of the rest are in the middle, the two tests have about the same power.

1. Introduction

Let $\boldsymbol{X}_i$ be a $p\times 1$ random vector, $i=1, 2, \cdots, k$, which has a multivariate normal distribution with mean $\boldsymbol{\mu}_i$ and covariance matrix $\boldsymbol{\Sigma}_i = \sigma_i^2[(1-\rho_i)\boldsymbol{I}+\rho_i\boldsymbol{e}\boldsymbol{e}']$, where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{e}'=(1, \cdots, 1)$. This model is referred to as the intraclass correlation model. It is desired to test the hypothesis H_0 : $\boldsymbol{\Sigma}_i$ are all equal, against the alternative H_1 : at least one $\boldsymbol{\Sigma}_i$ is different. This testing problem was considered by Krishnaiah and Pathak [7] and Srivastava [10]. Krishnaiah and Pathak derived a test by using Roy's union-intersection principle, which will be referred to as the union-intersection (UI) test. They have noted that their test is superior to the test procedure proposed by Srivastava which is a product of two $F_{\max}$ statistics. But both of these tests are restricted by equal sample sizes. Further, the percentage points of the UI test statistic are only available for some special values of the levels of significance. Srivastava gave a different asymptotic test to deal with the case of unequal sample sizes. In this paper, Section 2 derives the likelihood ratio test modified by substituting the sample sizes by degrees of freedom (Bartlett [2]). The modified likelihood ratio (MLR) test can be used when the sample sizes are equal or unequal. Asymptotic distributions of the test statistic are derived in Section 3 and percentage points are obtained by using the general inverse expansion formula of Hill and Davis [6]. Since the UI test is better than the procedure by Srivastava [10], in Section 4 the MLR test is compared with the UI test by a Monte Carlo study.

2. Modified likelihood ratio test

Suppose independent random samples of size N_i are taken from the ith population, $i=1, 2, \cdots, k$. Following Olkin and Pratt [8], let $\boldsymbol{\Gamma}$ be a $p\times p$ orthogonal matrix with the first row $e'/p^{1/2}$, then $\boldsymbol{\Gamma\Sigma}_i\boldsymbol{\Gamma}'=\operatorname{diag}(\alpha_i, \beta_i, \cdots, \beta_i)=\boldsymbol{\Sigma}_i^*$, say, where $\alpha_i=\sigma_i^2[1+(p-1)\rho_i]$ and $\beta_i=\sigma_i^2(1-\rho_i)$. The null hypothesis of the equality of covariance matrices is equivalent to $H_0: \alpha_1=\cdots=\alpha_k$ and $\beta_1=\cdots=\beta_k$. We shall derive the likelihood ratio test for this hypothesis.

Let $\boldsymbol{Y}_{iu}=\boldsymbol{\Gamma X}_{iu}$, $u=1, \cdots, N_i$, then $\boldsymbol{Y}_{iu}\sim N(\boldsymbol{\nu}_i, \boldsymbol{\Sigma}_i^*)$ where $\boldsymbol{\nu}_i=\boldsymbol{\Gamma}\mu_i$. The likelihood is

$$L=\prod_{i=1}^{k}(2\pi)^{-N_ip/2}\alpha_i^{-N_i/2}\beta_i^{-N_i(p-1)/2} \cdot \exp\left\{-(1/2)\sum_{u=1}^{N_i}(\boldsymbol{Y}_{iu}-\boldsymbol{\nu}_i)'\boldsymbol{\Sigma}_i^{*-1}(\boldsymbol{Y}_{iu}-\boldsymbol{\nu}_i)\right\}. \tag{1}$$

Under the null hypothesis $\omega=\{-\infty<\boldsymbol{\nu}_i<\infty,\ \alpha_i=\alpha>0 \text{ and } \beta_i=\beta>0\}$ the maximum likelihood estimators of $\boldsymbol{\nu}_i$, α and β are

$$\hat{\boldsymbol{\nu}}_i=\frac{1}{N_i}\sum_{u=1}^{N_i}\boldsymbol{Y}_{iu}, \quad \hat{\alpha}=\frac{1}{N}W_1 \quad\text{and}\quad \hat{\beta}=\frac{1}{N(p-1)}W_2$$

where

$$N=\sum_{i=1}^{k}N_i, \quad W_1=\sum_{i=1}^{k}\sum_{u=1}^{N_i}(y_{i1u}-\hat{\nu}_{i1})^2 \quad\text{and}\quad W_2=\sum_{i=1}^{k}\sum_{j=2}^{p}\sum_{u=1}^{N_i}(y_{iju}-\hat{\nu}_{ij})^2,$$

y_{iju} and $\hat{\nu}_{ij}$ are the jth components of $\boldsymbol{Y}_{iu}$ and $\hat{\boldsymbol{\nu}}_i$ respectively. Hence

$$\max_{\omega} L=(2\pi)^{-Np/2}N^{N/2}[N(p-1)]^{N(p-1)/2}W_1^{-N/2}W_2^{-N(p-1)/2}e^{-Np/2}.$$

Under $\Omega=\{-\infty<\boldsymbol{\nu}_i<\infty,\ \alpha_i>0 \text{ and } \beta_i>0,\ i=1, \cdots, k\}$, the maximum likelihood estimators are

$$\hat{\boldsymbol{\nu}}_i=\frac{1}{N_i}\sum_{u=1}^{N_i}\boldsymbol{Y}_{iu}, \quad \hat{\alpha}_i=\frac{1}{N_i}W_{1i} \quad\text{and}\quad \hat{\beta}_i=\frac{1}{N_i(p-1)}W_{2i}$$

where

$$W_{1i}=\sum_{u=1}^{N_i}(y_{i1u}-\hat{\nu}_{i1})^2 \quad\text{and}\quad W_{2i}=\sum_{j=2}^{p}\sum_{u=1}^{N_i}(y_{iju}-\hat{\nu}_{ij})^2.$$

We note that $W_1=\sum W_{1i}$ and $W_2=\sum W_{2i}$. Then

$$\max_{\Omega} L=(2\pi)^{-Np/2}\left\{\prod_{i=1}^{k} N_i^{N_i/2}[N_i(p-1)]^{N_i(p-1)/2}W_{1i}^{-N_i/2}W_{2i}^{-N_i(p-1)/2}\right\}e^{-Np/2}.$$

The likelihood ratio statistic is

$$(2)\qquad \lambda=\max_{\omega} L/\max_{\Omega} L=N^{N/2}[N(p-1)]^{N(p-1)/2}$$

$$\cdot\left\{\prod_{i=1}^{k} N_i^{-N_i/2}[N_i(p-1)]^{-N_i(p-1)/2}\left(\frac{W_{1i}}{W_1}\right)^{N_i/2}\left(\frac{W_{2i}}{W_2}\right)^{N_i(p-1)/2}\right\}.$$

Following Bartlett [2], we replace sample sizes by degrees of freedom; then the modified likelihood ratio statistic is

$$(3)\qquad \lambda'=K\prod_{i=1}^{k}\left(\frac{W_{1i}}{W_1}\right)^{n_i/2}\left(\frac{W_{2i}}{W_2}\right)^{n_i(p-1)/2}$$

where

$$K=n^{n/2}[n(p-1)]^{n(p-1)/2}\left\{\prod_{i=1}^{k}(n_i)^{-n_i/2}[n_i(p-1)]^{-n_i(p-1)/2}\right\},$$

$$n_i=N_i-1 \text{ and } n=\sum n_i.$$

It was shown by Pitman [9] that the Bartlett test of the equality of variances is unbiased. The test λ' can be viewed as a combination of two independent tests. By a similar procedure it can be seen that the modified likelihood ratio test is also unbiased. The null hypothesis H_0 is rejected if $\lambda'<d$ where d is determined by the distribution of λ' and the level of significance α. The exact distribution of λ' is not easy to obtain, hence we shall derive its asymptotic distribution.

3. Asymptotic distribution of the modified likelihood ratio statistic

We shall first consider the asymptotic distribution of λ' under H_0. Since $\boldsymbol{\Sigma}_i^*$ is a diagonal matrix, W_{1i}/α_i and W_{2i}/β_i, $i=1,\cdots,k$, are independently distributed as $\chi^2_{n_i}$ and $\chi^2_{n_i(p-1)}$, respectively. Using the results in Anderson ([1], p. 253), we obtain the hth moment of λ' to be

$$\mathrm{E}(\lambda'^h)=K^h\frac{\Gamma\left[\frac{n}{2}\right]\Gamma\left[\frac{1}{2}n(p-1)\right]\prod_{i=1}^{k}\Gamma\left[\frac{1}{2}n_i(1+h)\right]\Gamma\left[\frac{1}{2}n_i(p-1)(1+h)\right]}{\Gamma\left[\frac{1}{2}n(1+h)\right]\Gamma\left[\frac{1}{2}n(p-1)(1+h)\right]\prod_{i=1}^{k}\Gamma\left[\frac{n_i}{2}\right]\Gamma\left[\frac{1}{2}n_i(p-1)\right]}.$$

By the Box [3] method of expansion (see also Anderson [1], p. 203), we found the correction factor for $M=-2\log\lambda'$ to be

$$c=1-\frac{p}{6(k-1)(p-1)}\left[\sum_{i=1}^{k}\frac{1}{n_i}-\frac{1}{n}\right]$$

and the asymptotic expansion of the null distribution cM as

$$(4)\quad \Pr\{cM \leqq cM_0\} = \Pr\{\chi_f^2 \leqq cM_0\} + \omega_2(\Pr\{\chi_{f+4}^2 \leqq cM_0\} - \Pr\{\chi_f^2 \leqq cM_0\}) + \omega_3(\Pr\{\chi_{f+6}^2 \leqq cM_0\} - \Pr\{\chi_f^2 \leqq cM_0\}) + O(n^{-4})$$

where

$$f = 2(k-1)\ ,$$

$$\omega_2 = -\left[\sum_{i=1}^{k} \frac{n}{n_i} - 1\right] p^2 \Big/ [72(cn)^2(k-1)(p-1)^2]\ ,$$

and

$$\omega_3 = \left\{ \left[\sum_{i=1}^{k} \frac{n}{n_i} - 1\right]^3 p^3 \Big/ [108(k-1)^2(p-1)^3] - \left[\sum_{i=1}^{k} \left(\frac{n}{n_i}\right)^3 - 1\right] [1 + (p-1)^3] \Big/ [15(p-1)] \right\} \Big/ 3(cn)^3 .$$

Applying the general inversion expansion of Hill and Davis [6] to (4), we can get the asymptotic formula for the percentage point of cM as

$$(5)\quad u + 2\{\omega_2[u^2 + (f+2)u][f(f+2)]^{-1} + \omega_3[u^3 + (f+4)u^2 + (f+2)(f+4)u][f(f+2)(f+4)]^{-1}\} + O(n^{-4})$$

where u is the percentage point of the χ^2 distribution with f degrees of freedom.

Under the alternative hypothesis, the asymptotic expansion of the distribution of cM can be obtained by using the method given in Sugiura and Nagao [11]. Since the technique is similar, we shall only derive the leading term of the expansion under a fixed alternative. Following Sugiura and Nagao, we put $cn_i = m_i$ and $\sum_{i=1}^{k} m_i = m$, then

$$cM = m \log \left(\sum_{i=1}^{k} W_{1i} \Big/ m \right) - \sum_{i=1}^{k} m_i \log (W_{1i}/m_i) + m(p-1) \log \left[\sum_{i=1}^{k} W_{2i} \Big/ m(p-1) \right] - \sum_{i=1}^{k} m_i(p-1) \cdot \log [W_{2i}/m_i(p-1)]\ .$$

Let $U_{1i} = [W_{1i}/\alpha_i - m_i]/(2m_i)^{1/2}$ and $U_{2i} = [W_{2i}/\beta_i - m_i(p-1)]/[2m_i(p-1)]^{1/2}$, then cM is expressed by U_{1i} and U_{2i} as

$$cM = m(\log \bar{\alpha} - \tilde{\alpha}) + m(p-1)(\log \bar{\beta} - \tilde{\beta}) + \sqrt{m} \sum_{i=1}^{k} [\sqrt{2g_i}\,(\eta_i - 1)U_{1i} + \sqrt{2g_i(p-1)}\,(\gamma_i - 1)U_{2i}] + O_p(1)$$

where

$$\bar{\alpha}=\sum_{i=1}^{k} g_i\alpha_i\ , \qquad \tilde{\alpha}=\sum_{i=1}^{k} g_i \log \alpha_i\ , \qquad \eta_i=\alpha_i/\bar{\alpha}\ , \qquad \bar{\beta}=\sum_{i=1}^{k} g_i\beta_i\ ,$$
$$\tilde{\beta}=\sum_{i=1}^{k} g_i \log \beta_i\ , \qquad \gamma_i=\beta_i/\bar{\beta} \qquad \text{and} \qquad g_i=n_i/n\ .$$

g_i is assumed to be fixed as n tends to infinity. Since U_{1i} and U_{2i} tend to the standard normal distribution when the sample sizes are large and they are independently distributed, we have that

$$(6) \qquad \{cM-m(\log \bar{\alpha}-\tilde{\alpha})-m(p-1)(\log \bar{\beta}-\tilde{\beta})\}\Big/ \left\{m \sum_{i=1}^{k} [2g_i(\eta_i-1)^2+2g_i(p-1)(\gamma_i-1)^2]\right\}^{1/2}$$

is distributed asymptotically as the standard normal distribution when n is large. As noted by Sugiura and Nagao [11], the non-null asymptotic distribution does not reduce to the null distribution when the null hypothesis is true since the asymptotic variance of the non-null distribution of cM vanishes under the null hypothesis. Therefore the expansion does not give a good approximation when the alternative hypothesis is near to the nullhypothesis.

4. Comparison of tests

As described in Krishnaiah and Pathak [7], the UI test is better than the test given by Srivastava [10]; we shall only compare the MLR test with the UI test. Since the non-null distribution of the UI test is unknown, the comparison is made by a Monte Carlo study. As mentioned in the introduction, the UI test can only be used when the sample sizes are all equal, hence we let $N_i=N$. The UI test is to accept H_0 if and only if

$$(7) \qquad F_{1ij}=\frac{W_{1i}}{W_{1j}}\leqq a \quad \text{and} \quad F_{2ij}=\frac{W_{2i}}{W_{2j}}\leqq b \qquad \text{for } i\neq j$$

where a and b are chosen such that

$$(8) \qquad \Pr\,[\max_{i\neq j} F_{1ij}\leqq a\,|\,H_0]\,\Pr\,[\max_{i\neq j} F_{2ij}\leqq b\,|\,H_0]=1-\alpha\ .$$

The values of a and b can be obtained from Hartley [5] or David [4] for some special values of α, $N-1$, and $(p-1)(N-1)$. David has tabulated the upper 5 and 1% point of the $F_{\max}$ test. Therefore we may let $1-\alpha=(.95)^2$ and $(.99)^2$, i.e., $\alpha=.0975$ and $.0199$. In the Monte Carlo study, $p=3$, $N=16, 31$ and $k=3, 5$ are used. The empirical powers of the MLR test and the UI test are computed from 200 statistics simulated on the computer, an IBM 360, at Iowa State University.

For $k=3$, the eigenvalues α_i and β_i of the three covariance matrices are set as

Population	1	2	3
α_i	2.0	2.5	3.0
β_i	1 1	1.5 1.5	2.0 2.0

The power comparisons of the two tests for $N=16$ and 31 are given in Table 1.

Table 1 Power comparison, $k=3$

	$N=16$		$N=31$	
	$\alpha=.0975$	$\alpha=.0199$	$\alpha=.0975$	$\alpha=.0199$
MLR	.55	.32	.79	.46
UI	.50	.31	.71	.40

It is seen that the power of the MLR test for this case is larger than the UI test.

We next consider $k=5$. Two sets of covariance matrices are used. The first set is

Population	1	2	3	4	5
α_i	2.0	2.5	3.0	3.5	4.0
β_i	1.0 1.0	1.5 1.5	2.0 2.0	2.5 2.5	3.0 3.0

The second set is

Population	1	2	3	4	5
α_i	2.0	3.0	3.0	3.0	4.0
β_i	1.0 1.0	2.0 2.0	2.0 2.0	2.0 2.0	3.0 3.0

In the second set of covariance structures, we let one population have small eigenvalues and one have large eigenvalues. The eigenvalues of the other three populations are in the middle. Such a situation is favorable to the UI test. The power comparisons are given in Table 2.

Table 2 Power comparison, $k=5$

	$N=16$		$N=31$	
	$\alpha=.0975$	$\alpha=.0199$	$\alpha=.0975$	$\alpha=.0199$
MLR (First set)	.80	.62	.99	.96
UI (First set)	.82	.51	.98	.90
MLR (Second set)	.74	.48	.96	.91
UI (Second set)	.78	.44	.96	.88

In the first set of covariance matrices the power of the MLR test is larger except when $N=16$ and $\alpha=.0975$. For the second set, the power of the two tests is about the same.

It is noticed, during the Monte Carlo study, that the adjustment of the percentage point u of the χ^2 distribution in (5) is small. For example, the 90.25% point of χ_4^2 is 7.8431, the adjusted values are 7.8424 and 7.8429 for $N=16$ and 31 respectively; the 90.25% point of χ_8^2 is 13.4427 and the adjusted values are 13.4422 and 13.4426 for $N=16$ and 31 respectively. Therefore the convergence is very fast even for moderate values of sample sizes.

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