BALANCED ARRAYS OF STRENGTH $2l$ AND BALANCED FRACTIONAL 2^m FACTORIAL DESIGNS

SUMIYASU YAMAMOTO, TERUHIRO SHIRAKURA AND MASAHIDE KUWADA

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Summary

A connection between a balanced fractional 2^m factorial design of resolution V and a balanced array of strength 4 with index set $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$ has been established by Srivastava [3]. The purpose of this paper is to generalize his results by investigating the combinatorial property of a fraction T and the algebraic structure of the information matrix of the fractional design. Main results are: A necessary and sufficient condition for a fractional 2^m factorial design T of resolution $2l+1$ to be balanced is that T is a balanced array of strength $2l$ with index set $\{\mu_0, \mu_1, \mu_2, \cdots, \mu_{2l}\}$ provided the information matrix M is non-singular.

1. Introduction

The theory of fractional factorial designs has found increasing use in various fields of experimental research. It is well known that an orthogonal fractional factorial design is desirable since the estimates of various effects are uncorrelated. Orthogonal fractions, however, are generally uneconomic in that they involve more than the desirable number of assemblies or treatment combinations. Non-orthogonal or irregular fractions, especially balanced fractions, have been investigated by Bose and Srivastava [1], [2], Srivastava [3] and Srivastava and Chopra [4], [5]. Among others, Srivastava [3] has established a connection between a balanced fractional 2^m factorial design of resolution V and a balanced array of strength 4 with index set $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$.

Even in these recent works, investigations have been restricted to effects up to two factors only. The study of the effects of three or more factors will eventually become necessary or desirable.

One of the purposes of this paper is to establish the relation between the elements of the information matrix of an irregular fractional 2^m factorial design T and the combinatorial properties of the array.

Necessary and sufficient conditions for the array to be an orthogonal array of strength t and for the array to be a balanced array of strength t with index set $\{\mu_0, \mu_1, \cdots, \mu_t\}$, are given in Section 3 in terms of the structures of the information matrices.

Another purpose of this paper is to establish in Section 5 a connection between a balanced fractional 2^m factorial design of resolution $2l+1$ and a balanced array of strength $2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$. This is a generalization of the results given by Srivastava [3].

For this purpose, some properties of a triangular type multidimensional partially balanced association scheme which is defined among the various effects of factors up to l are considered in Section 4.

In a separate paper, we shall investigate further the structure of multidimensional partially balanced association schemes. The decomposition of the triangular type multidimensional partially balanced association algebra into its two-sided ideals has provided us a powerful tool in obtaining the characteristic roots of the information matrix of a balanced fractional 2^m factorial design of resolution $2l+1$. The formula obtained includes the ingenious results given by Srivastava and Chopra [5].

2. Preliminaries

Consider a factorial experiment with m factors $F_1, F_2, \cdots, F_m$, each at two levels. The treatment combinations or assemblies will be represented by the vector $(j_1, j_2, \cdots, j_m)$ where $j_t=0$ or 1 represents the level of the factor F_t for each t. The observations and their expectations of corresponding assemblies will be denoted by $y(j_1, j_2, \cdots, j_m)$ and $\eta(j_1, j_2, \cdots, j_m)$, respectively. The totality of all types of $N=2^m$ assemblies arranged in the binary order will be denoted by

$$\text{(2.1)} \qquad \underset{(N\times m)}{Z} = \begin{bmatrix} 0 & 0\cdots 0 & 0 \\ 0 & 0\cdots 0 & 1 \\ 0 & 0\cdots 1 & 0 \\ 0 & 0\cdots 1 & 1 \\ \vdots & \vdots \quad \vdots & \vdots \\ 1 & 1\cdots 1 & 0 \\ 1 & 1\cdots 1 & 1 \end{bmatrix},$$

and the corresponding observation and expectation vectors will be denoted by

$$(2.2)\qquad \boldsymbol{y}(Z)=\begin{bmatrix} y(0,\cdots,0,0) \\ y(0,\cdots,0,1) \\ y(0,\cdots,1,0) \\ \vdots \\ y(1,\cdots,1,1) \end{bmatrix} \quad\text{and}\quad \boldsymbol{\eta}(Z)=\begin{bmatrix} \eta(0,\cdots,0,0) \\ \eta(0,\cdots,0,1) \\ \eta(0,\cdots,1,0) \\ \vdots \\ \eta(1,\cdots,1,1) \end{bmatrix}.$$

The vector of parameters or various effects, $\boldsymbol{\theta}(Z)$, of the 2^m factorial experiment will be defined in a usual manner as

$$(2.3)\qquad \boldsymbol{\theta}(Z)=\begin{bmatrix} \theta(0,\cdots,0,0) \\ \theta(0,\cdots,0,1) \\ \theta(0,\cdots,1,0) \\ \vdots \\ \theta(1,\cdots,1,1) \end{bmatrix}=\frac{1}{N}D_{(m)}\boldsymbol{\eta}(Z)\ ,$$

where $D_{(m)}=D\otimes D\otimes\cdots\otimes D$ (m times Kronecker products of D), and $D=\begin{bmatrix} d_0(0) & d_1(0) \\ d_0(1) & d_1(1)\end{bmatrix}=\begin{bmatrix} 1 & 1 \\ -1 & 1\end{bmatrix}$. Note that the columns as well as the rows of $D_{(m)}$ (Hadamard matrix of order 2^m) are mutually orthogonal, since $DD'=D'D=\begin{bmatrix} 2 & 0 \\ 0 & 2\end{bmatrix}$.

In particular, the general mean, denoted alternatively by θ_ϕ in this paper, is represented by $\theta(0,0,\cdots,0)$, the main effect of the factor F_1, denoted alternatively by θ_1, is represented by $\theta(1,0,\cdots,0)$, and the two factor interaction of the factors F_1 and F_2, denoted alternatively by θ_{12}, is represented by $\theta(1,1,0,\cdots,0)$. In general, the k-factor interaction of the factors $F_{t_1}, F_{t_2},\cdots, F_{t_k}$, denoted alternatively by $\theta_{t_1t_2\cdots t_k}$, is represented by $\theta(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)$ with $\varepsilon_{t_1}=\varepsilon_{t_2}=\cdots=\varepsilon_{t_k}=1$ and the remaining ε_j are all equal to zero. In other words, according as the weight $w(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)$, or the number of nonzero elements of the vector $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)$, is either $0, 1,\cdots$, or l, $\theta(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_m)$ represents either the general mean, the main effect of the factor corresponding to the nonzero element of the vector, $\cdots$, or the l-factor interaction of the l factors corresponding respectively to the l nonzero elements of the vector.

Solving (2.3) with respect to $\boldsymbol{\eta}(Z)$, we have

$$(2.4)\qquad \boldsymbol{\eta}(Z)=D'_{(m)}\boldsymbol{\theta}(Z)\ ,$$

since $1/N\cdot D_{(m)}D'_{(m)}=1/N\cdot D'_{(m)}D_{(m)}=I_N$, the unit matrix of order N. The $(i_12^{m-1}+i_22^{m-2}+\cdots+i_m+1)$th row of Z can conventionally be called the $(i_1,i_2,\cdots,i_m)$th row in the binary way starting from $(0,0,\cdots,0)$. The element of $D_{(m)}$ can, therefore, be represented by $d_{i_1i_2\cdots i_m}^{j_1j_2\cdots j_m}$ and be called the $((i_1,i_2,\cdots,i_m),(j_1,j_2,\cdots,j_m))$ element. Then, from (2.4) and the definition of $D_{(m)}$, we have

$$(2.5)\qquad \eta(j_1, j_2, \cdots, j_m) = \sum_{i_1 i_2 \cdots i_m} d_{i_1 i_2 \cdots i_m}^{j_1 j_2 \cdots j_m} \theta(i_1, i_2, \cdots, i_m) = \sum_{i_1 i_2 \cdots i_m} d_{j_1}(i_1) d_{j_2}(i_2) \cdots d_{j_m}(i_m) \theta(i_1, i_2, \cdots, i_m) .$$

Since $d_j(0)=1$ for all j, (2.5) can be expressed by rearranging $(i_1, i_2, \cdots, i_m)$ in the order of their weights as follows:

$$(2.6)\qquad \eta(j_1, j_2, \cdots, j_m) = \sum_{k=0}^{m} \sum_{\{t_1, t_2, \cdots, t_k\} \in \mathfrak{M}_k} d_{j_{t_1}}(1) d_{j_{t_2}}(1) \cdots d_{j_{t_k}}(1) \theta_{t_1 t_2 \cdots t_k} = \theta_\phi + \sum_{\{t_1\} \in \mathfrak{M}_1} d_{j_{t_1}}(1) \theta_{t_1} + \sum_{\{t_1, t_2\} \in \mathfrak{M}_2} d_{j_{t_1}}(1) d_{j_{t_2}}(1) \theta_{t_1 t_2} + \cdots ,$$

where $\mathfrak{M}_k$ denotes the collection of all subsets of $\{1, 2, \cdots, m\}$ with cardinality k.

In the case where $(l+1)$-factor or more interactions can be assumed negligible, we can assume the following model for the expectation of the observation corresponding to an assembly $(j_1, j_2, \cdots, j_m)$, i.e.,

$$(2.7)\qquad \eta(j_1, j_2, \cdots, j_m) = \theta_\phi + \sum_{\{t_1\} \in \mathfrak{M}_1} d_{j_{t_1}}(1) \theta_{t_1} + \sum_{\{t_1, t_2\} \in \mathfrak{M}_2} d_{j_{t_1}}(1) d_{j_{t_2}}(1) \theta_{t_1 t_2} + \cdots + \sum_{\{t_1, \cdots, t_l\} \in \mathfrak{M}_l} d_{j_{t_1}}(1) \cdots d_{j_{t_l}}(1) \theta_{t_1 \cdots t_l} .$$

3. Balanced array of strength t and information matrix

Let T be a fraction of n assemblies. Then, T can be expressed as a (0, 1) matrix of size $n \times m$ whose αth row constitutes the αth assembly $(j_1^{(\alpha)}, j_2^{(\alpha)}, \cdots, j_m^{(\alpha)})$ for every $\alpha = 1, 2, \cdots, n$.

The observation vector $\boldsymbol{y}(T)$ can be expressed as follows:

$$(3.1)\qquad \boldsymbol{y}(T) = \begin{bmatrix} y(j_1^{(1)}, \cdots, j_m^{(1)}) \\ \vdots \\ y(j_1^{(\alpha)}, \cdots, j_m^{(\alpha)}) \\ \vdots \\ y(j_1^{(n)}, \cdots, j_m^{(n)}) \end{bmatrix} = E\boldsymbol{\theta} + \boldsymbol{e} ,$$

where E is the design matrix of T, $\boldsymbol{\theta}' = (\theta_\phi, \{\theta_{t_1}\}, \{\theta_{t_1 t_2}\}, \cdots, \{\theta_{t_1 \cdots t_k}\}, \cdots)$ is the vector of various effects and $\boldsymbol{e}$ is the error vector of order n whose components are assumed to be uncorrelated and each has zero mean and the same variance σ^2.

The normal equation for estimating $\boldsymbol{\theta}$ can then be written as

$$(3.2)\qquad M\hat{\boldsymbol{\theta}} = E'\boldsymbol{y}(T) ,$$

where $M = E'E$.

The column of E corresponding to $\theta_{t_1 t_2 \cdots t_k}$ will be called the $(t_1, t_2, \cdots, t_k)$th column since the columns can be arranged in any way. The element of $E'E$ whose row and column correspond to $\theta_{t_1 \cdots t_u}$ and $\theta_{t'_1 \cdots t'_v}$, respectively, will therefore be called the $((t_1, \cdots, t_u), (t'_1, \cdots, t'_v))$ element.

With respect to the elements of the information matrix of a fractional 2^m factorial design T, we have the following theorem.

THEOREM 3.1. *In a fractional factorial design T, the $((t_1, \cdots, t_u), (t'_1, \cdots, t'_v))$ element $\varepsilon(t_1, \cdots, t_u; t'_1, \cdots, t'_v)$ of the information matrix $M = E'E$ depends on the set of indices $\{t_1, \cdots, t_u\}$ and $\{t'_1, \cdots, t'_v\}$ only through the symmetric difference $\{t_1, \cdots, t_u\} \ominus \{t'_1, \cdots, t'_v\}$ of them. The element, therefore, does not depend on those factors belonging to the intersection of the sets of indices.*

PROOF. Since

$$
\begin{aligned}
\eta(j_1^{(\alpha)}, j_2^{(\alpha)}, \cdots, j_m^{(\alpha)})
&= \theta_\phi + \sum_{\{t_1\} \in \mathfrak{M}_1} d_{j_{t_1}^{(\alpha)}}(1)\theta_{t_1} + \sum_{\{t_1, t_2\} \in \mathfrak{M}_2} d_{j_{t_1}^{(\alpha)}}(1) d_{j_{t_2}^{(\alpha)}}(1)\theta_{t_1 t_2} + \cdots \\
&\quad + \sum_{\{t_1, t_2, \cdots, t_k\} \in \mathfrak{M}_k} d_{j_{t_1}^{(\alpha)}}(1) d_{j_{t_2}^{(\alpha)}}(1) \cdots d_{j_{t_k}^{(\alpha)}}(1)\theta_{t_1 t_2 \cdots t_k} + \cdots ,
\end{aligned}
$$

we have

$$
\varepsilon(t_1, \cdots, t_u; t'_1, \cdots, t'_v) = \sum_\alpha d_{j_{t_1}^{(\alpha)}}(1) \cdots d_{j_{t_u}^{(\alpha)}}(1) d_{j_{t'_1}^{(\alpha)}}(1) \cdots d_{j_{t'_v}^{(\alpha)}}(1) .
$$

Since $d_j(1) = 1$ or -1 according as $j = 1$ or 0 and $d_{j_{t_1}}(\varepsilon) d_{j_{t'_1}}(\varepsilon) = 1$ if $t_1 = t'_1$, we have the theorem.

We, therefore, will denote $\varepsilon(t_1, \cdots, t_u; t'_1, \cdots, t'_v)$ by $\gamma_{(i_1, \cdots, i_k)}$ when $\{t_1, \cdots, t_u\} \ominus \{t'_1, \cdots, t'_v\} = \{i_1, \cdots, i_k\}$.

As an example we shall illustrate the case where $m = 4$ and three-factor or more interactions are negligible. The parameters to be considered are θ_ϕ, $\theta_1, \cdots, \theta_4$, $\theta_{12}, \theta_{13}, \cdots, \theta_{34}$. The information matrix of any fraction T can be expressed as follows:

	θ_ϕ	θ_1	θ_2	θ_3	θ_4	θ_{12}	θ_{13}	θ_{14}	θ_{23}	θ_{24}	θ_{34}
θ_ϕ	$\gamma_{(\phi)}$	$\gamma_{(1)}$	$\gamma_{(2)}$	$\gamma_{(3)}$	$\gamma_{(4)}$	$\gamma_{(12)}$	$\gamma_{(13)}$	$\gamma_{(14)}$	$\gamma_{(23)}$	$\gamma_{(24)}$	$\gamma_{(34)}$
θ_1		$\gamma_{(\phi)}$	$\gamma_{(12)}$	$\gamma_{(13)}$	$\gamma_{(14)}$	$\gamma_{(2)}$	$\gamma_{(3)}$	$\gamma_{(4)}$	$\gamma_{(123)}$	$\gamma_{(124)}$	$\gamma_{(134)}$
θ_2			$\gamma_{(\phi)}$	$\gamma_{(23)}$	$\gamma_{(24)}$	$\gamma_{(1)}$	$\gamma_{(123)}$	$\gamma_{(124)}$	$\gamma_{(3)}$	$\gamma_{(4)}$	$\gamma_{(234)}$
θ_3				$\gamma_{(\phi)}$	$\gamma_{(34)}$	$\gamma_{(123)}$	$\gamma_{(1)}$	$\gamma_{(134)}$	$\gamma_{(2)}$	$\gamma_{(234)}$	$\gamma_{(4)}$
θ_4					$\gamma_{(\phi)}$	$\gamma_{(124)}$	$\gamma_{(134)}$	$\gamma_{(1)}$	$\gamma_{(234)}$	$\gamma_{(2)}$	$\gamma_{(3)}$
θ_{12}						$\gamma_{(\phi)}$	$\gamma_{(23)}$	$\gamma_{(24)}$	$\gamma_{(13)}$	$\gamma_{(14)}$	$\gamma_{(1234)}$
θ_{13}			Sym.				$\gamma_{(\phi)}$	$\gamma_{(34)}$	$\gamma_{(12)}$	$\gamma_{(1234)}$	$\gamma_{(14)}$
θ_{14}								$\gamma_{(\phi)}$	$\gamma_{(1234)}$	$\gamma_{(12)}$	$\gamma_{(13)}$
θ_{23}									$\gamma_{(\phi)}$	$\gamma_{(34)}$	$\gamma_{(24)}$
θ_{24}										$\gamma_{(\phi)}$	$\gamma_{(23)}$
θ_{34}											$\gamma_{(\phi)}$

Let $T_{i_1 i_2 \cdots i_t}$ be an $n\times t$ subarray composed of i_1th, i_2th, $\cdots$, i_tth columns of the array T and let $\lambda_{i_1\cdots i_t}^{\eta_1\cdots\eta_t}$ be the number of times the row $(\eta_1, \cdots, \eta_t)$ occurs among the subarray $T_{i_1\cdots i_t}$. Denote $\gamma_{(u_1\cdots u_k)}$ by $\gamma_{i_1\cdots i_t}^{\varepsilon_1\cdots\varepsilon_t}$ for any subset $\{u_1, \cdots, u_k\}$ of $\{i_1, \cdots, i_t\}$, where $\varepsilon_j=1$ or 0 according as $i_j \in \{u_1, \cdots, u_k\}$ or not.

Arranging these γ and λ in the binary order as

$$\text{(3.3)} \qquad \boldsymbol{\lambda}_{i_1\cdots i_t}=\begin{bmatrix} \lambda_{i_1\cdots i_t}^{0\cdots 0} \\ \lambda_{i_1\cdots i_t}^{0\cdots 1} \\ \vdots \\ \lambda_{i_1\cdots i_t}^{1\cdots 1} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\gamma}_{i_1\cdots i_t}=\begin{bmatrix} \gamma_{i_1\cdots i_t}^{0\cdots 0} \\ \gamma_{i_1\cdots i_t}^{0\cdots 1} \\ \vdots \\ \gamma_{i_1\cdots i_t}^{1\cdots 1} \end{bmatrix},$$

we have the following theorem.

THEOREM 3.2. *For any $n\times t$ subarray $T_{i_1\cdots i_t}$ of T we have*

$$\text{(3.4)} \qquad \boldsymbol{\gamma}_{i_1\cdots i_t}=D_{(t)}\boldsymbol{\lambda}_{i_1\cdots i_t}\ .$$

where $D_{(t)}$ is the t times Kronecker products of D.

PROOF. Since $d_j(0)=1$, we have

$$\gamma_{i_1\cdots i_t}^{\varepsilon_1\cdots\varepsilon_t}=\gamma_{(u_1\cdots u_k)}=\sum_{\alpha=1}^{n} d_{j_{u_1}^{(\alpha)}}(1)\cdots d_{j_{u_k}^{(\alpha)}}(1)$$
$$=\sum_{\alpha=1}^{n} d_{j_{i_1}^{(\alpha)}}(\varepsilon_1)d_{j_{i_2}^{(\alpha)}}(\varepsilon_2)\cdots d_{j_{i_t}^{(\alpha)}}(\varepsilon_t)\ .$$

From the definition of $\lambda_{i_1\cdots i_t}^{\eta_1\cdots\eta_t}$, we have

$$\gamma_{i_1\cdots i_t}^{\varepsilon_1\cdots\varepsilon_t}=\sum_{\eta_1\cdots\eta_t} d_{\eta_1}(\varepsilon_1)\cdots d_{\eta_t}(\varepsilon_t)\lambda_{i_1\cdots i_t}^{\eta_1\cdots\eta_t}\ .$$

Hence we have (3.4).

DEFINITION 3.1. An $n\times m$ (0, 1) matrix T is said to be an orthogonal array (O-array) of strength t, size n, m constraints, 2 levels and index λ_t, if every $n\times t$ subarray $T_{i_1 i_2\cdots i_t}$ of T is such that every (0, 1) vector occurs exactly λ_t times as a row of $T_{i_1 i_2\cdots i_t}$.

DEFINITION 3.2. An $n\times$m (0, 1) matrix T is said to be a balanced array (B-array) of strength t, size n, m constraints, 2 levels and index set $\{\mu_0, \mu_1, \cdots, \mu_t\}$, if every subarray $T_{i_1 i_2\cdots i_t}$ is such that every (0, 1) vector with weight i $(i=0, 1, \cdots, t)$ occurs exactly μ_i times as a row of $T_{i_1 i_2\cdots i_t}$.

The following theorem is said to be well known. The authors, however, have never met with a general proof of it.

THEOREM 3.3. *A necessary and sufficient condition that every off-*

diagonal element $\gamma_{(u_1u_2\cdots u_k)}$ of the information matrix M of a fractional 2^m factorial design T vanishes for any $\{u_1,\cdots,u_k\}$ satisfying $1\leqq k\leqq t$, is that T is an orthogonal array of strength t.

PROOF. (Necessity) From the assumption and Theorem 3.2, we have

$$D_{(t)}\boldsymbol{\lambda}_{i_1\cdots i_t}=\begin{bmatrix}\gamma_\phi\\ \mathbf{0}\end{bmatrix}, \tag{3.5}$$

for every subarray $T_{i_1i_2\cdots i_t}$. Since $D_{(t)}$ is an Hadamard matrix of order 2^t, its rows are mutually orthogonal and every row except the first one is a contrast. Thus we have $\lambda_{i_1\cdots i_t}^{\eta_1\cdots\eta_t}=\lambda_t$ for every $\{i_1,i_2,\cdots,i_t\}$ and for every (0, 1) vector $(\eta_1,\eta_2,\cdots,\eta_t)$.

(Sufficiency) If T is an O-array of strength t with index λ_t, then we have $\boldsymbol{\lambda}_{i_1\cdots i_t}=\lambda_t\boldsymbol{j}_{2^t}$ for every $\{i_1,\cdots,i_t\}$, where $\boldsymbol{j}_n$ is an $n\times 1$ column vector whose elements are all unity. In this case, Theorem 3.2 shows that

$$\boldsymbol{\gamma}_{i_1\cdots i_t}=\lambda_t D_{(t)}\boldsymbol{j}_{2^t}=\begin{bmatrix}2^t\lambda_t\\ \mathbf{0}\end{bmatrix} \tag{3.6}$$

for every subset $\{i_1,\cdots,i_t\}$. This means that $\gamma_{(u_1\cdots u_k)}=0$ for any subset $\{u_1,\cdots,u_k\}$ satisfying $1\leqq k\leqq t$.

We may note that if the design T is an O-array of strength $2l$, and $(l+1)$-factor or more interactions are negligible, then the $\nu_l\times\nu_l$ $\left(\nu_l=1+\binom{m}{1}+\cdots+\binom{m}{l}\right)$ information matrix M is diagonal and hence all effects involving l or less factors are uncorrelatedly estimable.

THEOREM 3.4. *A necessary and sufficient condition that, for all $k\leqq t$, every element $\gamma_{(u_1\cdots u_k)}$ of the information matrix M of a fractional 2^m factorial design T depends on the set $\{u_1,u_2,\cdots,u_k\}$ only through the cardinality k of the set, i.e., $\gamma_{(u_1\cdots u_k)}=\gamma_k$, is that T is a balanced array of strength t with index set $\{\mu_0,\mu_1,\cdots,\mu_t\}$.*

PROOF. If the design T is a B-array of strength t with index set $\{\mu_0,\mu_1,\cdots,\mu_t\}$, then we have

$$\boldsymbol{\lambda}_{i_1i_2\cdots i_t}=K_{(t)}\boldsymbol{\mu}, \tag{3.7}$$

for every $\{i_1,i_2,\cdots,i_t\}$, where $\boldsymbol{\mu}'=(\mu_0,\mu_1,\cdots,\mu_t)$ and $K_{(t)}$ is a $2^t\times(t+1)$ matrix whose $((\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_t),j)$ element $k_{(\varepsilon_1,\ldots,\varepsilon_t),j}$ is equal to 1 or 0 according as the weight of $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_t)$, indicating the binary order of row, is equal to j or not. Substituting (3.7) into (3.4) we have

$$\boldsymbol{\gamma}_{i_1i_2\cdots i_t}=D_{(t)}K_{(t)}\boldsymbol{\mu}$$

or

$$\gamma_{i_1 i_2 \cdots i_t}^{\epsilon_1 \epsilon_2 \cdots \epsilon_t} = \sum_{j=0}^{t} \sum_{\epsilon_1' \cdots \epsilon_t'} d_{\epsilon_1'}(\epsilon_1) \cdots d_{\epsilon_t'}(\epsilon_t) k_{(\epsilon_1' \cdots \epsilon_t'), j} \mu_j . \tag{3.8}$$

Since $k_{(\epsilon_1' \cdots \epsilon_t'), j} = 1$ or 0 according as $w(\epsilon_1', \epsilon_2', \cdots, \epsilon_t') = j$ or not and $d_{\epsilon'}(\epsilon) = -1$ or 1 according as $(\epsilon, \epsilon') = (1, 0)$ or not, (3.8) can be reduced to

$$\gamma_{i_1 i_2 \cdots i_t}^{\epsilon_1 \epsilon_2 \cdots \epsilon_t} = \sum_{j=0}^{t} \sum_{p=0}^{i} (-1)^p \binom{i}{p} \binom{t-i}{j-i+p} \mu_j , \tag{3.9}$$

for any $(\epsilon_1, \cdots, \epsilon_t)$ with $w(\epsilon_1, \cdots, \epsilon_t) = i$ $(i = 0, 1, \cdots, t)$. Since (3.9) holds for every $\{i_1, \cdots, i_t\}$, $\gamma_{(u_1 \cdots u_k)}$ depends on the set $\{u_1, \cdots, u_k\}$ only through its cardinality k, provided $k \leqq t$. We can, therefore, write $\gamma_{(u_1 \cdots u_k)}$ as γ_k for $k = 0, 1, \cdots, t$.

Conversely, if $\gamma_{(u_1 \cdots u_k)}$ depends on the set $\{u_1, \cdots, u_k\}$ only through k, then we have

$$\boldsymbol{\gamma}_{i_1 \cdots i_t} = K_{(t)} \boldsymbol{\gamma} , \tag{3.10}$$

for every $\{i_1, i_2, \cdots, i_t\}$, where $\boldsymbol{\gamma}' = (\gamma_0, \gamma_1, \cdots, \gamma_t)$. Since we have from (3.4)

$$\boldsymbol{\lambda}_{i_1 \cdots i_t} = \frac{1}{2^t} D_{(t)}' \boldsymbol{\gamma}_{i_1 \cdots i_t} , \tag{3.11}$$

similar arguments show that $\lambda_{i_1 \cdots i_t}^{\eta_1 \cdots \eta_t}$ depends only on the weight $w(\eta_1, \cdots, \eta_t)$ irrespective of the set $\{i_1, \cdots, i_t\}$, i.e.,

$$\lambda_{i_1 \cdots i_t}^{\eta_1 \cdots \eta_t} = \frac{1}{2^t} \sum_{j=0}^{t} \sum_{p=0}^{j} (-1)^p \binom{i}{j-p} \binom{t-i}{p} \gamma_j , \tag{3.12}$$

for all $(\eta_1, \cdots, \eta_t)$ with $w(\eta_1, \cdots, \eta_t) = i$ $(i = 0, 1, \cdots, t)$.

The relation (3.12) holds for every $\{i_1, \cdots, i_t\}$. Thus $\lambda_{i_1 \cdots i_t}^{\eta_1 \cdots \eta_t}$ can be denoted by μ_i for every $\{i_1, \cdots, i_t\}$ provided $w(\eta_1, \cdots, \eta_t) = i$. This means that T is a B-array of strength t with index set $\{\mu_0, \cdots, \mu_t\}$.

We may note that if T is a balanced array of strength $2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$, and if $(l+1)$-factor or more interactions are negligible, then the $\nu_l \times \nu_l$ information matrix M has at most $2l+1$ possibly different elements $\gamma_0, \gamma_1, \cdots, \gamma_{2l}$. This fact has been pointed out by Srivastava [3] for the case $l=2$. Formulas (3.9) and (3.12) show general relationship between γ_i and μ_j, i.e.,

$$\gamma_i = \sum_{j=0}^{2l} \sum_{p=0}^{i} (-1)^p \binom{i}{p} \binom{2l-i}{j-i+p} \mu_j , \tag{3.13}$$

$$\mu_i = \frac{1}{2^{2l}} \sum_{j=0}^{2l} \sum_{p=0}^{j} (-1)^p \binom{i}{j-p} \binom{2l-i}{p} \gamma_j , \qquad i = 0, 1, \cdots, 2l . \tag{3.14}$$

Some special cases will be indicated in the following.

Case I. For $2l=4$, from (3.13) and (3.14) we get

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ -1 & -2 & 0 & 2 & 1 \\ 1 & 0 & -2 & 0 & 1 \\ -1 & 2 & 0 & -2 & 1 \\ 1 & -4 & 6 & -4 & 1 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} \tag{3.15}$$

and

$$\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \frac{1}{16} \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & 0 & -2 & -1 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \end{bmatrix}. \tag{3.16}$$

The formula (3.15) is given in Srivastava [3].

Case II. For $2l=6$, we have

$$\begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix} = \begin{bmatrix} 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ -1 & -4 & -5 & 0 & 5 & 4 & 1 \\ 1 & 2 & -1 & -4 & -1 & 2 & 1 \\ -1 & 0 & 3 & 0 & -3 & 0 & 1 \\ 1 & -2 & -1 & 4 & -1 & -2 & 1 \\ -1 & 4 & -5 & 0 & 5 & -4 & 1 \\ 1 & -6 & 15 & -20 & 15 & -6 & 1 \end{bmatrix} \begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{bmatrix} \tag{3.17}$$

and

$$\begin{bmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{bmatrix} = \frac{1}{64} \begin{bmatrix} 1 & -6 & 15 & -20 & 15 & -6 & 1 \\ 1 & -4 & 5 & 0 & -5 & 4 & -1 \\ 1 & -2 & -1 & 4 & -1 & -2 & 1 \\ 1 & 0 & -3 & 0 & 3 & 0 & -1 \\ 1 & 2 & -1 & -4 & -1 & 2 & 1 \\ 1 & 4 & 5 & 0 & -5 & -4 & -1 \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 \end{bmatrix} \begin{bmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \\ \gamma_6 \end{bmatrix}. \tag{3.18}$$

4. Multidimensional partially balanced association schemes

Multidimensional partially balanced (MDPB) association schemes have been introduced by Bose and Srivastava [2] as a generalization of partially balanced association schemes.

Let $S_1, S_2, \cdots, S_m$ be m mutually disjoint non-null finite sets with $|S_i|=n_i$ each, where $|S|$ denotes the cardinality of a set S. Suppose a relation of association being defined for each ordered pair of objects (x_{ia}, x_{jb}) and $x_{jb} \in S_j$ being called the αth associate of $x_{ia} \in S_i$ for some α belonging to a set of association index $\pi^{(i,j)}$.

Conventionally, every object will be called the zeroth associate of itself and $0 \notin \pi^{(i,i)}$ will be assumed in order to make the definition of MDPB consistent with that of the partially balanced association scheme with $n^{(i,i)}$ associate classes defined within each set S_i.

DEFINITION 4.1. The relation of association defined in a collection of sets $(S_1, S_2, \cdots, S_m)$ will be called an m sets multidimensional partially balanced (MDPB) association scheme if the following conditions are satisfied:

(i) The relation of association is symmetrical, i.e., if x_{jb} is the αth associate of x_{ia}, then x_{ia} is the αth associate of x_{jb}.

(ii) With respect to any $x_{ia} \in S_i$, the objects of S_j distinct from x_{ia}, can be divided into $n^{(i,j)}$ disjoint classes and the number of objects in the αth associate class $S_j(\alpha; x_{ia})$ is $n_\alpha^{(i,j)}$, for $i, j=1, 2, \cdots, m$, the numbers $n^{(i,j)}$ and $n_\alpha^{(i,j)}$ being independent of the particular object x_{ia} chosen in S_i.

(iii) Let S_i, S_j and S_k be any three sets where they are not necessarily distinct. Let $x_{jb} \in S_j$ be the αth associate of $x_{ia} \in S_i$ and consider the sets $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$. Then the number of objects common to the sets $S_k(\beta; x_{ia})$ and $S_k(\gamma; x_{jb})$ is a number $p(i, j, \alpha; k, \beta, \gamma)$ dependent on the pair (x_{ia}, x_{jb}) and S_k only through i, j, α, k, β and γ.

Note that the condition (i) implies $\pi^{(i,j)}=\pi^{(j,i)}$ and $n^{(i,j)}=n^{(j,i)}$. The number $n_0^{(i,i)}=1$ can be consistently defined for all i.

Consider a 2^m factorial design in which $(l+1)$-factor or more interactions are negligible, then the various parameters to be considered are $\{\theta_\phi\}, \{\theta_{t_1}\}, \{\theta_{t_1t_2}\}, \cdots, \{\theta_{t_1t_2\cdots t_l}\}$. Although the integer l can assume any value between 1 to m, we confine ourselves to some l satisfying $l \leqq m/2$ in order to simplify the description throughout this paper. Slight modifications are necessary for those $l > m/2$.

Let S_0 be $\{\theta_\phi\}$, the set of general mean with $|S_0|=1$; S_1 be $\{\theta_{t_1}\}$, the set of main effects with $|S_1|=m$; S_2 be $\{\theta_{t_1t_2}\}$, the set of two-factor interactions with $|S_2|=\binom{m}{2}$; and, in general, S_i be $\{\theta_{t_1\cdots t_i}\}$, the set of i-factor interactions with $|S_i|=\binom{m}{i}$ for all $i=0, 1, \cdots, l$. Then, a natural relation of association can be introduced for each pair of parameters in those $l+1$ sets by defining that $\theta_{t_1\cdots t_u} \in S_u$ and $\theta_{t_1'\cdots t_v'} \in S_v$ are the αth associates if and only if

$$|\{t_1, \cdots, t_u\} \cap \{t'_1, \cdots, t'_v\}| = \min(u, v) - \alpha \tag{4.1}$$

holds. Note that (4.1) defines an association not only for any pair of different parameters but also for any parameter which is the zero associate of itself.

THEOREM 4.1. *The relation of association defined by* (4.1) *among the* $l+1$ *sets of parameters* $\{\theta_\phi\}, \{\theta_{t_1}\}, \{\theta_{t_1t_2}\}, \cdots, \{\theta_{t_1t_2\cdots t_l}\}$ *is an* $l+1$ *sets* MDPB *association scheme with the following parameters*:

$$\pi^{(u,v)} = \begin{cases} \{0, 1, \cdots, \min(u, v)\} & u \neq v \\ \{1, 2, \cdots, u\} & u = v\,, \end{cases} \tag{4.2}$$

$$n^{(u,v)} = \begin{cases} \min(u, v)+1 & u \neq v \\ u & u = v\,, \end{cases} \tag{4.3}$$

$$n_\alpha^{(u,v)} = \binom{u}{\min(u, v)-\alpha}\binom{m-u}{v-\min(u, v)+\alpha}\,, \tag{4.4}$$

$$\begin{aligned} &p(u, v, \alpha; w, \beta, \gamma) \\ &\quad = \sum_{k=0}^{\min(u,v)-\alpha} \binom{\min(u, v)-\alpha}{k}\binom{u-\min(u, v)+\alpha}{\min(u, w)-\beta-k} \\ &\qquad \cdot \binom{v-\min(u, v)+\alpha}{\min(v, w)-\gamma-k}\binom{m-u-v+\min(u, v)-\alpha}{w-\min(u, w)+\beta-\min(v, w)+\gamma+k}\,. \end{aligned} \tag{4.5}$$

PROOF. The condition (i) of MDPB association scheme is satisfied, since the relation of association defined by (4.1) is symmetrical. The definition (4.1) shows that, for any $\theta_{t_1t_2\cdots t_u} \in S_u$, the objects in S_v, other than $\theta_{t_1t_2\cdots t_u}$, are divided into $\min(u, v)+1$ or u classes $S_v(\alpha; \theta_{t_1\cdots t_u})$ according as $v \neq u$ or $v = u$, since α can take any value in the set $\pi^{(u,v)}$ which is given by (4.2). The number of classes $n^{(u,v)}$ is clearly given by (4.3). The cardinality $n_\alpha^{(u,v)}$ of the set $S_v(\alpha; \theta_{t_1t_2\cdots t_u})$ in (4.4) will be given by counting the number of sets $\{t'_1, t'_2, \cdots, t'_v\}$ of cardinality v satisfying $|\{t_1, t_2, \cdots, t_u\} \cap \{t'_1, t'_2, \cdots, t'_v\}| = \min(u, v) - \alpha$. Since those numbers are independent of the particular choice of the object $\theta_{t_1\cdots t_u}$ in S_u, the condition (ii) of the MDPB association scheme is satisfied.

The condition (iii) of the MDPB association scheme can be verified by counting the number of those $\theta_{t''_1t''_2\cdots t''_w}$ in S_w satisfying that each $\theta_{t''_1t''_2\cdots t''_w}$ is respectively the βth associate and the γth associate of $\theta_{t_1t_2\cdots t_u}$ and $\theta_{t'_1t'_2\cdots t'_v}$, which are the αth associates of each other. It is equivalent to count the number of sets $\{t''_1, t''_2, \cdots, t''_w\}$ such that each has respectively $\min(u, w)-\alpha$ intersection with $\{t_1, t_2, \cdots, t_u\}$ and $\min(v, w)-\gamma$ intersection with $\{t'_1, t'_2, \cdots, t'_v\}$ for any given pairs of sets $\{t_1, t_2, \cdots, t_u\}$ and $\{t'_1, t'_2, \cdots, t'_v\}$ having $\min(u, v)-\alpha$ intersection. The number, therefore, is given by (4.5) and it can be seen that the number $p(u, v, \alpha; w, \beta, \gamma)$ is dependent on $\theta_{t_1t_2\cdots t_u}$, $\theta_{t'_1t'_2\cdots t'_v}$ and S_w only through u, v, α, w, β and γ.

The scheme thus defined is said to be a triangular type $l+1$ sets MDPB (TMDPB) association scheme, since, as will be seen later, it can be regarded as a generalization of triangular series of association schemes (see Yamamoto, Fujii and Hamada [6]).

First, we define *local* association matrices $A_\alpha^{(u,v)}$, ($\alpha=0, 1, \cdots, \min(u, v)$; $u, v=0, 1, \cdots, l$) in order to investigate the algebraic structure of TMDPB association schemes. Each matrix $A_\alpha^{(u,v)}=\|a_{t_1t_2\cdots t_u;\alpha}^{t'_1t'_2\cdots t'_v}\|$ of size $\binom{m}{u}\times\binom{m}{v}$ represents the ordered relation of association from $\binom{m}{u}$ parameters $\theta_{t_1t_2\cdots t_u}$ of S_u to $\binom{m}{v}$ parameters $\theta_{t'_1t'_2\cdots t'_v}$ of S_v, i.e.,

$$a_{t_1t_2\cdots t_u;\alpha}^{t'_1t'_2\cdots t'_v}=\begin{cases} 1: & \text{if } \theta_{t'_1t'_2\cdots t'_v} \text{ is the } \alpha\text{th associate of } \theta_{t_1t_2\cdots t_u}, \\ 0: & \text{otherwise}. \end{cases} \tag{4.6}$$

Clearly, we have

$$A_\alpha^{(u,v)\prime}=A_\alpha^{(v,u)}, \qquad A_\alpha^{(u,v)}\boldsymbol{j}_{\binom{m}{v}}=n_\alpha^{(u,v)}\boldsymbol{j}_{\binom{m}{u}}$$
$$\text{and} \qquad \sum_{\alpha=0}^{\min(u,v)} A_\alpha^{(u,v)}=G_{\binom{m}{u}\times\binom{m}{v}}, \tag{4.7}$$

where $G_{r\times s}$ and $\boldsymbol{j}_r$ are $r\times s$ matrix and $r\times 1$ column vector of all unities, respectively. It can also be seen that $A_\alpha^{(u,u)}$ ($\alpha=0, 1, \cdots, u$) are the association matrices of a T_u type (the uth order triangular) association scheme defined among $\binom{m}{u}$ objects (see, for example, Yamamoto, Fujii and Hamada [6]).

Next we define *ordered* association matrices $D_\alpha^{(u,v)}$ of size $\nu_l\times\nu_l$ each such that each matrix contains $(l+1)^2$ submatrices. Let $M^{(w,s)}$ be the $\binom{m}{w}\times\binom{m}{s}$ submatrix in the wth row block and sth column block corresponding respectively to S_w and S_s. Then $D_\alpha^{(u,v)}$ is such that $M^{(w,s)}=A_\alpha^{(u,v)}$ or 0 according as $(w, s)=(u, v)$ or not. Thus $D_\alpha^{(u,v)}$ is a $\nu_l\times\nu_l$ matrix in which all $(\{t_1, \cdots, t_u\}, \{t'_1, \cdots, t'_v\})$ elements corresponding to ordered pair $\theta_{t_1\cdots t_u}$ and $\theta_{t'_1\cdots t'_v}$ are unity if and only if $\theta_{t'_1\cdots t'_v}$ is the αth associate of $\theta_{t_1\cdots t_u}$, and the remaining elements are all zero.

The association matrices $B_\alpha^{(u,v)}$ which represent the relation of association of TMDPB association scheme can, therefore, be defined as follows:

$$B_\alpha^{(u,v)}=\begin{cases} D_\alpha^{(u,v)}+D_\alpha^{(v,u)} & (u\neq v) \\ D_\alpha^{(u,u)} & (u=v). \end{cases} \tag{4.8}$$

Now we have

THEOREM 4.2. *The relationship algebra* $\mathfrak{A}=\{B_\alpha^{(u,v)}: \alpha=0, 1, \cdots, u; 0\leqq u\leqq v\leqq l\}$ *generated by* $\binom{l+3}{3}$ *symmetric matrices* $B_\alpha^{(u,v)}$ *is a semi-simple, completely reducible matrix algebra containing the unit matrix* I_{ν_l}. *It can be represented by the linear closure* $[D_\alpha^{(u,v)}: \alpha=0, 1, \cdots, \min(u, v); u, v=0, 1, \cdots, l]$ *of all* $(l+1)(l+2)(2l+3)/6$ *ordered association matrices* $D_\alpha^{(u,v)}$.

PROOF. Since all generators of the matrix algebra $\mathfrak{A}$ are symmetric, the algebra $\mathfrak{A}$ is semi-simple and completely reducible. $\sum_{u=0}^{l} B_0^{(u,u)}=I_{\nu_l}$ shows $I_{\nu_l}\in\mathfrak{A}$. From definitions of $D_\alpha^{(u,v)}$ and $B_\alpha^{(u,v)}$ we have

$$D_\alpha^{(u,v)}=B_\alpha^{(u,v)}B_0^{(v,v)}, \qquad D_\alpha^{(v,u)}=B_0^{(v,v)}B_\alpha^{(u,v)}, \tag{4.9}$$

$$B_\alpha^{(u,v)}B_\beta^{(s,w)}=D_\alpha^{(u,v)}D_\beta^{(s,w)}+D_\alpha^{(u,v)}D_\beta^{(w,s)}+D_\alpha^{(v,u)}D_\beta^{(s,w)}+D_\alpha^{(v,u)}D_\beta^{(w,s)} \tag{4.10}$$

and from (4.5) we have

$$D_\beta^{(u,w)}D_\gamma^{(s,v)}=\begin{cases} 0 & \text{if } s\neq w, \\ \sum_\alpha p(u, v, \alpha; w, \beta, \gamma)D_\alpha^{(u,v)} & \text{if } s=w. \end{cases} \tag{4.11}$$

The relations (4.8), (4.9), (4.10) and (4.11) show that $\mathfrak{A}=[D_\alpha^{(u,v)}: \alpha=0, 1, \cdots, \min(u, v); u, v=0, 1, \cdots, l]$.

We may note that contrary to the ordinary association algebra (single set MDPB association algebra), the MDPB association algebra is not in general commutative as has been indicated by Bose and Srivastava [2]. The decomposition of TMDPB association algebra into its two-sided ideals has provided us an important rôle in obtaining the characteristic roots of the information matrix of a balanced fractional 2^m factorial design of resolution $2l+1$. Details will be seen in a succeeding paper.

5. Balanced fractional 2^m factorial designs

It has been established by Srivastava [3] that a necessary and sufficient condition of being a fractional 2^m factorial design T of resolution V balanced is that T is a balanced array of strength 4 provided the information matrix M of T is non-singular. The relation (3.15) between index set $\{\mu_0, \mu_1, \mu_2, \mu_3, \mu_4\}$ of the array and the possible five values γ_0, γ_1, γ_2, γ_3 and γ_4 of the information matrix M of a balanced design has also been given there. General formula for the latter is given in Theorem 3.4 of this paper. The following theorem is a generalization of the above mentioned results due to Srivastava [3].

DEFINITION 5.1. Consider the case where the parameters or various effects of a 2^m factorial design to be estimated are θ_ϕ, $\{\theta_{t_1}\}, \{\theta_{t_1t_2}\}, \cdots, \{\theta_{t_1t_2\cdots t_l}\}$, the remaining effects being assumed negligible. If a fractional 2^m factorial design T has a non-singular information matrix M $(=E'E)$ of size $\nu_l \times \nu_l$, then T is called a design of resolution $2l+1$.

DEFINITION 5.2. In a fractional 2^m factorial design of resolution $2l+1$, if all elements of the covariance matrix $M^{-1}\sigma^2$ is invariant under the group of permutation $\left\{\tau;\ \tau=\begin{pmatrix}1 & 2 & \cdots m \\ \tau(1) & \tau(2) & \cdots \tau(m)\end{pmatrix}\right\}$ of factors, then the design is called a balanced fractional 2^m factorial design.

THEOREM 5.1. *A necessary and sufficient condition for a fractional 2^m factorial design T of resolution $2l+1$ to be balanced is that T is a balanced array of strength $2l$ with index set $\{\mu_0, \mu_1, \cdots, \mu_{2l}\}$ provided the information matrix M of the design T is non-singular.*

PROOF. (Necessity) The condition that the design is to be balanced may be stated as

$$\text{(5.1)} \qquad \operatorname{Cov}(\hat{\theta}_{t_1t_2\cdots t_u}, \hat{\theta}_{t'_1t'_2\cdots t'_v}) = \operatorname{Cov}(\hat{\theta}_{\tau(t_1t_2\cdots t_u)}, \hat{\theta}_{\tau(t'_1t'_2\cdots t'_v)}) .$$

This means that each element of M^{-1} is a function of two subsets $\{t_1, t_2, \cdots, t_u\}$ and $\{t'_1, t'_2, \cdots, t'_v\}$ and is invariant under the permutation group $\left\{\tau;\ \tau=\begin{pmatrix}1 & 2 & \cdots m \\ \tau(1) & \tau(2) & \cdots \tau(m)\end{pmatrix}\right\}$. Since a maximal invariant of the function of two sets $\{t_1, t_2, \cdots, t_u\}$ and $\{t'_1, t'_2, \cdots, t'_v\}$ is $(u, v, |\{t_1, t_2, \cdots, t_u\} \ominus \{t'_1, t'_2, \cdots, t'_v\}|)$. This means that $\operatorname{Cov}(\hat{\theta}_{t_1t_2\cdots t_u}, \hat{\theta}_{t'_1t'_2\cdots t'_v})$ is a function of u, v and $|\{t_1, t_2, \cdots, t_u\} \ominus \{t'_1, t'_2, \cdots, t'_v\}|$. Hence M^{-1} can be written as a linear combination of $B_\alpha^{(u,v)}$, i.e.,

$$\text{(5.2)} \qquad M^{-1} = \sum_{\alpha, u, v} b_\alpha^{(u,v)} B_\alpha^{(u,v)} \in \mathfrak{A} .$$

As has been shown by Bose and Srivastava [2], (5.2) implies $M \in \mathfrak{A}$. Each element $\varepsilon(t_1, \cdots, t_u;\ t'_1, \cdots, t'_v)$ of M is, therefore, a function of u, v and $|\{t_1, \cdots, t_u\} \ominus \{t'_1, \cdots, t'_v\}|$. On the other hand, since M is the information matrix of a design T, Theorem 3.1 shows that $\varepsilon(t_1, \cdots, t_u;\ t'_1, \cdots, t'_v)$ is a function of $\{t_1, \cdots, t_u\} \ominus \{t'_1, \cdots, t'_v\}$ only. Thus Theorem 3.4 asserts that T is a balanced array of strength $2l$.

(Sufficiency) If T is a balanced array of strength $2l$, Theorem 3.4 asserts that M is a symmetric matrix belonging to the TMDPB association algebra $\mathfrak{A}$. Since M is assumed to be non-singular, M^{-1} can be written as (5.2). The design is, therefore, balanced.

COROLLARY 5.1. *A necessary and sufficient condition for a fractional 2^m factorial design T of resolution $2l+1$ to be orthogonal (M^{-1} is* *diagonal) is that T is an orthogonal array of strength $2l$ with index λ_{2l}.*

HIROSHIMA UNIVERSITY
MARITIME SAFETY ACADEMY

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