ASYMPTOTIC NON-NULL DISTRIBUTIONS OF TWO TEST CRITERIA FOR EQUALITY OF COVARIANCE MATRICES UNDER LOCAL ALTERNATIVES

HISAO NAGAO

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1. Introduction and summary

In a previous paper [6], the author has derived the asymptotic expansion of the distribution of the likelihood ratio (=LR) criterion for testing the equality of covariance matrices under the fixed alternative, the first term of which is a normal distribution. On the other hand Box [2] gave the asymptotic expansion of the null distribution in terms of χ^2-variates. In fact this limiting non-null distribution degenerates at the null hypothesis. Therefore we shall investigate the distribution under local alternatives. In the univariate case the author [7] obtained the asymptotic expansion of the distribution under local alternatives by using the normal approximation of $\log \chi^2$. In this paper, giving a natural extension of the asymptotic normality of $\log \chi^2$ to the multivariate case, we shall derive the asymptotic expansions of the non-null distributions under local alternatives for the LR criterion and the test proposed by the author [8], the first terms of which are non-central χ^2.

Recently based on the solution of a differential equation with respect to the hypergeometric function ${}_2F_1$ type due to Muirhead [5], the asymptotic distribution of the LR criterion for the two sample case has been derived by Sugiura [10] under local alternatives.

2. Preliminaries

Let the $p\times 1$ vectors $X_{\alpha 1}, X_{\alpha 2}, \cdots, X_{\alpha N_\alpha}$ be a random sample from a p-variate normal distribution with mean vector μ_α and covariance matrix Σ_α $(\alpha=1, 2, \cdots, k)$. For testing the hypothesis H: $\Sigma_1=\Sigma_2=\cdots=\Sigma_k$ $(=\Sigma)$ against all alternatives K: $\Sigma_\alpha\neq\Sigma_\beta$ for some α and β $(\alpha\neq\beta)$ with unspecified μ_α, the modified LR test is given by

$$(2.1)\qquad \lambda_1=\prod_{\alpha=1}^{k} |S_\alpha/n_\alpha|^{n_\alpha/2}\Big/\left|\sum_{\alpha=1}^{k} S_\alpha/n\right|^{n/2},$$

where $S_\alpha=\sum_{\beta=1}^{N_\alpha}(X_{\alpha\beta}-\bar{X}_\alpha)(X_{\alpha\beta}-\bar{X}_\alpha)'$, $\bar{X}_\alpha=N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha}X_{\alpha\beta}$, $n_\alpha=N_\alpha-1$ and $n=\sum_{\alpha=1}^{k}n_\alpha$. Especially in the case $k=2$, this test was shown to be unbiased by our previous paper [9]. For the above problem the author [8] proposed the test statistic

$$\lambda_2=\frac{1}{2}\sum_{\alpha=1}^{k}n_\alpha \operatorname{tr}\left\{\frac{S_\alpha}{n_\alpha}\left(\frac{1}{n}\sum_{\alpha=1}^{k}S_\alpha\right)^{-1}-I\right\}^2, \tag{2.2}$$

and derived an asymptotic expansion for the null distribution of λ_2.

As in Chevally ([3], p. 14) we can define the logarithm for a real positive definite matrix. So we shall give a multivariate extension of the normal approximation of $\log\chi^2$.

LEMMA 2.1. *Let S $(p\times p)$ be distributed according to the Wishart distribution $W(\Sigma, n)$. Then under the assumption $\Sigma=I+n^{-1/2}\theta$, the $p(p+1)/2$ random variables y_{ij} $(i\leqq j)$ of the statistic $Y=(y_{ij})=\sqrt{n/2}(\log S/n-\log\Sigma)$ are stochastically independent as $n\to\infty$. y_{ii} converges in law to $N(0,1)$ and y_{ij} $(i<j)$ converges in law to $N(0,1/2)$.*

PROOF. Let $U=(\Sigma^{-1/2}S\Sigma^{-1/2}-nI)/\sqrt{2n}$. Then the statistic U has an asymptotically normal distribution. (See Nagao [6], p. 201.) Since $\Sigma=I+n^{-1/2}\theta$, we have $\log S/n=n^{-1/2}\theta+(n/2)^{-1/2}U+O_p(n^{-1})$. Thus $Y=U+O_p(n^{-1/2})$. Hence we can obtain the desired conclusion.

Finally we shall consider the distribution of the statistic $Z_\alpha=\sqrt{m_\alpha/2}\cdot\log S_\alpha/m_\alpha$, where S_α $(p\times p)$ has the Wishart distribution $W(\Sigma_\alpha, n_\alpha)$ and $m_\alpha=\rho n_\alpha$ with $\rho=1+o(1)$. Since $S_\alpha=m_\alpha\exp(\sqrt{2/m_\alpha}Z_\alpha)$, expressing the characteristic roots of the matrix Z_α as $ch_i(Z_\alpha)$, the Jacobian as in Jack [4] is given by

$$\left|\frac{\partial S_\alpha}{\partial Z_\alpha}\right|=(2m_\alpha)^{p(p+1)/4}\operatorname{etr}\left[\sqrt{\frac{2}{m_\alpha}}Z_\alpha\right]\prod_{i>j}^{p}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}, \tag{2.3}$$

where $f(\lambda_i)=e^{\lambda_i}$ with $\lambda_i=\sqrt{2/m_\alpha}ch_i(Z_\alpha)$. Since we are interested in obtaining asymptotic expansions, we note that the last term in (2.3) can be expanded for large m_α as

$$\begin{aligned}\prod_{i>j}^{p}\frac{f(\lambda_i)-f(\lambda_j)}{\lambda_i-\lambda_j}=1&+\frac{1}{2}(p-1)\sqrt{\frac{2}{m_\alpha}}\operatorname{tr}Z_\alpha+\frac{1}{12m_\alpha}\\&\cdot\{(3p^2-6p+2)(\operatorname{tr}Z_\alpha)^2+p\operatorname{tr}Z_\alpha^2\}+O(m_\alpha^{-3/2}).\end{aligned} \tag{2.4}$$

Since $|e^A|=\operatorname{etr}A$ for any square matrix A, the "asymptotic" distribution of Z_α can be expressed as

$$c_\alpha^*\cdot\operatorname{etr}\left[\frac{1}{2}(m_\alpha-p+1+2\Delta_\alpha)\sqrt{\frac{2}{m_\alpha}}Z_\alpha-\frac{m_\alpha}{2}\Sigma_\alpha^{-1}\exp\left(\sqrt{\frac{2}{m_\alpha}}Z_\alpha\right)\right] \tag{2.5}$$

$$\cdot \left[1+\frac{1}{2}(p-1)\sqrt{\frac{2}{m_\alpha}}\operatorname{tr} Z_\alpha+\frac{1}{12m_\alpha}\{(3p^2-6p+2)(\operatorname{tr} Z_\alpha)^2 +p\operatorname{tr} Z_\alpha^2\}+O(m_\alpha^{-3/2})\right],$$

where $\Delta_\alpha=(n_\alpha-m_\alpha)/2$ and

$$c_\alpha^* = \left\{\prod_{\beta=1}^{p}\Gamma\left[\frac{1}{2}(m_\alpha+1-\beta+2\Delta_\alpha)\right]\right\}^{-1}\left(\frac{m_\alpha}{2}\right)^{p(2m_\alpha-p-1+4\Delta_\alpha)/4} \cdot \pi^{-p(p-1)/4}|\Sigma_\alpha|^{-(m_\alpha+2\Delta_\alpha)/2}. \tag{2.6}$$

3. Asymptotic distribution of the modified LR criterion

From the statistic (2.1), we have

$$-2\rho\log\lambda_1=m\log\left|\sum_{\alpha=1}^{k}S_\alpha/m\right|-\sum_{\alpha=1}^{k}m_\alpha\log|S_\alpha/m_\alpha|, \tag{3.1}$$

where $m_\alpha=\rho n_\alpha$, $m=\sum_{\alpha=1}^{k}m_\alpha$ and a correction factor ρ as in Anderson ([1], p. 255) is given by

$$\rho=1-\left(\sum_{\alpha=1}^{k}n_\alpha^{-1}-n^{-1}\right)\frac{2p^2+3p-1}{6(p+1)(k-1)}. \tag{3.2}$$

Since the statistic $-2\rho\log\lambda_1$ remains invariant by the transformation $S_\alpha\to AS_\alpha A'$ ($\alpha=1,2,\cdots,k$) for any non-singular matrix A, we may assume $\Sigma_1=I$ without loss of generality. We consider the distribution of $-2\rho\log\lambda_1$ under the sequence of alternatives K_m : $\Sigma_\alpha=I+m^{-1/2}\theta_\alpha$ ($\alpha=1,2,\cdots,k$). Put $Y_\alpha=\sqrt{m_\alpha/2}(\log S_\alpha/m_\alpha-\log\Sigma_\alpha)$ ($\alpha=1,2,\cdots,k$), then by Lemma 2.1 Y_α is asymptotically normal. Thus we can express the statistic (3.1) in terms of the Y's with the fixed $\rho_\alpha=m_\alpha/m$ ($\alpha=1,2,\cdots,k$) as

$$-2\rho\log\lambda_1=q_0(Y)+m^{-1/2}q_1(Y)+m^{-1}q_2(Y)+O_p(m^{-3/2}), \tag{3.3}$$

where

$$q_0(Y)=\operatorname{tr}\sum_{\alpha=1}^{k}\left(Y_\alpha+\sqrt{\frac{\rho_\alpha}{2}}\theta_\alpha\right)^2-\operatorname{tr}\left\{\sum_{\alpha=1}^{k}\left(\sqrt{\rho_\alpha}Y_\alpha+\frac{1}{\sqrt{2}}\rho_\alpha\theta_\alpha\right)\right\}^2, \tag{3.4}$$

$$\begin{aligned} q_1(Y)=&-\frac{1}{3}\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^3+\operatorname{tr}\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^2+\frac{\sqrt{2}}{3}\operatorname{tr}\sum_{\alpha=1}^{k}Y_\alpha^3/\sqrt{\rho_\alpha}\\ &-\operatorname{tr}\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}Y_\alpha+\rho_\alpha\theta_\alpha)\sum_{\alpha=1}^{k}(Y_\alpha^2+\sqrt{2\rho_\alpha}\theta_\alpha Y_\alpha)\\ &+\frac{1}{3}\operatorname{tr}\left\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}Y_\alpha+\rho_\alpha\theta_\alpha)\right\}^3, \end{aligned} \tag{3.5}$$

$$(3.6)\quad q_2(Y)=\frac{1}{4}\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^4+\frac{1}{6}\operatorname{tr}\sum_{\alpha=1}^{k}(\theta_\alpha Y_\alpha)^2-\frac{1}{6}\operatorname{tr}\sum_{\alpha=1}^{k}\theta_\alpha^2 Y_\alpha^2$$
$$+\frac{\sqrt{2}}{3}\operatorname{tr}\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^3/\sqrt{\rho_\alpha}+\frac{1}{6}\operatorname{tr}\sum_{\alpha=1}^{k}Y_\alpha^4/\rho_\alpha$$
$$-\operatorname{tr}\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\Big\{\frac{\sqrt{2}}{6}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Y_\alpha\theta_\alpha$$
$$-\frac{\sqrt{2}}{12}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2 Y_\alpha-\frac{\sqrt{2}}{12}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,Y_\alpha\theta_\alpha^2+\frac{1}{3}\sum_{\alpha=1}^{k}Y_\alpha\theta_\alpha Y_\alpha$$
$$+\frac{1}{3}\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^2+\frac{1}{3}\sum_{\alpha=1}^{k}Y_\alpha^2\theta_\alpha+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}Y_\alpha^3/\sqrt{\rho_\alpha}\Big\}$$
$$+\frac{1}{2}\operatorname{tr}\Big\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\Big\}^2\Big\{\sqrt{2}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Y_\alpha$$
$$+\sqrt{2}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,Y_\alpha\theta_\alpha+2\sum_{\alpha=1}^{k}Y_\alpha^2\Big\}$$
$$-\frac{1}{2}\operatorname{tr}\Big\{\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Y_\alpha+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,Y_\alpha\theta_\alpha+\sum_{\alpha=1}^{k}Y_\alpha^2\Big\}^2$$
$$-\frac{1}{4}\operatorname{tr}\Big\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\Big\}^4 .$$

Thus we can express the characteristic function of $-2\rho\log\lambda_1$ as follows:

$$(3.7)\quad C(t)=\mathrm{E}\Big[\exp(itq_0(Y))\Big[1+m^{-1/2}(it)q_1(Y)+m^{-1}$$
$$\cdot\Big\{(it)q_2(Y)+\frac{(it)^2}{2}q_1(Y)^2\Big\}\Big]\Big]+O(m^{-3/2}) .$$

Then using the distribution of Z_α in (2.5), we can evaluate each term in (3.7). To explain the method of its calculation, the matrix $\exp(\sqrt{2/m_\alpha}\cdot Z_\alpha)$ in (2.5) can be expanded asymptotically as

$$(3.8)\quad \exp\Big(\sqrt{\frac{2}{m_\alpha}}Z_\alpha\Big)=I+\sqrt{\frac{2}{m_\alpha}}Z_\alpha+\frac{1}{m_\alpha}Z_\alpha^2+\frac{\sqrt{2}}{3m_\alpha\sqrt{m_\alpha}}Z_\alpha^3$$
$$+\frac{1}{6m_\alpha^2}Z_\alpha^4+O(m^{-5/2}) .$$

Rewrite $q_0(Y)$ in terms of $Z_\alpha=(z_{ij}^{(\alpha)})$ under K_m. Furthermore, arranging the resulting expression in the exponential part in (2.5) according to the power of $m^{-1/2}$ and combining the term $(it)q_0(Y)$, the first term containing Z_α in the exponential part is given by

$$(3.9)\quad (1-2it)\operatorname{tr}\sum_{\alpha=1}^{k}Z_\alpha^2+2(it)\operatorname{tr}\Big(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,Z_\alpha\Big)^2-\sqrt{2}\operatorname{tr}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Z_\alpha$$
$$=(z-\mu)'[\{(1-2it)^{-1}(I_k-\gamma\gamma')+\gamma\gamma'\}\otimes\tilde{I}_{p(p+1)/2}]^{-1}(z-\mu)$$

$$-\frac{1}{2}\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2-\frac{it}{(1-2it)}\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2 ,$$

where $z'=(z_{11}^{(1)},\cdots,z_{pp}^{(1)},z_{12}^{(1)},\cdots,z_{p-1,p}^{(1)},\cdots,z_{11}^{(k)},\cdots,z_{pp}^{(k)},z_{12}^{(k)},\cdots,z_{p-1,p}^{(k)})$, $\gamma'=(\sqrt{\rho_1},\sqrt{\rho_2},\cdots,\sqrt{\rho_k})$, $\tilde{I}_{p(p+1)/2}=\operatorname{diag}(1,1,\cdots,1,1/2,1/2,\cdots,1/2)$ having multiplicity p of 1 and multiplicity $p(p-1)/2$ of 1/2 and the symbol $\otimes$ denotes the Kronecker product. Also the component $\mu_{ij}^{(\alpha)}$ of $kp(p+1)/2\times 1$ vector μ corresponding to the vector z is given by $\mu_{ij}^{(\alpha)}=\sqrt{\rho_\alpha/2}(\theta_{ij}^{(\alpha)}-2it\tilde{\theta}_{ij})$ with $\theta_\alpha=(\theta_{ij}^{(\alpha)})$ and $\tilde{\theta}=(\tilde{\theta}_{ij})=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha$. Hence from (3.9), we may regard the variable z as having a $kp(p+1)/2$ dimensional normal distribution with mean vector μ and covariance matrix $V=(\sigma_{ij,kl}^{(\alpha,\beta)})$ with $\sigma_{ij,kl}^{(\alpha,\beta)}=(1-2it)^{-1}(\delta_{\alpha\beta}-2it\sqrt{\rho_\alpha\rho_\beta})(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})/2$. The terms of order $m^{-1/2}$ and m^{-1} are performed by calculating moments. Also the two terms in (3.7) can be obtained by tedious calculations. Thus we can obtain the characteristic function of $-2\rho\log\lambda_1$ under the sequences of alternatives K_m: $\Sigma_\alpha=\Sigma+m^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$ as follows:

$$\text{(3.10)}\quad C(t)=(1-2it)^{-f/2}\exp\left[\frac{it}{2(1-2it)}\operatorname{tr}A_2\right]\left[1+m^{-1/2}\left\{\frac{(t)_2}{6}\operatorname{tr}A_3+(t)_1\cdot\left(-\frac{1}{2}\operatorname{tr}A_3-\frac{1}{2}\operatorname{tr}\tilde{\theta}A_2\right)+\frac{1}{3}\operatorname{tr}A_3+\frac{1}{2}\operatorname{tr}\tilde{\theta}A_2\right\}+m^{-1}\sum_{\alpha=0}^{4}h_{2\alpha}(t)_\alpha\right]+O(m^{-3/2}) ,$$

where $(t)_\alpha=(1-2it)^{-\alpha}$, $f=(k-1)p(p+1)/2$ and the coefficients $h_{2\alpha}$ with $C_\alpha=(\theta_\alpha-\tilde{\theta})\Sigma^{-1}$, $A_\alpha=\sum_{\beta=1}^{k}\rho_\beta C_\beta^\alpha$, $\tilde{\theta}=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha$, $\bar{\theta}=\tilde{\theta}\Sigma^{-1}$ and $B=\sum_{\alpha=1}^{k}\rho_\alpha(\bar{\theta}C_\alpha)^2$ are given by

$$\text{(3.11)}\quad h_8=\frac{1}{72}(\operatorname{tr}A_3)^2 ,$$

$$h_6=-\frac{1}{12}(\operatorname{tr}A_3)(\operatorname{tr}A_3+\operatorname{tr}\bar{\theta}A_2)+\frac{1}{8}\operatorname{tr}A_4-\frac{1}{4}\operatorname{tr}A_2^2+\frac{1}{16}\operatorname{tr}\bar{\theta}^2A_2-\frac{1}{16}\operatorname{tr}B ,$$

$$h_4=\frac{1}{72}(\operatorname{tr}A_3)(13\operatorname{tr}A_3+24\operatorname{tr}\bar{\theta}A_2)-\frac{1}{2}\operatorname{tr}A_4+\frac{3}{4}\operatorname{tr}A_2^2+\frac{1}{8}(\operatorname{tr}\bar{\theta}A_2)^2-\frac{1}{2}\operatorname{tr}\bar{\theta}A_3-\frac{3}{8}\operatorname{tr}\bar{\theta}^2A_2+\frac{3}{8}\operatorname{tr}B-\frac{1}{4}(2\Delta+1)\operatorname{tr}A_2-\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr}C_\alpha)^2 ,$$

$$h_2=-\frac{1}{12}(\operatorname{tr} A_3)(2\operatorname{tr} A_3+5\operatorname{tr}\bar{\theta}A_2)+\frac{3}{4}\operatorname{tr} A_4-\frac{3}{4}\operatorname{tr} A_2^2$$
$$-\frac{1}{4}(\operatorname{tr}\bar{\theta}A_2)^2+\frac{3}{2}\operatorname{tr}\bar{\theta}A_3+\frac{9}{16}\operatorname{tr}\bar{\theta}^2A_2+\frac{3}{16}\operatorname{tr} B$$
$$+\frac{1}{4}(4\Delta+1)\operatorname{tr} A_2+\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr} C_\alpha)^2\,,$$

$$h_0=\frac{1}{18}(\operatorname{tr} A_3)(\operatorname{tr} A_3+3\operatorname{tr}\bar{\theta}A_2)-\frac{3}{8}\operatorname{tr} A_4+\frac{1}{4}\operatorname{tr} A_2^2$$
$$+\frac{1}{8}(\operatorname{tr}\bar{\theta}A_2)^2-\operatorname{tr}\bar{\theta}A_3-\frac{1}{4}\operatorname{tr}\bar{\theta}^2A_2-\frac{1}{2}\operatorname{tr} B-\frac{\Delta}{2}\operatorname{tr} A_2\,,$$

where $\Delta=(n-m)/2$. Inverting this characteristic function, we have the following theorem:

THEOREM 3.1. *Under the sequences of alternatives* $K_m:\Sigma_\alpha=\Sigma+m^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$, *the distribution of* $-2\rho\log\lambda_1$ *with the correction factor* ρ *given by* (3.2) *can be expanded asymptotically for large* m $(=\rho n)$ *as*

(3.12) $\Pr(-2\rho\log\lambda_1\leqq x)$
$$=P_f(\delta^2)+m^{-1/2}\Big\{\frac{1}{6}(\operatorname{tr} A_3)P_{f+4}(\delta^2)+\Big(-\frac{1}{2}\operatorname{tr} A_3-\frac{1}{2}\operatorname{tr}\bar{\theta}A_2\Big)P_{f+2}(\delta^2)$$
$$+\Big(\frac{1}{3}\operatorname{tr} A_3+\frac{1}{2}\operatorname{tr}\bar{\theta}A_2\Big)P_f(\delta^2)\Big\}$$
$$+m^{-1}\sum_{\alpha=0}^{4}h_{2\alpha}P_{f+2\alpha}(\delta^2)+O(m^{-3/2})\,,$$

where the symbol $P_f(\delta^2)$ *stands for the distribution function of noncentral* χ^2 *variate with* $f=(k-1)p(p+1)/2$ *degrees of freedom, and noncentrality parameter* $\delta^2=\frac{1}{4}\operatorname{tr}\sum_{\alpha=1}^{k}\rho_\alpha\{(\theta_\alpha-\tilde{\theta})\Sigma^{-1}\}^2$ *and the coefficients* $h_{2\alpha}$ *by* (3.11).

We remark that this formula (3.12) agrees, in the case $k=2$, with the expansion obtained previously by Sugiura [10]. Also it may be interesting to note that each sum of the coefficients of order $m^{-1/2}$ and m^{-1} is zero.

4. Asymptotic distribution of the test criterion λ_2

Using the same method as above, the asymptotic distribution of λ_2 can be obtained. The final result is given in the following:

THEOREM 4.1. *Under the sequences of alternatives* $K_n:\Sigma_\alpha=\Sigma+n^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$, *the distribution of* λ_2 *given by* (2.2) *can be expanded asymptotically for large* n *as*

$$\begin{aligned}(4.1)\quad \Pr(\lambda_2 \leqq x) = P_f(\delta^2) + n^{-1/2}\Big\{\frac{1}{6}(\operatorname{tr} A_3)P_{f+6}(\delta^2) + \frac{1}{2}(p+1)(\operatorname{tr} E_1)P_{f+4}(\delta^2) \\ + \Big[-\frac{1}{2}\operatorname{tr} A_3 - \frac{1}{2}\operatorname{tr}\bar{\theta}A_2 - \frac{1}{2}(p+1)\operatorname{tr} E_1\Big]P_{f+2}(\delta^2) \\ + \Big(\frac{1}{3}\operatorname{tr} A_3 + \frac{1}{2}\operatorname{tr}\bar{\theta}A_2\Big)P_f(\delta^2)\Big\} \\ + n^{-1}\sum_{\alpha=1}^{6} g_{2\alpha}P_{f+2\alpha}(\delta^2) + O(n^{-3/2}),\end{aligned}$$

where $f=(k-1)p(p+1)/2$ *and* $\delta^2=(1/4)\operatorname{tr} A_2$. *The coefficients* $g_{2\alpha}$ *with* $E_\alpha = \sum_{\beta=1}^{k} C_\beta^\alpha$, $\tilde{\rho}=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}$ *and the same notation as Theorem* 3.1 *are given by*

$$(4.2)\quad g_{12} = \frac{1}{72}(\operatorname{tr} A_3)^2,$$

$$g_{10} = \frac{1}{4}\operatorname{tr} A_4 - \frac{1}{4}\operatorname{tr} A_2^2 + \frac{1}{12}(p+1)\operatorname{tr} E_1 \operatorname{tr} A_3,$$

$$\begin{aligned}g_8 = -\frac{1}{12}(\operatorname{tr} A_3)(\operatorname{tr} A_3 + \operatorname{tr}\bar{\theta}A_2) + \frac{1}{24}(p+1)(\operatorname{tr} E_1) \\ \cdot\{3(p+1)\operatorname{tr} E_1 - 2\operatorname{tr} A_3\} - \frac{1}{8}\operatorname{tr} A_4 - \frac{1}{2}\{k(p+1)+p+2\}\operatorname{tr} A_2 \\ -\frac{1}{2}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr} C_\alpha)^2 + \frac{1}{4}(3p+4)\operatorname{tr} E_2 + \frac{1}{4}\sum_{\alpha=1}^{k}(\operatorname{tr} C_\alpha)^2,\end{aligned}$$

$$\begin{aligned}g_6 = \frac{1}{36}(\operatorname{tr} A_3)(2\operatorname{tr} A_3 + 3\operatorname{tr}\bar{\theta}A_2) - \frac{1}{4}(p+1)(\operatorname{tr} E_1) \\ \cdot\{\operatorname{tr} A_3 + \operatorname{tr}\bar{\theta}A_2 + (p+1)\operatorname{tr} E_1\} - \frac{1}{2}\operatorname{tr} A_4 + \frac{1}{2}\operatorname{tr} A_2^2 \\ -\frac{1}{2}\operatorname{tr}\bar{\theta}A_3 + \frac{9}{16}\operatorname{tr}\bar{\theta}^2A_2 - \frac{9}{16}\operatorname{tr} B + \frac{1}{4}\{2k(p+1)+(2p+3)\}\operatorname{tr} A_2 \\ +\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr} C_\alpha)^2 - \frac{1}{4}(4p+5)\operatorname{tr} E_2 - \frac{1}{4}\sum_{\alpha=1}^{k}(\operatorname{tr} C_\alpha)^2 \\ -\frac{p}{12}\{3k^2(p+1)^2 + (3k-2)(p^2+3p+4) - \tilde{\rho}(4p^2+9p+7)\},\end{aligned}$$

$$\begin{aligned}g_4 = \frac{1}{8}(\operatorname{tr} A_3)(\operatorname{tr} A_3 + 2\operatorname{tr}\bar{\theta}A_2) + \frac{1}{24}(p+1)(\operatorname{tr} E_1) \\ \cdot\{3(p+1)\operatorname{tr} E_1 + 12\operatorname{tr}\bar{\theta}A_2 + 10\operatorname{tr} A_3\} + \frac{1}{4}\operatorname{tr} A_2^2 \\ +\frac{1}{8}(\operatorname{tr}\bar{\theta}A_2)^2 - \frac{9}{8}\operatorname{tr}\bar{\theta}^2A_2 + \frac{9}{8}\operatorname{tr} B + \frac{1}{2}k(p+1)\operatorname{tr} A_2\end{aligned}$$

$$-\frac{1}{4}(p+1)\operatorname{tr} E_2-\frac{1}{2}(p+1)\operatorname{tr}\bar{\theta}E_1$$
$$+\frac{1}{8}p\{4k^2(p+1)^2+(2k-1)(2p^2+5p+5)-\tilde{\rho}(6p^2+13p+9)\}\ ,$$

$$g_2=-\frac{1}{12}(\operatorname{tr} A_3)(2\operatorname{tr} A_3+5\operatorname{tr}\bar{\theta}A_2)-\frac{1}{12}(p+1)(\operatorname{tr} E_1)$$
$$\cdot(2\operatorname{tr} A_3+3\operatorname{tr}\bar{\theta}A_2)+\frac{3}{4}\operatorname{tr} A_4-\frac{3}{4}\operatorname{tr} A_2^2-\frac{1}{4}(\operatorname{tr}\bar{\theta}A_2)^2$$
$$+\frac{3}{2}\operatorname{tr}\bar{\theta}A_3+\frac{17}{16}\operatorname{tr}\bar{\theta}^2A_2-\frac{5}{16}\operatorname{tr} B-\frac{1}{4}\{2k(p+1)-1\}\operatorname{tr} A_2$$
$$+\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha(\operatorname{tr} C_\alpha)^2+\frac{1}{2}(p+1)\operatorname{tr} E_2+\frac{1}{2}(p+1)\operatorname{tr}\bar{\theta}E_1$$
$$-\frac{1}{4}(k^2+k-2\tilde{\rho})p(p+1)^2\ ,$$

$$g_0=\frac{1}{18}(\operatorname{tr} A_3)(\operatorname{tr} A_3+3\operatorname{tr}\bar{\theta}A_2)+\frac{1}{4}\operatorname{tr} A_2^2-\frac{3}{8}\operatorname{tr} A_4+\frac{1}{8}(\operatorname{tr}\bar{\theta}A_2)^2$$
$$-\operatorname{tr}\bar{\theta}A_3-\frac{1}{2}\operatorname{tr}\bar{\theta}^2A_2-\frac{1}{4}\operatorname{tr} B+\frac{1}{24}(1-\tilde{\rho})p(2p^2+3p-1)\ .$$

From Theorems 3.1 and 4.1, we note that Pitman's asymptotic relative efficiency of the λ_1 test with respect to the λ_2 is equal to 1.

KUMAMOTO UNIVERSITY

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