ON THE ERROR EVALUATION OF THE JOINT NORMAL APPROXIMATION FOR SAMPLE QUANTILES

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Summary

This paper is concerned with non-asymptotic error evaluation for the uniform (or $(\boldsymbol{B})_d$-type) joint normal approximation for k sample quantiles. The evaluation generalizes Reiss' single sample quantile case in [6], where the type of approximation is a little weaker than that of the present article.

Some improvements of the previous results [2] are also given for the uniform asymptotic joint normality of increasing number of sample quantiles as the sample size increases.

1. Introduction

Let, for each n, $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics based on a random sample of size n drawn from a continuous distribution on the real line, whose pdf and cdf are $f_n(x)$ and $F_n(x)$, respectively. Let us select k sample quantiles from the order statistics and denote the joint random variable by $X^n_{n(k)}=(X_{nn_1}, X_{nn_2},\cdots, X_{nn_k})'$ where $0<n_1<n_2<\cdots<n_k<n$ and k may or may not be dependent on n. Note that in the notation $X^n_{n(k)}$ the subscript n stands for the underlying sample size, whereas, the superscript n indicates that the situation concerned is the case of unequal basic distributions, that is, the population distribution function $F_n(x)$ may vary with n. In the case of equal basic distributions, where $F_n(x)=F(x)$ and hence $f_n(x)=f(x)$ for all n, we denote the corresponding vector by $X_{n(k)}$ without the superscript.

It is well-known that under certain conditions $X^n_{n(k)}$ is asymptotically normally distributed, and based on this fact the asymptotic theory of statistical inferences is developed by many authors. Ikeda and Matsunawa [2] proved that under some conditions $X^n_{n(k)}$ is asymptotically equivalent $(\boldsymbol{B})_d$ to a certain k-dimensional normal random vector, $Y^n_{n(k)}=(Y_{n1},\cdots, Y_{nk})'$ say, whose mean vector and dispersion matrix are respectively $s_{n(k)}=(s_{n1},\cdots, s_{nk})'$ and $S_{n(k)}=||l_{ni}(1-l_{nj})/f_{ni}f_{nj}||/(n+2)$, $1\leqq i\leqq j\leqq k$, with $s_{ni}=F_n(l_{ni})$ and $f_{ni}=f_n(s_{ni})$, $i=1,\cdots,k$, where, the authors showed that the order of magnitude for the approximate error defined by

$$(1.1) \qquad \delta_d(X^n_{n(k)}, Y^n_{n(k)} : \boldsymbol{B})=\sup_{E\in\boldsymbol{B}} |P^{X^n_{n(k)}}(E)-P^{Y^n_{n(k)}}(E)|$$

is at most $O(k(n)^{1/2}/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})^{1/4})$ as $n\to\infty$, where $\boldsymbol{B}$ denotes the usual Borel field of the k-dimensional Euclidean space $R_{(k)}$, and $P^{X^n_{n(k)}}$ and $P^{Y^n_{n(k)}}$ designate the probability measures corresponding to $X^n_{n(k)}$ and $Y^n_{n(k)}$, respectively. Related results to the above in censored cases have been also obtained by the present author [4].

From the practical point of view, however, the above asymptotic evaluation seems to be unsatisfactory when n is finite or moderately large; it is desirable to find sharper bounds for the error of normal approximation. For $k=1$ Reiss [6] recently considered this problem for the sample p-quantile $X_{n,[np]+1}$ (or sometimes $X_{n,[np]}$) based on the order statistics for an equal basic distribution: Let ξ_p be the population p-quantile, $\sigma_p=(p(1-p))^{1/2}$, $q_n=[1-\sigma_p^{-2}(|1-2p|((\log n)/n)^{1/2}+(\log n)/n)]^{1/2}$ and $\|f'\|=\sup\{|f'(x)|: x\in R_{(1)}\}$. Let, further, Φ be the standard normal distribution function, P and P^n be the probability measure with cdf $F(x)$ and the n independent product of P for each n, respectively. Then, under some regularity conditions on $F(x)$ and for $n\geqq 9$ Reiss gave the following estimate by resorting to the Berry-Essen theorem:

$$(1.2) \qquad \sup_{t\in R_{(1)}}\left|P^n\left\{\frac{n^{1/2}f(\xi_p)}{\sigma_p}(X_{n,[np]+1}-\xi_p)<t\right\}-\Phi(t)\right|$$
$$\leqq n^{-1/2}\left[\frac{3\|f'\|\sigma_p}{5f^2(\xi_p)}+\frac{\|f'\|^2}{f^4(\xi_p)n^{1/2}}+R_{p,n}\right],$$

where $R_{p,n}=C(1-2\sigma_p^2q_n^2)/(\sigma_pq_n)+3(|1-2p|+((\log n)/n)^{1/2})/(10\sigma_pq_n^2)$ with $(3+\sqrt{10})/(6/\sqrt{2\pi})\leqq C<0.7975$. It should be remarked that the approximation in (1.2) is apparently weaker than type $(\boldsymbol{B})_d$-approximation, and that it seems to be unsatisfactory for some cases of practical application.

The purpose of this article is to give an exact error bound of the joint normal approximation in the sense of $(\boldsymbol{B})_d$ for $k=k(n)$ sample quantiles in more general situations than Reiss'. In the following section some inequalities are presented. The main result is stated in Theorem 3.1. The mathematical tools used there are almost the same as those in [2], but derivations are a little simple.

2. Preliminary lemmas

In this section some useful inequalities are presented, which play important roles in our error estimation.

Let $\{X_s\}$ $(s=1,2,\cdots)$ and $\{Y_s\}$ $(s=1,2,\cdots)$ be two sequences of random variables distributed over a measurable space $(R_s, \boldsymbol{B}_s)$ for each s, where $\boldsymbol{B}$ denotes a σ-field of subsets of any given abstract space R_s. Put

$$\delta_d(X_s, Y_s: \boldsymbol{B}_s) = \sup_{E \in \boldsymbol{B}_s} |P^{X_s}(E) - P^{Y_s}(E)| . \tag{2.1}$$

Suppose further that for each s both X_s and Y_s are absolutely continuous with respect to a σ-finite measure μ_s over $(R_s, \boldsymbol{B}_s)$ and denote their gpdf's (μ_s) by f_s and g_s, respectively. Now, as in [2], we can define some measures of discrepancy between two probability distributions:

$$\rho(X_s, Y_s) = \int_{R_s} (f_s g_s)^{1/2} d\mu_s , \tag{2.2}$$

$$I_f(X_s, Y_s) = \int_{R_s} f_s \log (f_s/g_s) d\mu_s , \tag{2.3}$$

and

$$I_g(X_s, Y_s) = \int_{R_s} g_s \log (g_s/f_s) d\mu_s . \tag{2.4}$$

We can state the following

LEMMA 2.1. *Let* $I(X_s, Y_s) = \min [I_f(X_s, Y_s), I_g(X_s, Y_s)]$, *then for each* s *it holds that*

$$\delta_d(X_s, Y_s: \boldsymbol{B}_s) \leqq \min [\eta_1(s), \eta_2(s)] , \tag{2.5}$$

where

$$\eta_1(s) = \frac{3}{2}\left[\left(1 + \frac{4}{9} I(X_s, Y_s)\right)^{1/2} - 1\right]^{1/2} (\leqq [I(X_s, Y_s)/2]^{1/2}) , \tag{2.6}$$

and

$$\eta_2(s) = [1 - \exp(-I(X_s, Y_s))]^{1/2} . \tag{2.7}$$

The equality in (2.5) *holds iff* $f_s = g_s$ [a.e. μ_s], *for each* s, *in which case the value of* $\delta_d(X_s, Y_s: \boldsymbol{B}_s) = 0$.

PROOF. Without loss of generality we can assume that $I(X_s, Y_s) = I_f(X_s, Y_s)$, then we have

$$I(X_s, Y_s) = -2 \int_{R_s} f_s \log (g_s/f_s)^{1/2} d\mu_s \geqq -2 \log \rho(X_s, Y_s) ,$$

that is,

$$\rho(X_s, Y_s) \geqq \exp(-I(X_s, Y_s)/2) . \tag{2.8}$$

Noticing the fact $\delta_d(X_s, Y_s: \boldsymbol{B}_s) = \int_{R_s} |f_s - g_s| d\mu_s/2$, it is then easily seen that

$$\delta_d(X_s, Y_s: \boldsymbol{B}_s) \leqq [1-\rho^2(X_s, Y_s)]^{1/2} \leqq \eta_2(s) . \tag{2.9}$$

To find another bound in (2.5) we make use of the following delicate inequality due to Kraft [3]:

$$x \log \frac{x}{y} + (1-x) \log \frac{1-x}{1-y} \geqq 2(x-y)^2 + \frac{4}{9}(x-y)^4 , \tag{2.10}$$

for $0<y\leqq x<1$. Now, consider a simple partition of R_s such that

$$A = \{x: f_s \geqq g_s\} \qquad \text{and} \qquad B = \{x: f_s < g_s\} ,$$

then we have

$$\begin{aligned} \delta_d(X_s, Y_s: \boldsymbol{B}_s) &= \frac{1}{2}\left\{\int_A (f_s-g_s)d\mu_s + \int_B (g_s-f_s)d\mu_s\right\} \\ &= P^{X_s}(A) - P^{Y_s}(A) . \end{aligned} \tag{2.11}$$

Hence, the substitution $x=P^{X_s}(A)$ and $y=P^{Y_s}(A)$ into (2.10) yields

$$\begin{aligned} &P^{X_s}(A) \log [P^{X_s}(A)/P^{Y_s}(A)] + P^{X_s}(B) \log [P^{X_s}(B)/P^{Y_s}(B)] \\ &\qquad \geqq 2[\delta_d(X_s, Y_s: \boldsymbol{B}_s)]^2 + \frac{4}{9}[\delta_d(X_s, Y_s: \boldsymbol{B}_s)]^4 . \end{aligned} \tag{2.12}$$

On the other hand, in view of a property of the K-L information number, the LHS of (2.12) can not exceed the corresponding $I(X_s, Y_s)$. Thus, we have the following inequality

$$I(X_s, Y_s) \geqq 2[\delta_d(X_s, Y_s: \boldsymbol{B}_s)]^2 + \frac{4}{9}[\delta_d(X_s, Y_s: \boldsymbol{B}_s)]^4 . \tag{2.13}$$

Solving the inequality in $\delta_d(\cdot)$ under the condition $0\leqq\delta_d(\cdot)\leqq1$ and combining the resulting inequality with (2.9), we obtain (2.5), which completes the proof of the lemma.

Next, we shall state two inequalities which are helpful to evaluate the amount of the K-L information number in the following section. They can be obtained by estimating the corresponding infinite series of inverse factorials. The detailed investigations have been done in [5], so their proofs will be omitted.

LEMMA 2.2. (i) *For positive integer* $p\geqq 2$

$$\sum_{i=1}^{p} \frac{1}{i} = C + \log p + \frac{1}{2p} - \frac{1}{p} T(p) , \tag{2.14}$$

wheer C denotes the Euler constant and

(2.15) $$T(p)=\sum_{i=1}^{\infty}\frac{a_{i+1}}{(p+1)(p+2)\cdots(p+i)}$$

with

(2.16) $$a_r=\frac{1}{r}\int_0^1 z(1-z)(2-z)\cdots(r-1-z)dz\ , \qquad (r\geqq 2)\ ,$$

and further that

(2.17) $$\underline{T}(p)<T(p)<\bar{T}(p)\ ,$$

here

(2.18) $$\underline{T}(p)=\frac{1}{12(p-1)}-\frac{1}{6(p-1)(p+1)}-\frac{1}{6(p-1)(p+1)(p+2)}>\frac{1}{6}\bar{T}(p)\ ,$$

and

(2.19) $$\bar{T}(p)=\frac{1}{12(p-1)}\ .$$

(ii) *For positive integer* $p\geqq 2$,

(2.20) $$\log p!=\frac{1}{2}\log 2\pi+\left(p+\frac{1}{2}\right)\log p-p+\frac{1}{12p}-R(p)\ ,$$

where

(2.21) $$R(p)=\sum_{i=1}^{\infty}\frac{b_{i+1}}{p(p+1)(p+2)\cdots(p+i)}$$

with

(2.22) $$b_r=\frac{1}{r}\int_0^1 z(1-z)(2-z)\cdots(r-1-z)\left(\frac{1}{2}-z\right)dz\ , \qquad (r\geqq 2)\ ,$$

and further

(2.23) $$\underline{R}(p)<R(p)<\bar{R}(p)\ ,$$

where

(2.24) $$\underline{R}(p)=\frac{1}{360p(p-1)(p+1)}-\frac{1}{120p^2(p-1)(p+1)}$$

and

(2.25) $$\underline{R}(p)=\frac{1}{360p(p-1)(p+1)}+\frac{11}{480p^2(p-1)(p+1)}\ .$$

3. Exact error evaluation of the normal $(B)_d$-approximation for k sample quantiles

Let, for each n, $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a random sample of size n from a continuous distribution over the real line with pdf $f_n(x)$ and cdf $F_n(x)$.

Throughout this section we assume that

(A.1) for each n, $D(f_n)=\{x: f_n(x)>0\}$ is an open interval over the real line, say (a_n, b_n), where a_n and b_n are extended real numbers such that $a_n<b_n$, and

(A.2) for each n, $f_n(x)$ is differentiable once over $D(f_n)$.

The assumption (A.1) assures the existence of the exact inverse function of F_n, say F_n^{-1}, for each n, which is clearly a nonsingular transformation from the interval (0, 1) onto the interval (a_n, b_n), for each n.

Let, as in Section 1, $X^n_{n(k)}$ be the joint random variable of $k=k(n)$ sample quantiles based on the order statistics, and denote by $Y^n_{n(k)}$ the k-dimensional normal random vector with the mean vector $s_{n(k)}$ and the dispersion matrix $S_{n(k)}$. Then, for each n, the pdf's of $X^n_{n(k)}$ and $Y^n_{n(k)}$ are respectively given by

$$h_n^*(x_{(k)})=\left\{n!\Big/\prod_{i=1}^{k+1}(d_i!)\right\}\prod_{i=1}^{k+1}[F_n(x_i)-F_n(x_{i-1})]^{d_i}f(x_i)\,, \tag{3.1}$$
$$(a_n=x_0<x_1<\cdots<x_k<x_{k+1}=b_n)\,,$$

and

$$g_n^*(x_{(k)})=(2\pi)^{-k/2}|S_{n(k)}|^{-1/2}\exp\left[-\frac{1}{2}(x_{(k)}-s_{n(k)})'S_{n(k)}^{-1}(x_{(k)}-s_{n(k)})\right] \tag{3.2}$$
$$(-\infty<x_i<\infty,\ i=1,\cdots,k)\,,$$

where $d_i=n_i-n_{i-1}-1$, $i=1,\cdots,k+1$ with $n_0=0$, $n_{k+1}=n+1$, $F(x_0)=0$, $F(x_{k+1})=1$, and $x_{(k)}=(x_1,\cdots,x_k)'$.

Now, making use of the transformations $U_{ni}=F_n(X_{ni})$, $i=1,\cdots,n$, we shall reduce the unequal basic case to the special equal basic situation $U_{n1}<U_{n2}<\cdots<U_{nn}$ which are regarded as order statistics from a uniform distribution over (0, 1), for each n. Thus, for our purpose it suffices to consider the $k=k(n)$ sample quantiles $U_{nn_1}<U_{nn_2}<\cdots<U_{nn_k}$ in this simplified situation. We denote the joint variable by $(U_{nn_1},\cdots, U_{nn_k})'$. Then, for each n, the pdf of $U_{n(k)}$ is given by

$$h_n(z_{(k)})=\left\{n!\Big/\prod_{i=1}^{k+1}(d_i!)\right\}\prod_{i=1}^{k+1}(z_i-z_{i-1})^{d_i}\,, \tag{3.3}$$
$$(0=z_0<z_1<\cdots<z_k<z_{k+1}=1)\,,$$

where $z_{(k)}=(z_1,\cdots,z_k)'$.

Further, for each n, consider a random vector $V_{n(k)}=(v_{n1},\cdots,v_{nk})'$ whose pdf are given by

$$(3.4)\qquad p_n(z_{(k)})=(2\pi)^{-k/2}|S_{n(k)}|^{-1/2}\prod_{i=1}^{k}f_n(F_n^{-1}(z_i))^{-1}\times\exp\left[-\frac{1}{2}(F_n^{-1}(z_{(k)})-s_{n(k)})'S_{n(k)}^{-1}(z_{(k)}-s_{n(k)})\right],\quad (0<z_i<1,\ i=1,\cdots,k),$$

where we have put $F_n^{-1}(z_{(k)})=(F_n^{-1}(z_1),\cdots,F_n^{-1}(z_k))'$. It can be easily verified that for each n the pdf of the transformed random vector $F_n^{-1}(V_{n(k)})=(F_n^{-1}(V_{n1}),\cdots,F_n^{-1}(V_{nk}))'$ coincides with that of $Y_{n(k)}^n$, namely (3.2). Applying Taylor's expansion we can rewrite (3.4) as

$$(3.5)\qquad p_n(z_{(k)})=(2\pi)^{-k/2}|L_{n(k)}|^{-1/2}\exp\left[-\frac{1}{2}(z_{(k)}-l_{n(k)})'L_{n(k)}^{-1}(z_{(k)}-l_{n(k)})\right]\times\exp\left[\frac{1}{2}w'_{n(k)}L_{n(k)}^{-1}(z_{(k)}-l_{n(k)})-\frac{1}{8}w'_{n(k)}L_{n(k)}^{-1}w_{n(k)}\right]\times\prod_{i=1}^{k}f_n(F_n^{-1}(l_{ni}))/f_n(F_n^{-1}(z_i)),$$

where $L_{n(k)}=(1/(n+2))\|l_{ni}(1-l_{nj})\|$, $1\leqq i\leqq j\leqq k$, and $w_{n(k)}=(w_{n1},\cdots,w_{nk})'$ with

$$(3.6)\qquad w_{ni}=\varphi_n(z_{ni}^*)\phi_n(z_{ni}^*;l_{ni})(z_i-l_{ni})^2,\qquad i=1,\cdots,k,$$

$$(3.7)\qquad \varphi_n(z)=f_n'(F_n^{-1}(z))/f_n^2(F_n^{-1}(z)),\qquad (0<z<1),$$

$$(3.8)\qquad \phi_n(z;l)=f_n(F_n^{-1}(l))/f_n(F_n^{-1}(z)),\qquad (0<z,\ l<1),$$

where z_{ni}^* is a certain function of z_i lying between z_i and l_{ni} (denoted by $z_{ni}^*\in((z_i,l_{ni}))$ in what follows), for each i.

We are now in a position to prove the following

THEOREM 3.1. *Under the assumptions* (A.1) *and* (A.2), *assume that there exists some positive constant M independent of n such that*

$$(3.9)\qquad \sup_{z_{(k)}\in\Delta_k}\ \max_{1\leqq i\leqq k}\ \sup_{z_{ni}^*\in((z_i,l_{ni}))}\max\{|\varphi_n(z_{ni}^*)|,\ \phi_n(z_{ni}^*;l_{ni})\}\leqq M$$

is satisfied uniformly for all n, where $\Delta_k=\{z_{(k)}:\ 0<z_1<\cdots<z_k<1\}$.

Then, the condition

$$(3.10)\qquad d_{n0}\equiv\min_{1\leq i\leq k+1}(n_i-n_{i-1}-1)\geqq 2$$

implies that

$$(3.11)\qquad \delta_d(X_{n(k)}^n,Y_{n(k)}^n:\boldsymbol{B})<\min[\eta_1(n;k),\eta_2(n;k)],$$

where

$$(3.12)\qquad \eta_1(n;k)=\frac{3}{2}\left[\left(1+\frac{4}{9}I(n;k)\right)^{1/2}-1\right]^{1/2}\leqq[I(n;k)/2]^{1/2},$$

$$(3.13)\qquad \eta_2(n;k)=[1-\exp(-I(n;k))]^{1/2},$$

here $I(n;k)\equiv I_0(n;k)+I_1(n;k)$ *with*

$$(3.14)\qquad I_0(n;k)=\frac{1}{2}\left\{\sum_{i=1}^{k+1}\frac{1}{n_i-n_{i-1}}-\frac{k}{n}\right\}$$

and

$$(3.15)\qquad I_1(n;k)=\frac{M^2}{(n+1)(n+3)}\sum_{i=1}^{k+1}|n+1-2n_i|+\frac{M^4}{2(n+1)^2}\left(3+\frac{7}{n+4}\right)\times\max_{1\leqq i\leqq k}n_i(n+1-n_i)\sum_{i=1}^{k+1}\frac{1}{n_i-n_{i-1}}.$$

PROOF. By means of Lemma 2.1 it is only required to evaluate the K-L information. Noticing that the transformation F_n^{-1} is nonsingular for each n, we can easily calculate the information as

$$(3.16)\qquad \begin{aligned} I(X^n_{n(k)}:Y^n_{n(k)})&=\mathcal{E}\left[\log\frac{h_n^*(X^n_{n(k)})}{g_n^*(X^n_{n(k)})}\right]=\mathcal{E}\left[\log\frac{h_n(U_{n(k)})}{p_n(U_{n(k)})}\right]\\ &=\log\left[(2\pi)^{k/2}n!\,|L_{n(k)}|^{1/2}\Big/\prod_{i=1}^{k+1}(d_i!)\right]\\ &\quad+\sum_{i=1}^{k+1}d_i\mathcal{E}[\log(U_{nn_i}-U_{nn_{i-1}})]\\ &\quad+\frac{1}{2}\mathcal{E}[(U_{n(k)}-l_{n(k)})'L_{n(k)}(U_{n(k)}-l_{n(k)})]\\ &\quad+\sum_{i=1}^{k}\mathcal{E}^*[\varphi_n(z_{ni}^{**})(z_i-l_{ni})]\\ &\quad-\frac{1}{2}\mathcal{E}^*[w'_{n(k)}L^{-1}_{n(k)}(z_{(k)}-l_{n(k)})]\\ &\quad+\frac{1}{8}\mathcal{E}^*[w'_{n(k)}L^{-1}_{n(k)}w_{n(k)}],\end{aligned}$$

where $\mathcal{E}^*[\cdot]$ designates the integral operator $\int_{\Delta_k}[\cdot]h_n(z_{(k)})dz_{(k)}$ and $z_{ni}^{**}\in((z_i,l_{ni}))$, for each i.

With the aid of Lemma 2.2 and the method used in [2] the sum of the first three terms in the last expression of (3.16), say $I_0(X^n_{n(k)}:Y^n_{n(k)})$, becomes

$$(3.17)\qquad I_0(X^n_{n(k)}:Y^n_{n(k)})=\frac{k}{2}\log\left(1-\frac{2}{n+2}\right)+\frac{k+1}{2}\log\left(1-\frac{1}{n+1}\right)+\frac{k}{2n}+\frac{1}{12n}+\left(1-\frac{k}{n}\right)T(n)-\sum_{i=1}^{k+1}T(d_i)+\frac{1}{2}\sum_{i=1}^{k+1}\log\left(1+\frac{1}{d_i}\right)$$

$$-\frac{1}{12}\sum_{i=1}^{k+1}\frac{1}{d_i}+\sum_{i=1}^{k+1}R(d_i)-R(n)$$
$$<\frac{1}{2}\left\{\sum_{i=1}^{k+1}\frac{1}{n_i-n_{i-1}}-\frac{k}{n}\right\}=I_0(n,k)\ ,$$

where $T(\cdot)$ and $R(\cdot)$ are the infinite series defined by (2.15) and (2.21), respectively.

For other terms of (3.16) the following formulae are available (Cf. David and Johnson [1]):

$$\mathcal{E}^*[z_i-l_{ni}]=0\ ,\qquad \mathcal{E}^*[(z_i-l_{ni})^2]=\frac{l_{ni}(1-l_{ni})}{n+2}\ ,$$

$$\mathcal{E}^*[(z_i-l_{ni})^3]=\frac{2l_{ni}(1-2l_{ni})(1-l_{ni})}{(n+2)(n+3)}\ ,$$

$$\mathcal{E}^*[(z_i-l_{ni})^2(z_{i+1}-l_{ni+1})]=\frac{2l_{ni}(1-2l_{ni})(1-l_{ni+1})}{(n+2)(n+3)}\ ,$$

$$\mathcal{E}^*[(z_i-l_{ni})(z_{i+1}-l_{ni+1})^2]=\frac{2l_{ni}(1-2l_{ni+1})(1-l_{ni+1})}{(n+2)(n+3)}\ ,$$

(3.18)
$$\mathcal{E}^*[(z_i-l_{ni})^4]=\frac{6l_{ni}(1-l_{ni})}{(n+2)(n+3)(n+4)}\left\{(1-2l_{ni})^2-\frac{n+3}{n+2}l_{ni}(1-l_{ni})\right\}+3\left\{\frac{l_{ni}(1-l_{ni})}{n+2}\right\}^2,$$

$$\mathcal{E}^*[(z_i-l_{ni})^2(z_{i+1}-l_{ni+1})^2]$$
$$=\frac{l_{ni}(1-l_{ni})}{(n+2)(n+3)(n+4)}\left\{6(1-2l_{ni})(1-2l_{ni+1})+2\frac{n+1}{n+2}l_{ni+1}(1-l_{ni})-4\frac{2n+5}{n+2}l_{ni}(1-l_{ni+1})\right\}$$
$$+2\left\{\frac{l_{ni}(1-l_{ni+1})}{n+2}\right\}^2+\frac{l_{ni}l_{ni+1}(1-l_{ni})(1-l_{ni+1})}{(n+2)^2}\ .$$

Then, under the condition (3.9) we can give the following estimates, which are more accurate than the corresponding ones given in [2];

(3.19)
$$\sum_{i=1}^{k}\mathcal{E}^*[\varphi_n(z_{ni}^{**})(z_i-l_{ni})]=\Sigma^+-\Sigma^-\leqq 0\ ,$$

where

$$\Sigma^+=\sum_{i\in I_1}\mathcal{E}^*[\varphi_n(z_{ni}^{**})(z_i-l_{ni})]\leqq M\sum_{i\in I_1}\mathcal{E}^*[z_i-l_{ni}]=0$$

and

$$\Sigma^- = \sum_{i\in I_2} \mathcal{E}^*[\varphi_n(z_{ni}^{**})(z_i-l_{ni})] \geqq M \sum_{i\in I_2} \mathcal{E}^*[z_i-l_{ni}]=0 ,$$

with $I_1=\{i:\ (z_i-l_{ni})\geqq 0,\ i=1,\cdots,k\}$ and $I_2=\{i:\ (z_i-l_{ni})<0,\ i=1,\cdots,k\}$,

$$\begin{aligned}(3.20)\quad &|\mathcal{E}^*[w'_{n(k)}L^{-1}_{n(k)}(z_{(k)}-l_{n(k)})]| \\ &=(n+2)\Big\{\sum_{i=1}^{k}\frac{l_{ni+1}-l_{ni-1}}{(l_{ni+1}-l_{ni})(l_{ni}-l_{ni-1})}\mathcal{E}^*[\varphi_n(z_{ni}^*)\phi_n(z_{ni}^*;l_{ni})(z_i-l_{ni})^3] \\ &\quad -\sum_{i=1}^{k-1}\frac{1}{l_{ni+1}-l_{ni}}\mathcal{E}^*[\varphi_n(z_{ni}^*)\phi_n(z_{ni}^*;l_{ni})(z_{ni}-l_{ni})^2(z_{i+1}-l_{ni+1})] \\ &\quad -\sum_{i=1}^{k-1}\frac{1}{l_{ni+1}-l_{ni}}\mathcal{E}^*[\varphi_n(z_{ni+1}^*)\phi_n(z_{ni+1}^*;l_{ni+1})(z_i-l_{ni})(z_{i+1}-l_{ni+1})^2]\Big\} \\ &\leqq \frac{2M^2}{n+2}\sum_{i=1}^{k}|1-2l_{ni}| = \frac{2M^2}{(n+1)(n+2)}\sum_{i=1}^{k}|n+1-2n_i| ,\end{aligned}$$

and further

$$\begin{aligned}(3.21)\quad &\mathcal{E}^*[w'_{n(k)}L^{-1}_{n(k)}w_{n(k)}] \\ &\leqq 4M^4\Big(3+\frac{7}{n+4}\Big)\max_{1\leqq i\leqq k} l_{ni}(1-l_{ni})\sum_{i=1}^{k+1}\frac{1}{n_i-n_{i-1}} \\ &= \frac{4M^4}{(n+1)^2}\Big(3+\frac{7}{n+4}\Big)\max_{1\leqq i\leqq k} n_i(n+1-n_i)\sum_{i=1}^{k+1}\frac{1}{n_i-n_{i-1}} .\end{aligned}$$

Thus, the sum of the last three terms in the last expression of (3.16), say $I_1(X^n_{n(k)}:Y^n_{n(k)})$, can be estimated by (3.19)–(3.21) as

$$(3.22)\qquad I_1(X^n_{n(k)}:Y^n_{n(k)}) \leqq I_1(n;k) .$$

Consequently, by Lemma 2.1 together with (3.16), (3.17) and (3.22), we immediately obtain the target inequality (3.11), which completes the proof of the theorem.

From the above theorem we immediately obtain the following asymptotic result which is an improvement for Theorems 4.1, 5.2 and others in [2].

COROLLARY 3.1. *Under the same assumptions as those in Theorem* 3.1, *the condition*

$$(3.23)\qquad k(n)/\min_{1\leqq i\leqq k+1}(n_i-n_i)\to 0 , \qquad (n\to\infty) ,$$

implies that

$$(3.24)\qquad X^n_{n(k)} \sim Y^n_{n(k)}\ (\boldsymbol{B})_d , \qquad (n\to\infty) .$$

Remark 3.1. In [2], $k(n)^2/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})\to 0$, $(n\to\infty)$ is required for (3.24) to hold, whereas in the above theorem the requirement is weakened as (3.23).

Remark 3.2. In case of equal basic distributions, parallel results with respect to $X_{n(k)}$ can be obtained as the special cases of those in this section. In addition, we can state analogous theorems to Theorem 3.1, when a set of spacings is chosen first and then the corresponding sample quantiles (cf. [2]).

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