A MULTIVARIATE MODEL WITH INTRA-CLASS COVARIANCE STRUCTURE*

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Summary

Fraser [3], [4] and Fraser and Haq [5] discussed a comprehensive multivariate model: a model with an error variable internal to the system with a known multivariate distribution and a positive affine transformation which generates a response vector from an error vector. Here a multivariate model, with the error variable having a multivariate normal distribution with intra-class covariance structure, has been considered. The analysis of the responses has been carried on in the framework of a transformed structural model and it produces structural distribution for the location parameters and the scale parameter, and the marginal likelihood function for the intra-class correlation coefficient.

1. Introduction

Consider the set of responses

$$X_\alpha=\mu_\alpha+\sigma U_\alpha \qquad (\alpha=1, 2, \cdots, p)\,, \tag{1.1}$$

where the responses are assumed to have been obtained from the error variable U_α by different location change, but the same scale change. The U_α's are assumed to have a p-variate normal distribution with $\mathrm{E}(U_\alpha)=0$ and covariance matrix of U_α as

$$\mathrm{Cov}(U_\alpha, U_\alpha)=\Omega=\begin{bmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \cdot & \cdot & \cdots & \cdot \\ \rho & \rho & \cdots & 1 \end{bmatrix}, \qquad |\rho|<1\,, \tag{1.2}$$

which has an intra-class covariance structure. For a sample of size n the responses constitute a sample from $\mathcal{N}(\boldsymbol{\mu}, \sigma^2\Omega)$. Inferences concerning the parameters of such a statistical model, mostly, have been based on

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standard statistical procedures. The works of De Lury [1], Olkin and Prat [9], Geisser [6], Srivastava [10], Mehta and Gurland [8] may be mentioned in this respect.

Fraser [3], [4] and Fraser and Haq [5] discussed a comprehensive multivariate model and made structural probability statements about the parameters of the model. But when the covariance matrix of the model is of intra-class covariance structure the results are not applicable. However, by a suitable transformation the error variables may be transformed into uncorrelated variables, and the responses can be analysed in the framework of a transformed structural model. The method has been discussed by Fraser [2], [4]. Haq [7] followed the method to make inference about the parameters of a first order autoregressive stochastic model.

2. The transformed structural model

Consider n responses from the multivariate model described in Section 1, which may be written as:

$$X_{ij}=\mu_i+\sigma u_{ij}\,, \qquad i=1,2,\cdots,p,\ j=1,2,\cdots,n \tag{2.1}$$
$$\text{or} \qquad X=\boldsymbol{\mu}\mathbf{1}'+\sigma U\,;$$

where X and U are $p\times n$ matrices, $\boldsymbol{\mu}'=(\mu_1,\mu_2,\cdots,\mu_p)$, and $\mathbf{1}$ is the n-dimensional unit vector.

Let Γ be an orthogonal transformation with initial row$=(p^{-1/2}, p^{-1/2}, \cdots, p^{-1/2})$ such that $\Gamma\Omega\Gamma'=D$, where D is a diagonal matrix with first diagonal element $\eta=1+(p-1)\rho$ and the remaining diagonal elements $\xi=(1-\rho)$. Applying the transformation Γ to (2.1) one gets

$$\Gamma X=\Gamma\boldsymbol{\mu}\mathbf{1}'+\sigma\Gamma U \qquad \text{or} \qquad Y=\boldsymbol{m}\mathbf{1}'+\sigma E\,. \tag{2.2}$$

Under the transformation, the probability distribution of E reduces to

$$P(E)dE=(2\pi)^{-np/2}(\eta)^{-n/2}(\xi)^{-n(p-1)/2} \times\exp\left[-\frac{1}{2\eta}\sum_{\alpha=1}^{n}e_{1\alpha}^2-\frac{1}{2\xi}\sum_{j=2}^{p}\sum_{\alpha=1}^{n}e_{j\alpha}^2\right]\prod_{j=1}^{p}\prod_{\alpha=1}^{n}de_{j\alpha}\,. \tag{2.3}$$

Clearly expectation $(E)=0$ and covariance matrix of $E=I_n\otimes\Gamma\Omega\Gamma'$ where I_n is the identity matrix with n elements and $\otimes$ denotes the direct product of two matrices.

Let

$$\theta=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_1 & \sigma & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ m_p & 0 & \cdots & \sigma \end{bmatrix} \tag{2.4}$$

be a transformation belonging to a group of transformations

$$G=\left\{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & c & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ a_p & 0 & \cdots & c \end{bmatrix} \middle| \begin{matrix} -\infty<a_i<\infty \\ c>0 \end{matrix} \right\} \tag{2.5}$$

and suppose that the transformed response Y is obtained from the transformed error variable E by the transformation θ to E in the following manner:

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ Y_{11} & Y_{12} & \cdots & Y_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ Y_{p1} & Y_{p2} & \cdots & Y_{pn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_1 & \sigma & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ m_p & 0 & \cdots & \sigma \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e_{11} & e_{12} & \cdots & e_{1n} \\ \cdot & \cdot & \cdots & \cdot \\ e_{p1} & e_{p2} & \cdots & e_{pn} \end{bmatrix}$$

$$\text{or} \quad \begin{bmatrix} 1' \\ Y \end{bmatrix} = \theta \begin{bmatrix} 1' \\ E \end{bmatrix} \quad \text{or} \quad \bar{Y}=\theta\bar{E} . \tag{2.6}$$

For a sample of size $n \geqq p+1$ the group of transformations G on the sample space $\mathcal{Y}$ of the transformed response Y is unitary (G is unitary on Y if for g_1 and $g_2 \in G$ and $y \in \mathcal{Y}$, $g_1 y = g_2 y$ implies $g_1 = g_2$). The transformed model has two parts: (i) the structural equation (2.6) and (ii) the probability element of the error variable (2.3). Thus for a known value of ρ the model is a structural model. For a detailed discussion of structural models see Fraser [4].

An element g of G puts the transformed response into its orbit $G\bar{Y} = \{gy \mid g \in G\}$. Let $D(\bar{Y})$ be a reference point on the orbit. The point $\bar{Y}$ on the orbit can be obtained from the reference point $D(\bar{Y})$ by a suitable transformation usually known as a transformation variable. Let $T(\bar{Y})$ be such a transformation variable. Then

$$T(\bar{Y})D(\bar{Y}) = \bar{Y} . \tag{2.7}$$

Let $\bar{Y}^*$ be another point on the orbit. Then there exists a g in G such that $g\bar{Y} = \bar{Y}^*$. Also $\bar{Y}^*$ can be reached from $D(\bar{Y})$ by the transformation $T(\bar{Y}^*) = T(g\bar{Y})$. Thus we have

$$g\bar{Y} = gT(\bar{Y})D(\bar{Y}) = T(g\bar{Y})D(\bar{Y}) = \bar{Y}^* . \tag{2.8}$$

Thus a transformation variable is a mapping from $\mathcal{Y}$ to G such that $T(g\bar{Y}) = gT(\bar{Y})$ for all g in G and all $\bar{Y}$ in $\mathcal{Y}$. Any such transformation variable will determine an unique reference point on the orbit:

$$\begin{aligned} D(\bar{Y}) &= T^{-1}(\bar{Y})\bar{Y} \\ &= T^{-1}(\bar{Y})g^{-1}g\bar{Y} \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
&= (gT(\bar{Y}))^{-1} g\bar{Y} \\
&= T^{-1}(g\bar{Y}) g\bar{Y} \\
&= T^{-1}(\bar{Y}^*)\bar{Y}^* .
\end{aligned}
$$

Consider the following transformation variable:

$$
T(\bar{Y}) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \bar{y}_1 & s_y & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \bar{y}_p & 0 & \cdots & s_y \end{bmatrix} \tag{2.10}
$$

where

$$
\bar{y}_j = \sum_{\alpha=1}^{n} y_{j\alpha}/n , \qquad j = 1, 2, \cdots, p ,
$$

and

$$
s_y^2 = \sum_{j=1}^{p} \sum_{\alpha=1}^{n} (y_{j\alpha} - \bar{y}_j)^2 .
$$

The transformation variable (2.10) induces the unique reference point on the orbit:

$$
D(Y) = T^{-1}(\bar{Y})\bar{Y} = \begin{bmatrix} 1' \\ \dfrac{y_{ij} - \bar{y}_i}{s_y} \end{bmatrix} = \begin{bmatrix} 1' \\ \dfrac{e_{ij} - \bar{e}_i}{s_e} \end{bmatrix} \tag{2.11}
$$

$$
i = 1, 2, \cdots, p ; \; j = 1, 2, \cdots, n .
$$

Let dm be the invariant differential on $\mathcal{Y}$ and $d\mu$ and $d\nu$ be the left and right invariant differential on G respectively and Δ be the corresponding modular function: $d\mu(g) = \Delta(g) d\nu(g)$. Then using the Jacobian results of the effect of transformations G on $\mathcal{Y}$ and G one obtains

$$
\begin{aligned}
&dm(\bar{E}) = dE/s_e^{np} , \\
&d\mu(g) = dg/c^{p+1} , \\
&d\nu(g) = dg/c , \\
&\Delta(g) = d\mu(g)/d\nu(g) = c^{-p} .
\end{aligned} \tag{2.12}
$$

Using the results of these invariant differentials and following Fraser ([4], Chapter 2), the conditional distribution of $T(\bar{E})$ (for known ρ) on the orbit is obtained as

$$
\begin{aligned}
&\psi_\rho(D(\bar{E}))(2\pi)^{-np/2} \eta^{-n/2} \xi^{-n(p-1)/2} s_e^{np-p-1} \\
&\quad \times \exp\left[-\frac{1}{2} \left\{ \frac{n}{n} \bar{e}_1^2 + \frac{n}{\xi} \sum_{j=2}^{p} \bar{e}_j^2 + s_e^2 R_\rho(d) \right\} \right] dT(E) ;
\end{aligned} \tag{2.13}
$$

where, $R_\rho(d)=\frac{1}{\eta}\sum_{\alpha=1}^{n} d_{1\alpha}^2+\frac{1}{\xi}\sum_{j=2}^{p}\sum_{\alpha=1}^{n} d_{j\alpha}^2$;

$dT(E)=d\bar{e}_1, d\bar{e}_2, \cdots, d\bar{e}_p, ds_e$;

and $\psi_\rho(D)$ is the normalising constant.

It is observed that $\bar{e}_1, \bar{e}_2, \cdots, \bar{e}_p$ and s_e are independent and also independent of $D(\bar{E})$, for a known value of ρ. Integrating over the variables of $T(E)$ the normalising content $\psi_\rho(D(\bar{E}))$ is obtained as

$$\psi_\rho(D)=\eta^{p/2}A_{(n-1)p}\eta^{(n-1)/2}\xi^{(n-1)(p-1)/2}[R_\rho(d)]^{p(n-1)/2} \tag{2.14}$$

where $A_f=2\pi^{f/2}/\Gamma(f/2)$ is the area of the unit sphere in R^f.

Further, we have the structural equation

$$T(\bar{Y})=T(\theta\bar{E})=\theta T(\bar{E})\,. \tag{2.15}$$

This structural equation along with the probability element (2.13) describes a structural model. Since it gives a reduction from R^{pn} to R^{p+1} it is known as reduced structural model. The derivation of these results is completely based on the information contained in the transformed structural model.

3. Structural inference for the parameters

The reduced structural model obtained in Section 2 yields through the structural equation $T(\bar{E})=\theta^{-1}T(\bar{Y})$ the structural distribution θ for given value of Y. On using the results of the right invariant differential on G and the modular function Δ obtained at (2.12) and following Fraser ([4], Chapter 2), the structural distribution of θ is obtained as

$$\begin{aligned}&\frac{n^{p/2}}{(2\pi)^{np/2}\sigma^{np+1}}\frac{A_{(n-1)p}s_y^{p(n-1)}}{\eta^{1/2}\xi^{(p-1)/2}}[R_\rho(d)]^{p(n-1)/2}\\ &\quad\times\exp\left[-\frac{1}{2\sigma^2}\left\{\frac{n}{\eta}(\bar{y}_1-m_1)^2+\frac{n}{\xi}\sum_{j=2}^{n}(\bar{y}_j-m_j)^2+s_y^2R_\rho(d)\right\}\right]d\theta\end{aligned} \tag{3.1}$$

where $d\theta=dm_1\,dm_2\cdots dm_p\,d\sigma$. This structural distribution is derived on the assumption that ρ is known.

3.1. *Inference about* μ

On integrating (3.1) with respect to σ the marginal distribution of $m_1, m_2, \cdots, m_p$ is obtained as

$$\frac{n^{p/2}A_{(n-1)p}}{A_{np}(s_y^2R_\rho(d))^{p/2}}(\eta)^{-1/2}(\xi)^{-(p-1)/2} \tag{3.2}$$

$$\times\left[1+\frac{n}{\eta s_y^2 R_\rho(d)}(\bar{y}_1-m_1)^2+\frac{n}{\xi s_y^2 R_\rho(d)}\sum_{j=2}^{p}(\bar{y}_j-m_j)^2\right]^{-np/2} d\boldsymbol{m}\ .$$

The inverse of the orthogonal transformations Γ described in Section 2 can be used to obtain the marginal structural distribution of $\mu_1,\cdots,\mu_p$, and is obtained as

$$\text{(3.3)}\qquad \frac{n^{p/2}A_{(n-1)p}}{A_{np}\phi_\rho^p(x)}(\eta)^{-1/2}(\xi)^{-(p-1)/2}$$
$$\times\left[1+\frac{n}{\eta\phi_\rho^2(x)}(\bar{x}_1-\mu_1)^2+\frac{n}{\xi\phi_\rho^2(x)}\sum_{j=2}^{p}(\bar{x}_j-\mu_j)^2\right]^{-np/2} d\boldsymbol{\mu}\ ;$$

where $\phi_\rho^2(x)=\frac{p}{\eta}\sum_{\alpha=1}^{n}(\bar{x}_{\cdot\alpha}-\bar{x})^2+\frac{1}{\xi}\sum_{j=1}^{p}\sum_{\alpha=1}^{n}(x_{j\alpha}-\bar{x}_{j\cdot}-\bar{x}_{\cdot\alpha}+\bar{x})^2$,

$$\bar{x}_{\cdot\alpha}=\sum_{j=1}^{p}x_{j\alpha}/p\ ,\qquad \alpha=1,2,\cdots,n\ ;$$

$$\bar{x}_{j\cdot}=\sum_{\alpha=1}^{n}x_{j\alpha}/n\ ,\qquad j=1,2,\cdots,p\ ;$$

$$\bar{x}=\sum_{j=1}^{p}\sum_{\alpha=1}^{n}x_{j\alpha}/pn\ .$$

The derivation of (3.3) uses the following relations:

$$\sum_{\alpha=1}^{n}(y_{1\alpha}-\bar{y}_1)^2=p\sum_{\alpha=1}^{n}(\bar{x}_{\cdot\alpha}-\bar{x})^2\ ,$$

$$\sum_{j=2}^{p}\sum_{\alpha=1}^{n}(y_{j\alpha}-\bar{y}_j)^2$$
$$=\sum_{j=1}^{p}\sum_{\alpha=1}^{n}(x_{j\alpha}-\bar{x})^2-p\sum_{\alpha=1}^{n}(\bar{x}_{\cdot\alpha}-\bar{x})^2-n\sum_{j=1}^{p}(\bar{x}_{j\cdot}-\bar{x})^2$$
$$=\sum_{j=1}^{p}\sum_{\alpha=1}^{n}(x_{j\alpha}-\bar{x}_{j\cdot}-\bar{x}_{\cdot\alpha}+\bar{x})^2\ .$$

For $\rho=0$, the structural distribution of $\mu_1,\mu_2,\cdots,\mu_p$ reduces to

$$\text{(3.4)}\qquad \frac{n^{p/2}}{s_{x(*)}^p}\frac{A_{(n-1)p}}{A_{np}}\left\{1+\frac{n}{s_{x(*)}^2}\sum_{j=1}^{p}(\bar{x}_{j\cdot}-\mu_j)^2\right\}^{-np/2} d\boldsymbol{\mu}$$

where $s_{x(*)}^2=\sum_{j=1}^{p}\sum_{\alpha=1}^{n}(x_{j\alpha}-\bar{x})^2-n\sum_{j=1}^{p}(\bar{x}_{j\cdot}-\bar{x})^2$.

3.2. *Inference about σ*

On integrating over $\boldsymbol{m}$ the expression (3.1) the marginal structural distribution for σ for a known value of ρ is obtained as

$$\text{(3.5)}\qquad \frac{A_{(n-1)p}s_y^{p(n-1)}(R_\rho(d))^{p(n-1)/2}}{(2\pi)^{p(n-1)/2}\sigma^{p(n-1)+1}}\exp\left\{-\frac{1}{2\sigma^2}s_y^2R_\rho(d)\right\}d\sigma\ ;$$

which when expressed in terms of the observed response X reduces to

$$\text{(3.6)}\qquad \frac{A_{(n-1)p}(\phi_\rho^2(x))^{p(n-1)/2}}{(2\pi)^{p(n-1)/2}\sigma^{p(n-1)+1}}\exp\left[-\frac{1}{2\sigma^2}\phi_\rho^2(x)\right]d\sigma\ .$$

For $\rho=0$, the structural distribution of σ is obtained as

$$\frac{A_{(n-1)p}s_{x(*)}^{p(n-1)}}{(2\pi)^{p(n-1)/2}\sigma^{p(n-1)+1}}\exp\left[-\frac{1}{2\sigma^2}s_{x(*)}^2\right]d\sigma\ .$$

3.3. *Inference about* ρ

In Section 2, the conditional distribution of $T(E)$ on the orbit indexed by $D(\bar{E})$ has been obtained by using the property of invariant differential and not by integrating (2.3) with respect to the orbital variables $D(\bar{E})$. So division of (2.3) by the conditional probability element (2.13) adjusted by the factor $n^{p/2}$ necessary to measure the Euclidean volume in R^{pn} must yield the probability distribution of $D(\bar{E})$. Thus the probability distribution of the orbital variables $D(\bar{E})$ is obtained as

$$\text{(3.7)}\qquad A_{(n-1)p}^{-1}\eta^{-(n-1)/2}\xi^{-(n-1)(p-1)/2}s_e^{-(np-p-1)}\frac{[R_\rho(d)]^{-p(n-1)/2}dE}{d(\sqrt{n}\,\bar{e}_1)\cdots d(\sqrt{n}\,\bar{e}_p)ds_e}$$
$$=A_{(n-1)p}^{-1}\eta^{-(n-1)/2}\xi^{-(n-1)(p-1)/2}s_e^{-(np-p-1)}[R_\rho(d)]^{-p(n-1)/2}\ .$$

This marginal probability element at Y becomes

$$\text{(3.8)}\qquad A_{(n-1)p}^{-1}\eta^{-(n-1)/2}(\xi)^{-(p-1)(n-1)/2}s_y^{-(pn-p-1)}[R_\rho(d)]^{-p(n-1)/2}dv\ .$$

The Jacobian of the inverse transformation from Y to X is unity. The same transformation changes the differential $d(\sqrt{n}\,\bar{y}_1),\cdots,d\sqrt{n}\,\bar{y}_p ds_y$ along the inverse image of the orbit to $d(\sqrt{n}\,\bar{x}_{1\cdot}),\cdots,d(\sqrt{n}\,\bar{x}_{p\cdot})d\phi_\rho(x)$. So in terms of the observed response the probability element cross-sectional to the inverse image of the orbit is obtained as

$$\text{(3.9)}\qquad A_{(n-1)p}^{-1}(\eta)^{-(n-1)/2}(\xi)^{-(p-1)(n-1)/2}[\phi_\rho^2(x)]^{-p(n-1)/2}\ .$$

This probability element depends on ρ and thus yields the marginal likelihood function of ρ as

$$L(\rho\,|\,X)=R^+(X)\eta^{-(n-1)/2}(\xi)^{-(p-1)(n-1)/2}\phi_\rho^{-p(n-1)}(x)$$
$$=R^+(X)\eta^{-(n-1)/2}(\xi)^{-(p-1)(n-1)/2}[\eta^{-1}P(x)+\xi^{-1}Q(x)]^{-p(n-1)/2}$$

where

$$P(x)=p\sum_{\alpha=1}^{n}(x_{\cdot\alpha}-x)^2\ ,\qquad Q(x)=\sum_{j=1}^{n}\sum_{\alpha=1}^{n}(x_{j\alpha}-\bar{x}_{j\cdot}-\bar{x}_{\cdot\alpha}+\bar{x})^2$$

and $R^+(X)$ is the map that carries that point X into the single entity $R^+=(0,\infty)$. The marginal likelihood function thus obtained provides the basis of inference about ρ. Olkin and Prat [9] derived an unbiased estimate of ρ which is a function of $P(x)$ and $Q(x)$. It is interesting to note that the likelihood function here depends on $P(x)$ and $Q(x)$ as a function of X.

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