ASYMPTOTIC EXPANSIONS OF THE NON-NULL DISTRIBUTIONS OF THREE STATISTICS IN GMANOVA*

YASUNORI FUJIKOSHI

(Received Nov. 2, 1972)

1. Introduction

Let us consider the GMANOVA model introduced by Potthoff and Roy [11]. Matrix Y is an $N\times p$ observation with independent rows each distributed as $N_p(\cdot, \Psi)$ and $\mathrm{E}Y=A\Xi B$, where A: $N\times k$ and B: $q\times p$ are known matrices of ranks k and q respectively and Ξ: $k\times q$ is a matrix of unknown parameters. The hypothesis to be tested is H_0: $U\Xi V=0$ against H_1: $U\Xi V\neq 0$, where U: $u\times k$ and V: $q\times v$ are known matrices of ranks u and v, respectively. We assume that $n=N-k-p+q\geqq v$. The likelihood ratio (=LR) statistic for this problem has been obtained by Gleser and Olkin [5], Khatri [8] and Rao [12] in different forms. In this paper we use the following canonical forms for this problem due to [5]: Let X: $N\times p$ be a random matrix with independent rows distributed as $N_p(\cdot, \Sigma)$ and

$$(1.1)\qquad \mathrm{E}\,X=\mathrm{E}\begin{bmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{bmatrix}\begin{matrix} u \\ k-u \\ N-k \end{matrix} = \begin{bmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} . \qquad \begin{matrix} q-v & v & p-q \end{matrix}$$

The hypotheses are H_0: $\xi_{12}=0$ and H_1: $\xi_{12}\neq 0$. Then, the LR statistic can be expressed as an increasing function of

$$(1.2)\qquad T_1=|S_e|/|S_h+S_e|$$

where the matrices of S_e and S_h, corresponding to the sums of products matrices due to error and due to the hypothesis respectively, are defined by

$$(1.3)\qquad \begin{aligned} S_e &= W_{22}-W_{23}W_{33}^{-1}W_{32}\,, \\ S_h &= (X_{12}-RW_{33}^{-1}W_{32})'(I+RW_{33}^{-1}R')^{-1}(X_{12}-RW_{33}^{-1}W_{32}) \end{aligned}$$

* This research was supported in part by the Sakkokai Foundation.

where $R=X_{13}$ and $W_{ij}=X_{3i}'X_{3j}$. In addition to the LR statistic, the following test statistics have been proposed as in the usual MANOVA models:

$$
\begin{aligned}
&T_2=\operatorname{tr} S_hS_e^{-1}\,,\\
(1.4)\qquad &T_3=\operatorname{tr} S_h(S_h+S_e)^{-1}\,,\\
&T_4=\text{the largest root of } S_hS_e^{-1}\,.
\end{aligned}
$$

The exact non-null distributions of these statistics have not been studied, except for $u=1$ (c.f. Olkin and Shrikhande [10]). Generally, the non-null distribution of the characteristic roots of $S_hS_e^{-1}$ depends only on n, u, v, $p-q$ and the characteristic roots $d_1\geqq d_2\geqq\cdots\geqq d_v\geqq 0$ of

$$
\Omega=\frac{1}{2}\Sigma_{22\cdot 3}^{-1/2}\xi_{12}'\xi_{12}\Sigma_{22\cdot 3}^{-1/2}\,,
$$

where

$$
(1.5)\qquad \Sigma=\begin{bmatrix}\Sigma_{11} & \Sigma_{12} & \Sigma_{13}\\ \Sigma_{21} & \Sigma_{22} & \Sigma_{23}\\ \Sigma_{31} & \Sigma_{32} & \Sigma_{33}\end{bmatrix},\qquad \Sigma_{22\cdot 3}=\Sigma_{22}-\Sigma_{23}\Sigma_{33}^{-1}\Sigma_{32}\,,
$$

Σ_{11}: $(q-v)\times(q-v)$ and Σ_{22}: $v\times v$. In this paper we study the asymptotic $(n\to\infty)$ non-null distributions of T_1, T_2 and T_3 under the assumption that u, v and $p-q$ are fixed and $\Omega=O(1)$ or $O(n)$. When $\Omega=O(1)$, asymptotic expansions of the non-null distribution of these statistics are derived in terms of noncentral χ^2-distributions up to the order n^{-2}. Under $\Omega=O(n)$ we derive asymptotic expansions for these statistics in terms of normal distribution function and its derivatives up to the order n^{-1}. We note that in the situation when $B=I$ and $V=I$, the statistics T_i are reduced to the ordinary test statistics in a MANOVA model and the asymptotic distributions have been treated by Sugiura and Fujikoshi [14], Siotani [12], Lee [9], Fujikoshi [2], [3] and Sugiura [15].

2. Preliminaries

First we state that the distributions of test statistics in a GMANOVA model are treated as those of test statistics in a MANOVA model by considering the conditional distributions of S_e and S_h given R and $W_{33}=W$.

LEMMA 1 (Fujikoshi [4]). *If one considers the distributions of the tests based on the roots of $S_hS_e^{-1}$, we may assume that given R and W,*

(i) *S_e has a central Wishart distribution $W_v(I, n)$,*

(ii) *S_h has a noncentral (possibly singular) Wishart distribution $W_v(I, u, \Delta)$ with the noncentrality matrix given by $\Delta=\eta'(I+RW^{-1}R')^{-1}\eta$, where η is a $u\times v$ matrix whose i-th diagonal element is $\sqrt{d_i}$ for $i=1, 2, \cdots, \min(u, v)$ and other elements are zero,*

(iii) *S_e and S_h are independent,*

(iv) *The rows of R are independently distributed as $N_{p-q}(0, I)$ and W has a central Wishart distribution $W_{p-q}(I, N-k)$.*

Khatri ([8]; p. 79, (26)) has obtained the similar conditional set-up as in Lemma 1. However, the distribution of Y_3 in his reduction is more complicated than that of R in the present result. The following lemma is a slight modification of a well known result due to James [7] and Ito [6].

LEMMA 2. *Let S have a central Wishart distribution $W_p(\Lambda, n)$ and let $f(S)$ be a real valued function of S and analytic about $S=\Lambda$, where $\Lambda=\mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_p)$. Put $m+d=n$, where d is a fixed constant. Then, for large n we have*

$$\mathrm{E}\left[f\left(\frac{1}{m}S\right)\right]=\left[1+\frac{1}{m}\{d\,\mathrm{tr}\,\Lambda\partial+\mathrm{tr}\,(\Lambda\partial)^2\}+O(m^{-2})\right]f(\Sigma)\bigg|_{\Sigma=\Lambda}, \tag{2.1}$$

where ∂ denotes the matrix of differential operators having $\{(1+\delta_{rs})/2\}\partial/\partial\sigma_{rs}$ as (r, s) element for a symmetric matrix $\Sigma=[\sigma_{rs}]$ with Kronecker's delta δ_{rs}.

3. Asymptotic expansions in the situation when $\Omega=O(1)$

Under the assumption that u, v, $p-q$ and Ω are fixed we derive asymptotic expansions of the non-null distributions of T_1, T_2 and T_3 with respect to $m=\rho n$. The correction factor $\rho=O(1)$ may be chosen as in a MANOVA model, since from Lemma 1 the null distributions are the same as in a MANOVA model. The final result is as follows:

THEOREM 1. *Under $\Omega=O(1)$ the following asymptotic formulas for the non-null distributions of T_1, T_2 and T_3 hold:*

$$\begin{aligned}\mathrm{P}(\lambda<x)=\mathrm{P}(\chi^2_f(\delta^2)<x)&+\frac{1}{4m}\sum_{\alpha=0}^{4}A_\alpha(\Omega)\,\mathrm{P}(\chi^2_{f+2\alpha}(\delta^2)<x)\\&+\frac{1}{96m^2}\sum_{\alpha=0}^{8}B_\alpha(\Omega)\,\mathrm{P}(\chi^2_{f+2\alpha}(\delta^2)<x)\\&+\frac{\mu}{m}\,\mathrm{tr}\,\Omega\{\mathrm{P}(\chi^2_f(\delta^2)<x)-\mathrm{P}(\chi^2_{f+2}(\delta^2)<x)\}\end{aligned} \tag{3.1}$$

$$+\frac{\mu}{4m^2}\sum_{\alpha=0}^{5} G_\alpha(\Omega)\,\mathrm{P}\,(\chi^2_{f+2\alpha}(\delta^2)<x)+O(m^{-3})\,,$$

where $f=uv$, $\delta^2=\operatorname{tr}\Omega$, λ *and* m *are defined as follows:*

	T_1	T_2	T_3
λ	$-m\log T_1$	mT_2	mT_3
m	$n+(u-v-1)/2$	$n-v-1$	$n+u$

The coefficients $A_\alpha(\Omega)$, $B_\alpha(\Omega)$ *and* $G_\alpha(\Omega)$ *are given in Appendix* 1 *by using the notations* $\beta=u^2+v^2-5$, $\gamma=u+v+1$, $\mu=p-q$ *and* $\omega_j=\operatorname{tr}\Omega^j$.

An appropriateness for such a definition of m in the case of T_2 and T_3 has been shown in Fujikoshi [3]. We shall only prove the formula (3.1) in the case of LR statistic, since the expansions of T_2 and T_3 are similarly derived. From Sugiura and Fujikoshi [14], Lemma 1 and the fact that $\Delta=O_p(1)$, the conditional characteristic function $C_\lambda(t|R, W)$ of $\lambda=-m\log T_1$ when R and W are given can be expressed as

$$(3.2)\qquad C_\lambda(t|R, W)=\phi^{f/2}e^{\{2it/(1-2it)\}\operatorname{tr}\Delta}\cdot\left[1+\frac{1}{4m}\sum_{\alpha=0}^{4}A_\alpha(\Delta)\phi^\alpha+\frac{1}{96m^2}\sum_{\alpha=0}^{8}B_\alpha(\Delta)+O_p(m^{-3})\right],$$

where $\phi=(1-2it)^{-1}$. Noting that

$$(3.3)\qquad \Delta=\Omega-\frac{1}{m}\eta' R\left(\frac{1}{m}W\right)^{-1}R'\eta+\frac{1}{m^2}\eta'\left\{R\left(\frac{1}{m}W\right)^{-1}R'\right\}^2\eta +O(m^{-3})f_1\left(\frac{1}{m}W, R\right),$$

we can write (3.2) as

$$(3.4)\qquad \phi^{f/2}e^{\{2it/(1-2it)\}\operatorname{tr}\Omega}\Bigg[1+\frac{1}{4m}\sum_{\alpha=0}^{4}A_\alpha(\Omega)\phi^\alpha +\frac{1}{96m^2}\sum_{\alpha=0}^{8}B_\alpha(\Omega)\phi^\alpha+\frac{1}{m}(1-\phi)\operatorname{tr}\eta' R\left(\frac{1}{m}W\right)^{-1}R'\eta +\frac{1}{4m^2}\Bigg\{\left(\sum_{\alpha=0}^{4}A_\alpha(\Omega)\phi^\alpha-2\gamma\phi\right)(1-\phi)\operatorname{tr}\eta' R\left(\frac{1}{m}W\right)^{-1}R'\eta +8\phi^2(\phi-1)\operatorname{tr}\Omega\eta' R\left(\frac{1}{m}W\right)^{-1}R'\eta+2(\phi-1)^2\left(\operatorname{tr}\eta' R\left(\frac{1}{m}W\right)^{-1}R'\eta\right)^2 +4(\phi-1)\operatorname{tr}\eta'\left(R\left(\frac{1}{m}W\right)^{-1}R'\right)^2\eta\Bigg\}+O(m^{-3})f_2\left(\frac{1}{m}W, R\right)\Bigg].$$

The unconditional characteristic function of λ is obtained by taking the expectation of (3.4) with respect to R and W. First we evaluate the expectaion with respect to W, which can be obtained by Lemma 2 and using the result that

$$\text{(3.5)}\qquad \begin{aligned} &\operatorname{tr}\partial \operatorname{tr}\eta' R\Sigma^{-1}R'\eta\Big|_{\Sigma=I} = -\operatorname{tr}\eta' RR'\eta\,, \\ &\operatorname{tr}\partial^2 \operatorname{tr}\eta' R\Sigma^{-1}R'\eta\Big|_{\Sigma=I} = (\mu+1)\operatorname{tr}\eta' RR'\eta\,. \end{aligned}$$

The expectation with respect to R can be evaluated by using the formulas,

$$\text{(3.6)}\qquad \begin{aligned} &\mathrm{E}\,[\operatorname{tr}\eta' RR'\eta] = \mu\omega_1\,, \\ &\mathrm{E}\,[\{\operatorname{tr}\eta' RR'\eta\}^2] = \mu(\mu\omega_1^2 + 2\omega_2\}\,, \\ &\mathrm{E}\,[\operatorname{tr}\eta'(RR')^2\eta] = \mu(\mu+u+1)\omega_1\,, \\ &\mathrm{E}\,[\operatorname{tr}\Omega\eta' RR'\eta] = \mu\omega_2\,, \end{aligned}$$

which are easily obtained by straight-forward computations. After much simplification, we obtain the expansion of the characteristic function of λ which may be inverted, using the well known fact that $\phi^{f/2}e^{\{2it/(1-2it)\}\delta^2}$ is the characteristic function of noncentral χ^2-variate with f degrees of freedom and noncentrality parameter δ^2, to yield (3.1) in the case of LR statistic.

When $B=I$ and $V=I$ (accordingly $p=q=v$) in (3.1), we obtain the asymptotic expansions of the non-null distributions of LR statistic, Hotelling's criterion and $\operatorname{tr} S_h(S_h+S_e)^{-1}$ in a MANOVA model, which have been obtained by Sugiura and Fujikoshi [14], Fujikoshi [3] and [2], respectively.

4. Asymptotic expansions in the situation when $\Omega=O(n)$

In this section we assume that $\eta=\sqrt{m}\,L$, where L: $u\times v$ is a fixed matrix and m is the same as in the previous section. Then, Ω is of the form $\Omega=mL'L=m\theta$. Under this assumption Sugiura [15] has derived asymptotic expansions of the non-null distributions of T_1, T_2 and T_3 in the usual MANOVA model. Using his result and the similar technique as in the previous section, we can derive asymptotic formulas in a GMANOVA model. For example, the conditional characteristic function of $\hat{\lambda}=-\sqrt{m}\,\{\log T_1+\log|I+2\theta|\}$ when R and W are given can be expressed as

$$\text{(4.1)}\qquad \mathrm{E}\,[e^{-\sqrt{m}\,it\{\log T_2+\log|I+2D|\}}|R,\, W]e^{it\sqrt{m}\{\log|I+2D|-\log|I+2\theta|\}}$$

with $D=L'(I+RW^{-1}R')^{-1}L$, which is $O_p(1)$. The first factor has been expanded in [15] as

$$(4.2)\quad e^{-t^2/2\sigma(D)^2}\left[1+\frac{1}{\sqrt{m}}\sum_{\alpha=1}^{2}(it)^{2\alpha-1}U_{2\alpha-1}(D)+\frac{1}{m}\sum_{\alpha=1}^{3}(it)^{2\alpha}V_{2\alpha}(D)+O_p(m^{-3/2})\right],$$

where $\sigma(D)^2=2\{v-\mathrm{tr}\,(I+2D)^{-2}\}$, the coefficients $U_{2\alpha-1}(D)$ and $V_{2\alpha}(D)$ are given in Appendix 2 with a little simplification. Noting that

$$D=\theta-m^{-1}L'R(m^{-1}W)^{-1}R'L+O(m^{-2})f(m^{-1}W, R),$$

and making similar calculations as in the expansion of (3.2), we can obtain the expansion of the unconditional characteristic function of $\hat{\lambda}$ which may be inverted, using the fact that

$$\int_{-\infty}^{\infty}\exp(itx)d\Phi^{(r)}(x)=(-it)^r e^{-t^2/2},$$

where $\Phi^{(r)}(x)$ is the r-th derivative of the standard normal distribution function $\Phi(x)$. It is clear that the above consideration holds also for T_2 and T_3. Hence we have the following:

THEOREM 2. *Under $\Omega=m\theta$, the following asymptotic formulas for the nonnull distributions of T_1, T_2 and T_3 hold:*

$$\begin{aligned}(4.3)\quad \mathrm{P}\,(\hat{\lambda}/\sigma<x)=\Phi(x)&-\frac{1}{\sqrt{m}}\sum_{\alpha=1}^{2}U_{2\alpha-1}(\theta)\Phi^{(2\alpha-1)}(x)\sigma^{-2\alpha+1}\\&+\frac{1}{m}\sum_{\alpha=1}^{3}V_{2\alpha}(\theta)\Phi^{(2\alpha)}(x)\sigma^{-2\alpha}-\frac{\mu}{\sqrt{m}}H(\theta)\Phi^{(1)}(x)\sigma^{-1}\\&+\frac{\mu}{m}\sum_{\alpha=1}^{2}W_{2\alpha}(\theta)\Phi^{(2\alpha)}(x)\sigma^{-2\alpha}+O(m^{-3/2}),\end{aligned}$$

with $\mu=p-q$ and $\sigma=\sqrt{\sigma^2}$, where $\hat{\lambda}$ and σ^2 are defined as follows:

	T_1	T_2	T_3
$\hat{\lambda}$	$-\sqrt{m}\{\log T_1+\log\|I+2\theta\|\}$	$\sqrt{m}\{T_2-\mathrm{tr}\,2\theta\}$	$\sqrt{m}\{T_3-\mathrm{tr}\,2\theta(I+2\theta)^{-1}\}$
σ^2	$2(v-c_2)$	$8(t_1+t_2)$	$2(c_2-c_4)$

with $c_j=\mathrm{tr}\,C^j=\mathrm{tr}\,(I+2\theta)^{-j}$ and $t_j=\mathrm{tr}\,\theta^j$. The coefficients $U_{2\alpha-1}(\theta)$, $V_{2\alpha}(\theta)$, $H(\theta)$ and $W_{2\alpha}(\theta)$ are given in Appendix 2 by using the notations $f=uv$, $\gamma=u+v+1$ and $s_j=\mathrm{tr}\,Q^j=\mathrm{tr}\,(I-C)^j$.

From Theorem 2 it follows that the limiting $(n\to\infty)$ distribution of $\hat{\lambda}$ is a normal distribution with mean 0 and variance σ^2. It is easily seen that this result holds also for the case when $m=\rho n$ and $\Omega=\tilde{m}\theta$ with $\tilde{m}=\tilde{\rho}n$, where $\rho=O(1)$ and $\tilde{\rho}=O(1)$. We note that Gleser and

Olkin [5] have treated the limiting distribution of LR statistic in the case of $\rho=\tilde{\rho}=1+(k+p-q)/n$, which is proved to be normal, but the asymptotic variance is incorrect.

Appendix 1

Table of the coefficients $A_\alpha(\Omega)$, $B_\alpha(\Omega)$ and $G_\alpha(\Omega)$ in (3.1)

	T_1	T_2	T_3
$A_0(\Omega)$	0	$f\gamma$	$-f\gamma$
$A_1(\Omega)$	$2\gamma\omega_1$	$-2\gamma(f-2\omega_1)$	$2f\gamma$
$A_2(\Omega)$	$-2\gamma\omega_1+4\omega_2$	$f\gamma-8\gamma\omega_1+4\omega_2$	$-f\gamma+4\gamma\omega_1+4\omega_2$
$A_3(\Omega)$	$-4\omega_2$	$4(\gamma\omega_1-2\omega_2)$	$-4\gamma\omega_1$
$A_4(\Omega)$	0	$4\omega_2$	$-4\omega_2$
$B_0(\Omega)$	$-2f\beta$	fl_0	fl_0
$B_1(\Omega)$	0	$l_1(f-2\omega_1)$	fl_1
$B_2(\Omega)$	$2\{f\beta-12\gamma^2\omega_1+6\gamma^2\omega_1^2$ $+24\gamma\omega_2\}$	$fl_2+2(l_1-2l_2)\omega_1+48\gamma^2\omega_1^2$ $+24(f+4)\gamma\omega_2$	$fl_2+2l_1\omega_1-24f\gamma\omega_2$
$B_3(\Omega)$	$8\{3\gamma^2\omega_1-3(\gamma^2+4)\omega_1^2$ $-12(2\gamma+1)\omega_2+6\gamma\omega_1\omega_2$ $+16\omega_3\}$	$fl_3+2(2l_2-3l_3)\omega_1$ $-192(\gamma^2+1)\omega_1^2$ $-96\{(f+8)\gamma+2\}\omega_2$ $+96\gamma\omega_1\omega_2+128\omega_3$	$fl_3+4l_2\omega_1+48(f+4)\gamma\omega_2$ $+128\omega_3$
$B_4(\Omega)$	$12\{(\gamma^2+8)\omega_1^2+4(3\gamma+2)\omega_2$ $-8\gamma\omega_1\omega_2-32\omega_3+4\omega_2^2\}$	$fl_4+2(3l_3-4l_4)\omega_1$ $+96(3\gamma^2+7)\omega_1^2$ $+48\{3(f+12)\gamma+14\}\omega_2$ $-384\gamma\omega_1\omega_2-768\omega_3+48\omega_2^2$	$fl_4+6l_3\omega_1+48(\gamma^2-2)\omega_1^2$ $-96(\gamma+1)\omega_2+96\gamma\omega_1\omega_2$ $+48\omega_2^2$
$B_5(\Omega)$	$16\{3\gamma\omega_1\omega_2+16\omega_3-6\omega_2^2\}$	$8[l_4\omega_1-24(\gamma^2+4)\omega_1^2$ $-12\{(f+16)\gamma+8\}\omega_2$ $+72\gamma\omega_1\omega_2+192\omega_3$ $-24\omega_2^2]$	$8[l_4\omega_1-12(\gamma^2+2)\omega_1^2$ $-6\{(f+12)\gamma+4\}\omega_2$ $-12\gamma\omega_1\omega_2-48\omega_3]$
$B_6(\Omega)$	$48\omega_2^2$	$8[6(\gamma^2+6)\omega_1^2+3\{(f+20)\gamma$ $+12\}\omega_2-48\gamma\omega_1\omega_2$ $-160\omega_3+36\omega_2^2]$	$8[6(\gamma^2+6)\omega_1^2+3\{(f+20)\gamma$ $+12\}\omega_2-12\gamma\omega_1\omega_2$ $-16\omega_3-12\omega_2^2]$
$B_7(\Omega)$	0	$96\{\gamma\omega_1\omega_2+4\omega_3-2\omega_2^2\}$	$96\{\gamma\omega_1\omega_2+4\omega_3\}$
$B_8(\Omega)$	0	$48\omega_2^2$	$48\omega_2^2$
$G_0(\Omega)$	$2\{-(2\mu+\gamma)\omega_1+\mu\omega_1^2+2\omega_2\}$	$\{(f-4)\gamma-4\mu\}\omega_1+2\mu\omega_1^2$ $+4\omega_2$	$-(f\gamma+4\mu)\omega_1+2\mu\omega_1^2+4\omega_2$
$G_1(\Omega)$	$2\{2\mu\omega_1+(\gamma-2\mu)\omega_1^2-4\omega_2\}$	$(-3f\gamma+4\mu)\omega_1+4(\gamma-\mu)\omega_1^2$ $-8\omega_2$	$(3f\gamma+4\mu)\omega_1-4\mu\omega_1^2-8\omega_2$

	T_1	T_2	T_3
$G_2(\Omega)$	$2\{\gamma\omega_1+(\mu-2\gamma)\omega_1^2-2\omega_2$ $+2\omega_1\omega_2\}$	$(3f+8)\gamma\omega_1+2(\mu-6\gamma)\omega_1^2$ $-4\omega_2+4\omega_1\omega_2$	$-(3f+4)\gamma\omega_1+2(\mu+2\gamma)\omega_1^2$ $-4\omega_2+4\omega_1\omega_2$
$G_3(\Omega)$	$2\{\gamma\omega_1^2+4\omega_2-4\omega_1\omega_2\}$	$-(f+4)\gamma\omega_1+12\gamma\omega_1^2+16\omega_2$ $-12\omega_1\omega_2$	$(f+4)\gamma\omega_1-8\gamma\omega_1^2-4\omega_1\omega_2$
$G_4(\Omega)$	$4\omega_1\omega_2$	$4(-\gamma\omega_1^2-2\omega_2+3\omega_1\omega_2)$	$4(\gamma\omega_1^2+2\omega_2-\omega_1\omega_2)$
$G_5(\Omega)$	0	$-4\omega_1\omega_2$	$4\omega_1\omega_2$

where l_α $(\alpha=0, 1, \cdots, 4)$ are given by

$$l_0=(3f-8)\gamma^2+4\gamma+4(f+2),\quad l_1=-12f\gamma^2,\quad l_2=6(3f+8)\gamma^2,$$
$$l_3=-4\{(3f+16)\gamma^2+4\gamma+4(f+2)\},\quad l_4=3\{(f+8)\gamma^2+4\gamma+4(f+2)\}.$$

Appendix 2

Table of the coefficients $U_{2\alpha-1}(\theta)$, $V_{2\alpha}(\theta)$, $H(\theta)$ and $W_{2\alpha}(\theta)$ in (4.3)

	T_1	T_2	T_3
$U_1(\theta)$	$\frac{1}{2}(2f-\gamma s_1+s_1^2+s_2)$	f	$-f+\gamma s_1-2s_1^2-2s_2+s_1s_2+s_3$
$U_3(\theta)$	$4s_1-8s_2+\frac{20}{3}s_3-2s_4$	$8\left(t_1+4t_2+\frac{8}{3}t_3\right)$	$\frac{4}{3}\operatorname{tr} QC^3(-3I+12Q-11Q^2$ $+3Q^3)$
$V_2(\theta)$	$\frac{1}{2}U_1(\theta)^2+f-2\gamma s_1+4s_1^2$ $+\frac{1}{2}(5\gamma+8)s_2-6s_1s_2$ $-(\gamma+6)s_3+\frac{1}{2}s_2^2+2s_1s_3$ $+\frac{5}{2}s_4$	$\frac{1}{2}U_1(\theta)^2+f+4\gamma t_1+4t_1^2+4t_2$	$\frac{1}{2}U_1(\theta)^2+f-6\gamma s_1+(11\gamma$ $+23)s_2-2(4\gamma+33)s_3$ $+2(\gamma+38)s_4-40s_5+8s_6$ $+23s_1^2-66s_1s_2+56s_1s_3$ $+20s_2^2-24s_1s_4-16s_2s_3$ $+3s_2s_4+4s_1s_5+s_3^2$
$V_4(\theta)$	$U_1(\theta)U_3(\theta)+8s_1-32s_2$ $+\frac{184}{3}s_3-62s_4+32s_5$ $-\frac{20}{3}s_6$	$U_1(\theta)U_3(\theta)+16(t_1+10t_2$ $+20t_3+10t_4)$	$U_1(\theta)U_3(\theta)+2\operatorname{tr} QC^4(4I$ $-40Q+112Q^2-127Q^3$ $+64Q^4-12Q^5)$
$V_6(\theta)$	$\frac{1}{2}U_3(\theta)^2$	$\frac{1}{2}U_3(\theta)^2$	$\frac{1}{2}U_3(\theta)^2$
$H(\theta)$	$-s_1$	$-2t_1$	$-(s_1-s_2)$
$W_2(\theta)$	$-(2+U_1(\theta))s_1+\frac{\mu}{2}s_1^2+5s_2$ $-2s_3$	$-2(2+U_1(\theta))t_1+2\mu t_1^2-4t_2$	$-(2+U_1(\theta))(s_1-s_2)+\frac{\mu}{2}s_1^2$ $+11s_2-\mu s_1s_2-24s_3$ $+\frac{\mu}{2}s_2^2+17s_4-4s_5$
$W_4(\theta)$	$-s_1U_3(\theta)$	$-2t_1U_3(\theta)$	$-(s_1-s_2)U_3(\theta)$

KOBE UNIVERSITY

REFERENCES

[1] Anderson, T. W. (1946). The noncentral Wishart distribution and certain problems of multivariate statistics, *Ann. Math. Statist.*, **17**, 409-431.

[2] Fujikoshi, Y. (1970). Asymptotic expansions of the distributions of test statistics in multivariate analysis, *J. Sci. Hiroshima Univ.*, Ser. A-I, **34**, 73-144.

[3] Fujikoshi, Y. (1973). Asymptotic formulas for the distributions of three statistics for multivariate linear hypothesis, *Ann. Inst. Statist. Math.*, **25**, 423-437.

[4] Fujikoshi, Y. (1973). Monotonicity of the power functions of some tests in general MANOVA models, *Ann. Statist.*, **1**, 388-391.

[5] Gleser, L. J. and Olkin, I. (1970). Linear models in multivariate analysis, *Essays in probability and statistics*, 267-292, The University of North Carolina Monograph Series (No. 3) in Probability and Statistics.

[6] Ito, K. (1956). Asymptotic formulae for the distribution of Hotelling's generalized T_0^2 statistics, *Ann. Math. Statist.*, **27**, 1091-1105.

[7] James, G. S. (1954). Test of linear hypothesis in univariate and multivariate analysis when the ratios of the population variances are unknown, *Biometrika*, **41**, 19-43.

[8] Khatri, C. G. (1966). A note on a MANOVA model applied to problems in growth curve, *Ann. Inst. Statist. Math.*, 18, 75-85.

[9] Lee, Y. S. (1971). Distribution of the canonical correlations and asymptotic expansions for the distributions of certain independence test statistics, *Ann. Math. Statist.*, **42**, 526-537.

[10] Olkin, I. and Shrikhande, S. S. (1954). On a modified T^2 problem, *Ann. Math. Statist.*, **25**, 808.

[11] Potthoff, R. F. and Roy, S. N. (1964). A general multivariate analysis of variance model useful especially growth curve problems, *Biometrika*, **51**, 313-326.

[12] Rao, C. R. (1966). Covariance adjustment and related problems in multivariate analysis, *Multivariate analysis*, edited by P. R. Krishnaiah, 87-102, Academic Press, New York.

[13] Siotani, M. (1971). An asymptotic expansion of the non-null distribution of Hotelling's T_0^2-statistic, *Ann. Math. Statist.*, **42**, 560-571.

[14] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence, *Ann. Math. Statist.*, **40**, 942-952.

[15] Sugiura, N. (1973). Further asymptotic formulas for the non-null distributions of three statistics for multivariate linear hypothesis, *Ann. Inst. Statist. Math.*, **25**, 153-163.