PAIRWISE AND VARIANCE BALANCED INCOMPLETE BLOCK DESIGNS*

A. HEDAYAT AND W. T. FEDERER

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Summary

The purpose of this paper is three-fold. The first purpose is to compile and to systematize published and dispersed results on two aspects of balancing in incomplete block designs, i.e., pairwise balance and variance balance. This was done in order to establish the status of these two concepts of balance in published literature and to put them in a form which is useful for further work in this area. Also, the results in this form are necessary for the development of the remainder of the paper.

The second purpose of this paper is to present a method of constructing unequal replicate and/or unequal block size experiment designs for which the variance balance property is achieved. The method of construction involves the union of blocks from two or more block designs and the augmentation of some of the blocks with additional treatments; the method is denoted as *unionizing block designs*. A straightforward extension of the method would produce a partially balanced block design with unequal replicate and/or unequal block designs. The enlargement of the concept and availability of variance balanced block designs to accommodate unequal replication and/or unequal block sizes is important to the researcher, the teacher, and the experimenter needing such designs. For example, an animal nutritionist or a psychologist is no longer required to have constant litter or family sizes for the blocks and may have unequal replication on the treatments for those treatments with insufficient material and still attain the goal of equal variances on all normalized treatment contrasts.

The third purpose of the paper is to apply the unionizing block designs method to construct a family of unequal replicate and unequal block size variance balanced designs. Some comments are given on the extension of the unionizing block designs method to construct other families of variance balanced or partially balanced block designs.

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1. Introduction and notation

Let $\Omega=\{a_1,\cdots,a_v\}$ be a set of v treatments. By a block design with parameters $v, b, k_1, k_2,\cdots, k_b, r_1, r_2,\cdots, r_v$ and incidence matrix N denoted by $\mathrm{BD}(v, b, r_1, r_2,\cdots, r_v; k_1, k_2,\cdots, k_b; N)$ on Ω we shall mean an allocation of elements of Ω one on each of the $m=\sum_{j=1}^{b} k_j$ experimental units arranged in b blocks or groups of experimental units of size k_1, $k_2,\cdots, k_b$ such that a_i is assigned into r_i experimental units. Thus, $\sum_{i=1}^{v} r_i=\sum_{j=1}^{b} k_j$. A block design is said to be *proper* if $k_j=k$, $j=1, 2,\cdots, b$. A block design in which $r_i=r$, $i=1, 2,\cdots, v$ holds will be said to be an *equireplicate* design. A block design is said to be *locally connected* when any two treatments can be connected by a chain consisting alternatively of treatments and blocks, so that if treatment a_i and block B_j are any two consecutive members of the chain, then treatment a_i occurs in block B_j. Hedayat [11] has given a generalization of the concept of local connectedness and has introduced *globally connected* designs, which are not the subject under discussion in this paper. In the following the term connectedness is equivalent to local connectedness in the Hedayat sense. Connectedness is an important and desirable property, as we shall see later. A block design is said to be *incomplete* if there exists a block which does not contain all the elements of Ω.

If we can assume that the position of experimental units in the blocks bears no information whatsoever, then the usual $v\times b$ incidence matrix $N=(n_{ij})$, where n_{ij} scores the number of experimental units in the jth block receiving the ith treatment, completely characterizes the combinatorial arrangement of the design. A block design is said to be an n-ary block design if the entries of N constitute n distinct integers (Das and Rao [9], Murty and Das [14], Rao and Das [16], and Tocher [19]). A block design is said to be *pairwise balanced* if $NN'=T+\lambda J$, where N' is the transpose of N, T a diagonal matrix, λ a scalar and J a matrix with unit entries everywhere. Note that our definition of pairwise balanced block designs covers the pairwise balanced block designs of index λ introduced by Bose and Shrikhande [6] as a special case.

A binary $(n_{ij}=0, 1)$ incomplete block design with v treatments each replicated r times and with b blocks each of size k such that $NN'=(r-\lambda)I+\lambda J$, I the identity matrix of order v, will be called the classical balanced incomplete block design (CBIB). Any such design will be designated by $\mathrm{BIB}(v, b, r, k, \lambda)$. Literature on this family of designs is vast. One can construct many families of nonproper (i.e., not proper) binary $(n_{ij}=0, 1)$ pairwise balanced incomplete block designs via CBIB designs either by deletion of some treatments from the known CBIB designs or simply by putting together several different CBIB designs. Note that all these generated designs will be equireplicated. This fact has led several authors (Adhikary [1], [2], Agrawal [3], Agrawal and Raghavachari [4], Calinski [7], Hedayat and Federer [12], and John [13]) to consider the construction of non-equireplicated pairwise balanced incomplete block designs.

Denote by R and K the diagonal matrices $\operatorname{diag}(r_1, r_2, \cdots, r_v)$ and $\operatorname{diag}(k_1, k_2, \cdots, k_b)$ respectively. The matrix $R-NK^{-1}N'$ is known as the coefficient matrix of the design and will be denoted by C. We shall shortly see the importance of the C matrix in block designs. Since each row (or column) of C adds to zero, the rank of C is at most $v-1$.

2. Preliminary results

The various relationships existing among connected designs, pairwise balanced designs, and variance balanced designs are given below.

LEMMA 2.1. *A block design is connected if and only if the rank of its coefficient matrix is* $v-1$.

Under the usual homoscedastic linear additive model and the standard intrablock analysis of the responses it is known that

LEMMA 2.2. *If the rank of* $C=v-\alpha$, *a set of* $\alpha-1$ *independent treatment contrasts is not estimable.*

A block design is said to be *variance balanced* if every normalized estimable linear function of the treatment effects can be estimated with the same variance. Vartak [20] proved that

THEOREM 2.1. *A block design is variance balanced if and only if the nonzero characteristic roots of its coefficient matrix are equal.*

In particular, if the design is connected the following is true.

THEOREM 2.2. *The following are equivalent.*
(i) *A connected block design is variance balanced if and only if its coefficient matrix has* $v-1$ *equal characteristic roots other than zero* (*Rao* [17]).
(ii) *A connected block design is variance balanced if and only if every treatment effect is estimated with the same variance and every two treatment effects with the same covariance* (*Atiqullah* [5]).
(iii) *A connected block design is variance balanced if and only if its coefficient matrix is of the form* c_1I+c_2J *where* I *is the identity matrix of order* v, J *a matrix with unit entries everywhere,* c_1 *and* c_2 *are scalars. It can be shown that* c_1 *is the nonzero characteristic root of* C *and* $c_2=$

c_1/v *where* v *is the number of treatments* (*Rao* [17]).

Note that Theorems 2.1 and 2.2 put no restriction whatsoever on block sizes, replications, and possible values of the n_{ij}.

COROLLARY 2.1. *Any binary* ($n_{ij}=0, 1$) *proper variance balanced block design is necessarily equireplicated* (*Rao* [17]).

However, if we relax the binary and/or the proper condition the conclusion of the above theorem is no longer true for all cases.

Example 1. Let $\Omega_1=\{1, 2, 3, 4, 5\}$. Then, $B_1=\{1, 2, 3\}$, $B_2=\{1, 2, 4\}$, $B_3=\{1, 3, 4\}$, $B_4=\{2, 3, 4\}$, $B_5=\{1, 5, 5\}$, $B_6=\{2, 5, 5\}$, $B_7=\{3, 5, 5\}$, $B_8=\{4, 5, 5\}$ is a trinary proper variance balanced block design which is not equireplicated.

Example 2. Let $\Omega_2=\{1, 2, 3, 4, 5\}$. Then, $B_1=\{1, 2, 3, 4\}$, $B_2=\{1, 2, 3, 4\}$, $B_3=\{1, 5\}$, $B_4=\{2, 5\}$, $B_5=\{3, 5\}$, $B_6=\{4, 5\}$ is a binary nonproper variance balanced block design which is not equireplicated.

Example 3. Let $\Omega_3=\{1, 2, 3, 4\}$. Then, $B_1=\{1, 2, 4, 4\}$, $B_2=\{1, 2, 4, 4\}$, $B_3=\{1, 3, 4, 4\}$, $B_4=\{1, 3, 4, 4\}$, $B_5=\{2, 3, 4, 4\}$, $B_6=\{2, 3, 4, 4\}$, $B_7=\{1, 2\}$, $B_8=\{1, 2\}$, $B_9=\{1, 2\}$, $B_{10}=\{1, 3\}$, $B_{11}=\{1, 3\}$, $B_{12}=\{1, 3\}$, $B_{13}=\{2, 3\}$, $B_{14}=\{2, 3\}$, $B_{15}=\{2, 3\}$ is a trinary nonproper variance balanced block design which is not equireplicated.

COROLLARY 2.2. *A binary* ($n_{ij}=0, 1$) *proper equireplicated block design is variance balanced if and only if it is a CBIB* (*Thompson* [18] *and Rao* [17]).

COROLLARY 2.3. *An equireplicate binary* ($n_{ij}=0, 1$) *variance balanced block design with* $v=b$ *is a CBIB* (*Rao* [15]).

Pairwise balancedness is neither necessary nor sufficient for a block design to be variance balanced as it is demonstrated in the following theorem.

THEOREM 2.3. *If a design is pairwise* (*variance*) *balanced this does not imply that it is variance* (*pairwise*) *balanced.*

PROOF. By counter examples:

Example 4. Let $\Omega_1=\{1, 2, \cdots, 6\}$. Then, $B_1=\{1, 2, 4\}$, $B_2=\{2, 3, 5\}$, $B_3=\{4, 5\}$, $B_4=\{3, 4, 6\}$, $B_5=\{1, 5, 6\}$, $B_6=\{6, 2\}$, and $B_7=\{1, 3\}$ is pairwise balanced. However, it can be shown that this design is not variance balanced.

Example 5. Let $\Omega_2=\{1, 2, 3, 4, 5\}$. Then, the following design is variance balanced but not pairwise balanced: $B_1=\{1, 2, 3, 4\}$, $B_2=\{1, 2, 3, 4\}$, $B_3=\{1, 5\}$, $B_4=\{2, 5\}$, $B_5=\{3, 5\}$, and $B_6=\{4, 5\}$.

Besides the classical balanced incomplete block designs, there are other families of incomplete block designs which are balanced in both senses, as illustrated in Example 6.

Example 6. Let $\Omega_3=\{1, 2, 3, 4\}$. Then, $B_1=\{1, 2, 3\}$, $B_2=\{1, 2, 4\}$, $B_3=\{1, 3, 4\}$, $B_4=\{2, 3, 4\}$, $B_5=\{1, 2\}$, $B_6=\{1, 3\}$, $B_7=\{1, 4\}$, $B_8=\{2, 3\}$, $B_9=\{2, 4\}$ and $B_{10}=\{3, 4\}$.

Some results given by John [13] may be reformulated in the following two theorems:

THEOREM 2.4. *If* $v\equiv 0$ *or* 1 (mod 3) *there exists a family of trinary* ($n_{ij}=0, 1, 2$) *proper variance balanced incomplete block designs for* $v+1$ *treatments with two different replications.*

THEOREM 2.5. *If there exists a* BIB (v, b, r, k, λ) *such that* $\lambda=k/2$, $r=k(v-1)/2(k-1)$, *then there exists a family of binary* ($n_{ij}=0, 1$) *variance balanced incomplete block designs for* $v+1$ *treatments with two different replications and block sizes.*

3. Unionizing block designs

Let D_i be a BD $(v_i, b_i; r_{i1}, \cdots, r_{iv_i}; k_{i1}, \cdots, k_{ib_i}; N_i)$ design on Ω_i, $i=1, 2, \cdots, t$ (the case $D_i=D_j$, $i\neq j$ is not ruled out). Let $D=\bigcup_{i=1}^{t} D_i$ where $\bigcup$ denotes the union sign. Then D is a block design on Ω where $\Omega=\bigcup_{i=1}^{t} \Omega_i$. An interesting and practically useful problem in this area is to find a set of necessary and sufficient conditions on the D_i which makes D a variance balanced design. The solution of this problem in its general form is unknown and indeed appears to be very difficult. Here we give a partial solution to this problem.

THEOREM 3.1. *D is variance balanced if the D_i are variance balanced on the same set of treatments.*

The proof follows from the fact that the coefficient matrix of D is the sum of the coefficient matrices of the D_i. Note that this theorem produces a family of nonproper equireplicated n-ary variance balanced block designs.

Example 3.1. Let $D_1=\{1, 2\}, \{2, 3\}, \{1, 3\}$ and $D_2=\{1, 1, 2, 2, 3, 3\}, \{1, 2, 3\}$. Then $D=D_1\cup D_2$ is a variance balanced block design.

As we shall see shortly, neither the variance balancedness of all D_i nor the identicality of the r_i is necessary for D to be variance balanced. To establish this let D_i be a BIB$(v, b_1, r_1, k_1, \lambda_1)$ design on Ω_1, $i=1, 2, \cdots, \alpha$. Let ω be a treatment not included in Ω_1. Utilizing the augmentation idea of Das [8], Federer [10], and John [13] we augment every block of D_i with ω and call the resulting design $\bar{D}_i$, $i=1, 2, \cdots, \alpha$. Note that in general $\bar{D}_i$ is not variance balanced. Also, let $\hat{D}_j$ be a BIB$(v, b_2, r_2, k_2, \lambda_2)$ on Ω_1, $j=1, 2, \cdots, \beta$. Now let

$$D=\left(\bigcup_{i=1}^{\alpha} \bar{D}_i\right) \cup \left(\bigcup_{j=1}^{\beta} \hat{D}_j\right) .$$

THEOREM 3.2. *D is variance balanced irrespective of the parameters of D_i and $\hat{D}_j$ for the choice of*

$$\alpha=\lambda_2(k_1+1)/d , \qquad \beta=k_2(r_1-\lambda_1)/d ,$$

where d is the greatest common divisor of $\lambda_2(k_1+1)$ and $k_2(r_1-\lambda_1)$. Moreover, these values are the minimum values that α and β can take to guarantee the variance balancedness of D.

PROOF. Let N_1 be the incidence matrix of D_i and N_2 be the incidence matrix of $\hat{D}_j$. Then, with no loss of generality, the incidence matrix of $\bar{D}_i$ can be written as $\left(\begin{array}{c} N_1 \\ \hdashline 1 \end{array}\right)$ where 1 denotes a row vector of ones. With this notation the incidence matrix of D can be written as

$$N=\left(\begin{array}{c:c:c:c:c:c:c:c} N_1 & N_1 & \cdots & N_1 & N_2 & N_2 & \cdots & N_2 \\ \hdashline 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}\right)$$
$$\underbrace{\qquad\qquad\qquad}_{\alpha \text{ times}} \qquad \underbrace{\qquad\qquad\qquad}_{\beta \text{ times}}$$

where 0 denotes a row vector of zeros. Therefore, the coefficient matrix of D will be

$$C=R-NK^{-1}N'$$
$$=\begin{bmatrix} \left(\frac{\alpha k_1 r_1}{k_1+1}+\frac{\beta(k_2-1)r_2}{k_2}+\frac{\alpha\lambda_1}{k_1+1}+\frac{\beta\lambda_2}{k_2}\right)I-\left(\frac{\alpha\lambda_1}{k_1+1}+\frac{\beta\lambda_2}{k_2}\right)J & \frac{-\alpha}{k_1+1}1' \\ \frac{-\alpha}{k_1+1}1 & \frac{\alpha b_1 k_1}{k_1+1} \end{bmatrix} .$$

Hence D is variance balanced if and only if

$$\begin{bmatrix} k_1(b_1-r_1)(k_1+1)^{-1} & -r_2(k_2-1)k_2^{-1} \\ (r_1-\lambda_1)(k_1+1)^{-1} & -\lambda_2 k_2^{-1} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} .$$

Thus our problem is solved if the above 2×2 matrix is not of full rank and the corresponding nontrivial solutions for α and β are nonnegative integers. Utilizing the combinatorial identities associated with D_i and $\hat{D}_j$ it is easy to see that the determinant of the above matrix is zero. Moreover, $\alpha=\lambda_2(k_1+1)/d$ and $\beta=k_2(r_1-\lambda_1)/d$ are the minimum integer solutions. Q.E.D.

We remark that for the choice of $k_2\neq k_1+1$, the design D has two different replications and two different block sizes, and thus we have

COROLLARY 3.1. *The existence of a* BIB$(v, b_1, r_1, k_1, \lambda_1)$ *and a* BIB$(v, b_2, r_2, k_2, \lambda_2)$, $k_2>k_1+1$ *implies the existence of a binary* $(n_{ij}=0, 1)$ *variance balanced incomplete block design for* $v+1$ *treatments with two different replications and two different block sizes (see also Hedayat and Federer* [12]).

Example 3.2. If we choose D_1 and $\hat{D}_1$ to be the following designs: $D_1=\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$, and $\hat{D}_1=\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}$, then, since $\alpha=2$ and $\beta=1$, we have for $\omega=5$ the following variance balanced block design with two different replications (8 and 9) and two different block sizes (2 and 4) for five treatments. $D=\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \{2, 3, 4, 5\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}$.

4. Concluding remarks

It is essential that the reader of published literature understand the kind of balance being discussed, as there are many kinds of balance. Unfortunately, the writers do not always make this clear. We have distinguished between two kinds of balance, pairwise and variance, in the preceding and have precisely defined both kinds.

The unionizing block designs method may be utilized to obtain pairwise balanced block designs which are unequally replicated and have unequal block sizes in much the same manner as utilized to obtain variance balanced designs. Also, $2, 3, \cdots$ additional treatments may be added to the v treatments; for example, using block designs D_1, D_2, D_3, and D_4 with treatment $v+1$ added to D_1, treatment $v+2$ to D_2, and both treatments $v+1$ and $v+2$ added to D_3, then select α_1, α_2, α_3, and α_4 such that $\alpha_1 D_1 \cup \alpha_2 D_2 \cup \alpha_3 D_3 \cup \alpha_4 D_4 = D$ is variance balanced. A third extension would be to utilize partially balanced incomplete block designs together with balanced incomplete block designs to obtain $2, 3, \cdots$ different variances for the treatment contrasts for unequally replicated block designs with unequal block sizes. Still another extension could be to write a computer program to obtain the values of α and β for unionizing all CBIB designs. Combining this process for a number of types of unionizing, a catalogue of unequally replicated variance balanced designs with unequal block sizes could be constructed.

The authors interest in pursuing these problems is marginal at the present time, as we are primarily interested in presenting the unionizing block designs method and not in its application. Students in design theory are urged to pursue some of these problems.

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* A. Hedayat is now at the Department of Statistics, Florida State University, Tallahassee, Fla.