A NOTE ON HOMOGENEOUS PROCESSES WITH INDEPENDENT INCREMENTS

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1. Introduction

Let $X(t)$, $t \geqq 0$, be a continuous (in probability) and homogeneous stochastic process with independent increments. Fix $t_1 > 0$. Then, given $X(t_1) = y$, the process $X(t)$, for $0 \leqq t \leqq t_1$, is called a tied-down process with end point equals y. Suppose $X(t)$ is a Poisson process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, for all $0 \leqq t \leqq t_1$, is binomial with parameter $(y, t/t_1)$, (see Karlin [1], p. 185) and therefore the conditional expectation and variance of $X(t)$ given $X(t_1) = y$ are linear functions of y. Suppose $X(t)$ is a Wiener process, then the conditional distribution of $X(t)$ given $X(t_1) = y$, (see Karlin [1], p. 275, Theorem 2.1) for all $0 \leqq t \leqq t_1$, is normal with parameter $((t/t_1)y, \sigma^2 t(1 - t/t_1))$, and hence the conditional expectation of $X(t)$ given $X(t_1) = y$ is a linear function of y and the conditional variance does not depend upon y.

In this note, we shall characterize a class of stochastic processes based on the property that the conditional mean and variance of $X(t)$, given $X(t_1) = y$, for some $0 < t < t_1$, are linear functions of y. It will be proved that if $\mathrm{E}(X(t) | X(t_1) = y) = \alpha_0 + \alpha_1 y$, then

1) $\mathrm{Var}(X(t) | X(t_1) = y) = \text{constant}$ a.e. if and only if $X(t) = W(t) + \mu t$, where $W(t)$ is a Wiener process and μ is a real constant,

2) $\mathrm{Var}(X(t) | X(t_1) = y) = \beta_0 + \beta_1 y$ $(\beta_1 \neq 0)$ if and only if $X(t) = cY(t) - \nu t$, where $Y(t)$ is a Poisson process and ν and c are real constants. To avoid trivial cases, we shall assume that $X(t)$ is not a degenerate process. Also all stochastic processes $X(t)$, $t \geqq 0$, considered in this note are assumed to be continuous, homogeneous, second-order and with independent increments. For a recent survey of the results on characterizations of stochastic processes see Lukacs [2].

2. The result

We need the following two lemmas.

LEMMA 1. *If* $\mathrm{E}(X(t) | X(t_1) = y) = \alpha_0 + \alpha_1 y$, *then* $\alpha_0 = 0$ *and* $\alpha_1 = t/t_1$, *for all* $0 \leqq t \leqq t_1 < \infty$.

PROOF. Since $X(t)$ is a continuous, homogeneous, second-order process with independent increments, it follows that $\mu(t) = \mathrm{E}(X(t)) = \mu t$, $\sigma^2(t) = \mathrm{Var}(X(t)) = \sigma^2 t$, for all $t \geqq 0$, and $\rho(t_1, t_2) = \text{Corr. Coeff.}(X(t_1), X(t_2)) = \min(t_1, t_2)/\sqrt{t_1}\sqrt{t_2}$, for all $t_1 > 0$ and $t_2 > 0$, where $\mu = \mathrm{E}(X(1))$ and $\sigma^2 = \mathrm{Var}(X(1))$. Therefore

$$\alpha_1 = \rho(t, t_1)\sigma(t)/\sigma(t_1) = t/t_1 ,$$

and

$$\alpha_0 = \mu(t) - \alpha_1 \mu(t_1) = 0 .$$

LEMMA 2. *Let* $g(s, t) = \mathrm{E}(e^{isX(t)})$ *be the characteristic function of* $X(t)$ *and* $g(s) = g(s, 1)$. *Then*

$$\mathrm{E}(X(t_0)e^{isX(t_1)}) = -it_0 g^{t_1 - 1}(s) g'(s)$$

and

$$\mathrm{E}(X^2(t_0)e^{isX(t_1)}) = -t_0(t_0 - 1) g^{t_1 - 2}(s)(g'(s))^2 - t_0 g^{t_1 - 1}(s) g''(s) ,$$

for all $0 < t_0 < t_1$ *and real* s.

PROOF. Because $X(t)$ is a continuous, homogeneous, second-order process with independent increments, $g(s, t) = g^t(s)$ and the second partial derivative of $g(s, t)$ w.r.t. s exists for all t and s. It then follows that

$$\mathrm{E}(X(t)e^{isX(t)}) = -itg^{t-1}(s)g'(s)$$

and

$$\mathrm{E}(X^2(t)e^{isX(t)}) = -t(t-1)g^{t-2}(s)(g'(s))^2 - tg^{t-1}(s)g''(s) ,$$

for all real s and $t > 0$. Then Lemma 2 follows from the fact that

$$\mathrm{E}(X^k(t_0)e^{isX(t_1)}) = \mathrm{E}(X^k(t_0)e^{isX(t_0)})g(s, t_1 - t_0)$$

for all k, $0 \leqq t_0 \leqq t_1$ and real s.

We now state and prove the main result of this note.

THEOREM. *Let* $0 < t_0 < t_1$. *Then the necessary and sufficient condition that*

$$\mathrm{E}(X(t_0) \mid X(t_1) = y) = \alpha_0 + \alpha_1 y \tag{1}$$

and

$$\mathrm{Var}(X(t_0) \mid X(t_1) = y) = \beta_0 + \beta_1 y \qquad a.e. , \tag{2}$$

where α_0, α_1, β_0 *and* β_1 *are constants w.r.t.* y, *is that*

1) $X(t)=W(t)+\mu t$, *if* $\beta_1=0$, *where* $W(t)$ *is a Wiener process and* μ *is a real constant,*

2) $X(t)=cY(t)-\nu t$, *if* $\beta_1\neq 0$, *where* $Y(t)$ *is a Poisson process and* ν *and* c *are real constants.*

PROOF. The sufficient condition can be verified by a straightforward calculation.

To prove the necessary condition. Suppose the conditions (1) and (2) hold. Then by Lemma 1, the conditions (1) and (2) imply

$$\text{(3)}\qquad \mathrm{E}\,(X^2(t_0)e^{isX(t_1)})-(t_0/t_1)^2\,\mathrm{E}\,(X^2(t_1)e^{isX(t_1)}) =\beta_0\,\mathrm{E}\,(e^{isX(t_1)})+\beta_1\,\mathrm{E}\,(X(t_1)e^{isX(t_1)})$$

for all s. By Lemma 2, equation (3) is equivalent to

$$\text{(4)}\qquad t_0(1-t_0/t_1)\{g^{t_1-2}(s)(g'(s))^2-g^{t_1-1}(s)g''(s)\}=\beta_0 g(s)-i\beta_1 t_1 g^{t_1-1}(s)g'(s)\ ,$$

for all real s. Because the characteristic functions of a homogeneous process with independent increments are infinitely divisible, and infinitely divisible characteristic functions never vanish, $g(s)\neq 0$ for all s. And we rewrite equation (4) in the form

$$\text{(5)}\qquad \frac{d}{ds}(g'(s)/g(s))=-B_0+iB_1(g'(s)/g(s))\ ,$$

where $B_0=\beta_0/(t_0(1-t_0/t_1))$ and $B_1=\beta_1 t_1/(t_0(1-t_0/t_1))$. Because the second derivative of $g(s)$ exists and does not vanish for the s in a neighbourhood N of the origin and $g'(s)/g(s)$ is independent of t_0 and t_1, B_0 and B_1 are independent of t_0 and t_1. In addition, it is easy to check that if $\beta_1=0$, then $B_0>0$. The solution of equation (4) is, if $\beta_1=0$,

$$g(s)=\exp\left\{i\mu s-\frac{1}{2}\sigma^2 s^2\right\}\ ,$$

where μ is a real constant and $\sigma^2=B_0>0$, and if $\beta_1\neq 0$,

$$g(s)=\exp\{-i\nu s+\lambda(e^{ics}-1)\}\ ,$$

where λ is a positive real constant independent of t and s, and $\nu=B_0/B_1$, $c=B_1$. Therefore, the characteristic function of $X(t)$ is, if $\beta_1=0$,

$$g(s,t)=\exp\left\{i\mu ts-\frac{1}{2}\sigma^2 ts^2\right\}\ ,$$

and, if $\beta_1\neq 0$,

$$g(s,t)=\exp\{-i\nu ts+\lambda t(e^{ics}-1)\}\ .$$

This completes our proof of the theorem.

The following two corollaries of the theorem are characterizations of the Wiener and the Poisson processes.

COROLLARY 1. *If* $\mathrm{E}(X(t))=0$ *for some* $t>0$, *then the necessary and sufficient condition that* $X(t)$ *is a Wiener process is that* $\mathrm{E}(X(t_0)|X(t_1)=y)$ *is a linear function of* y *and* $\mathrm{Var}(X(t_0)|X(t_1)=y)$ *is constant a.e. for some* $0<t_0<t_1$.

COROLLARY 2. *The necessary and sufficient condition that* $X(t)$ *is a Poisson process is that* $\mathrm{E}(X(t_0)|X(t_1)=y)$ *is a linear function of* y *and* $\mathrm{Var}(X(t_0)|X(t_1)=y)=(t_0/t_1)(1-t_0/t_1)y$ *a.e. for some* $0<t_0<t_1$.

It follows from Corollaries 1 and 2 that for some $t_1>t_0>0$; 1) the conditional distribution of $X(t_0)$ given $X(t_1)$ is binomial with parameter $(X(t_1), t_0/t_1)$ characterizes the Poisson process and 2) the conditional distribution of $X(t_0)$ given $X(t_1)$ is normal with parameter $((t_0/t_1)X(t_1), \sigma^2 t_0(1-t_0/t_1))$ characterizes the Wiener process.

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REFERENCES

[1] Karlin, S. (1969). *A First Course in Stochastic Processes*, Academic Press, Inc., New York.

[2] Lukacs, E. (1970). Characterization theorems for certain stochastic processes, *Rev. Int. Statist. Inst.*, **38**, 333-343.