ASYMPTOTIC EXPANSION OF THE NON-NULL DISTRIBUTION OF THE RATIO OF TWO CONDITIONALLY INDEPENDENT HOTELLING'S T_0^2-STATISTICS

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1. Introduction and summary

The T_0^2-statistic was introduced by Hotelling [1], [2] as a measure of multivariate dispersion in connection with the problem of testing the accuracy of bombsights. A problem of considerable importance is to work out the sampling distribution of the ratio of two values of T_0^2 which are conditionally independent for fixed values of the variance covariance estimates.

In general, let $\boldsymbol{Z}_1=\{\boldsymbol{z}_{11}, \boldsymbol{z}_{12}, \cdots, \boldsymbol{z}_{1m_1}\}$ be a $p\times m_1$ random matrix where $\boldsymbol{z}_{1i}$ are independently distributed according to p-variate normal distributions with means $\boldsymbol{\mu}_i$ and common covariance matrix $\boldsymbol{\Lambda}=(\lambda_{ij})$ (>0, positive definite) and let $\boldsymbol{Z}_2=\{\boldsymbol{z}_{21}, \boldsymbol{z}_{22}, \cdots, \boldsymbol{z}_{2m_2}\}$ be a $p\times m_2$ random matrix where $\boldsymbol{z}_{2j}$ are independently distributed according to p-variate normal distributions with zero means and common covariance matrix $\boldsymbol{\Lambda}=(\lambda_{ij})$ and $\boldsymbol{Z}_1$ is independent of $\boldsymbol{Z}_2$. Let $n\boldsymbol{S}_n=n(s_{ij})$ be a $p\times p$ matrix which is independent of $\boldsymbol{Z}_1$ and $\boldsymbol{Z}_2$ and is subject to a central Wishart distribution $W_p(n, \boldsymbol{\Lambda})$ with n degrees of freedom and covariance matrix $\boldsymbol{\Lambda}$. The statistic considered is the ratio of two Hotelling's generalized T_0^2-statistics and is defined by

$$F_0\equiv\frac{\operatorname{tr}\boldsymbol{S}_n^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'}{\operatorname{tr}\boldsymbol{S}_n^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'}=\frac{\sum_{i=1}^{m_1}\boldsymbol{z}_{1i}'\boldsymbol{S}_n^{-1}\boldsymbol{z}_{1i}}{\sum_{j=1}^{m_2}\boldsymbol{z}_{2j}'\boldsymbol{S}_n^{-1}\boldsymbol{z}_{2j}}, \tag{1.1}$$

which is a statistic proposed by Hotelling [1], [2] for testing the hypothesis $H:\boldsymbol{M}=\{\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots, \boldsymbol{\mu}_{m_1}\}=\boldsymbol{0}$ against $K:\boldsymbol{M}\neq\boldsymbol{0}$.

Percentage points of the distribution of F_0 when H is true has been

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treated by Siotani [3]. Even in the null case, the exact distribution of F_0 is not available.

In this paper an asymptotic expansion of the non-null distribution of F_0 is given up to the terms of order n^{-2}, in which the effect of the noncentrality is contained in powers of the form $\omega^2 = \mathrm{tr}\, \boldsymbol{\Lambda}^{-1}\boldsymbol{M}\boldsymbol{M}'$. The technique used in this paper for obtaining the asymptotic expansion of F_0 is an extension of the previous methods of Welch [4], [5] and of James [6], [7] who used them to solve the distribution problem of various statistics in connection with the Behrens-Fisher problem. The same technique has been used by several authors: for example, Siotani [3], [8]-[11], Ito [12], Okamoto [13], and Chattopadhyay and Pillai [14].

2. Expression of the distribution function of F_0 by the differential operator

Let $B_{\rho_1,\rho_2}(\eta, \omega^2)$ be the distribution function of

$$\mathscr{F}_0 = \frac{\mathrm{tr}\, \boldsymbol{\Lambda}^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'}{\mathrm{tr}\, \boldsymbol{\Lambda}^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'} ,$$

and $h(\boldsymbol{S}_n)$ be the conditional distribution of F_0 when $\boldsymbol{S}_n$ is fixed. According to the method due to Welch [4], [5] and James [6], [7], the distribution function of F_0 can be expressed by using the differential operator as follows:

$$(2.1)\qquad \begin{aligned} \Pr\{F_0 \leqq \eta\} &= E_{\boldsymbol{S}_n}[h(\boldsymbol{S}_n)] \\ &= E_{\boldsymbol{S}_n}[\exp\{\mathrm{tr}\,(\boldsymbol{S}_n - \boldsymbol{\Lambda})\boldsymbol{\partial}\}\, B_{\rho_1,\rho_2}(\eta; \omega^2)] \\ &= E_{\boldsymbol{S}_n}[\exp\{\mathrm{tr}\, \boldsymbol{S}_n\boldsymbol{\partial}\}]\exp\{-\mathrm{tr}\, \boldsymbol{\Lambda}\boldsymbol{\partial}\}\, B_{\rho_1,\rho_2}(\eta; \omega^2) \\ &= \Theta \cdot B_{\rho_1,\rho_2}(\eta; \omega^2) \end{aligned}$$

where $\rho_1 = m_1p/2$, $\rho_2 = m_2p/2$, $\boldsymbol{\partial} = (\partial_{ij}) = (((1+\delta_{ij})\partial/2)/\partial\lambda_{ij})$, ($\delta_{ij}$ is Kronecker's symbol), a $p\times p$ symmetric matrix of differential operators and

$$(2.2)\qquad \begin{aligned} \Theta &= \exp\left\{-\mathrm{tr}\, \boldsymbol{\Lambda}\boldsymbol{\partial} - \frac{n}{2}\ln\left|\boldsymbol{I} - \frac{2}{n}\boldsymbol{\Lambda}\boldsymbol{\partial}\right|\right\} \\ &= 1 + \frac{1}{n}\sum \lambda_{ur}\lambda_{st}\partial_{rs}\partial_{tu} + \frac{1}{n^2}\left\{\frac{4}{3}\sum \lambda_{wr}\lambda_{st}\lambda_{uv}\partial_{rs}\partial_{tu}\partial_{vw}\right. \\ &\quad \left. + \frac{1}{2}\sum \lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}\right\} + O(n^{-3}) . \end{aligned}$$

Symbol $\sum$ stands for the summation with respect to subscripts in the summand, each of which runs independently over $\{1, 2, \cdots, p\}$ and this simplification continued throughout the paper unless otherwise specified.

It is seen from (2.1) and (2.2) that in order to obtain an asymptotic expansion of the distribution of F_0, we need to evaluate the various derivatives, $\partial_{rs}B_{\rho_1,\rho_2}(\eta;\omega^2)$, $\partial_{rs}\partial_{tu}B_{\rho_1,\rho_2}(\eta;\omega^2)$, etc. This can be done by using the idea of perturbation in physics and consider

$$J=\Pr\left\{\frac{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'}{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'}\leqq\eta\right\} \tag{2.3}$$

where $\boldsymbol{\varepsilon}=(\varepsilon_{ij})$ is a $p\times p$ real symmetric matrix composed of small increments ε_{ij} to λ_{ij} such that $(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})$ is still positive definite. Then we have by Taylor's expansion

$$\begin{aligned} J=\Big[1&+\sum\varepsilon_{rs}\partial_{rs}+\frac{1}{2}\sum\varepsilon_{rs}\varepsilon_{tu}\partial_{rs}\partial_{tu}+\frac{1}{6}\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\partial_{rs}\partial_{tu}\partial_{vw} \\ &+\frac{1}{24}\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}+\cdots\Big]B_{\rho_1,\rho_2}(\eta;\omega^2)\,. \end{aligned} \tag{2.4}$$

On the other hand, we evaluate the right-hand side of (2.3) in the expanded form with respect to powers ε_{ij}'s. After that, if we correctly compare both expansions, we could obtain the derivatives.

3. Preliminary formulas

Let us use the following notations: $\boldsymbol{\Omega}=\boldsymbol{\Lambda}^{-1}\boldsymbol{M}\boldsymbol{M}'$, $\omega^2=\operatorname{tr}\boldsymbol{\Omega}$, $\boldsymbol{X}=(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{\Lambda}-\boldsymbol{I}$, $\chi_f^2(\eta;\omega^2)$ is the distribution function of the noncentral chi-square distribution with f degrees of freedom and noncentrality parameter ω^2. $\Delta_1=E_1-1$, $\Delta_2=E_2-1$, where E_1 and E_2 are two operators such that

$$E_1^r\chi_f^2(t_1;0)=\chi_{f+2r}^2(t_1;0)\,,\qquad E_2^r\chi_f^2(t_2;0)=\chi_{f+2r}^2(t_2;0)\,,\quad\text{for } r=0,1,2,\cdots.$$

LEMMA 3.1.

$$\begin{aligned} J_1&=\Pr\{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'\leqq t_1\} \\ &=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}e^{-\omega^2/2}\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}\chi_{m_1p}^2(t_1;0)\,. \end{aligned} \tag{3.1}$$

PROOF. For a detailed proof, the reader is referred to Siotani [9].

LEMMA 3.2.

$$J_2=\Pr\{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'\leqq t_2\}=|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}\chi_{m_2p}^2(t_2;0)\,. \tag{3.2}$$

PROOF. This is a special case of Lemma 3.1 with $\boldsymbol{M}=\boldsymbol{0}$.

LEMMA 3.3.

$$J=\Pr\left\{\frac{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'}{\operatorname{tr}(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'}\leqq\eta\right\} \tag{3.3}$$

$$=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}e^{-\omega^2/2}$$
$$\cdot\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}B_{\rho_1,\rho_2}(\eta;0)$$
$$=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}$$
$$\cdot\exp\left\{\frac{1}{2}E_1(\operatorname{tr}(\Delta_1\boldsymbol{X})\boldsymbol{\Omega}+\operatorname{tr}(\Delta_1\boldsymbol{X})^2\boldsymbol{\Omega}+\cdots)\right\}B_{\rho_1,\rho_2}(\eta;\omega^2)\,.$$

PROOF. According to Lemma 3.1, we obtain the density function $g_1(t_1)$ of $\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'$

$$g_1(t_1)=\frac{d}{dt_1}J_1=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}e^{-\omega^2/2}\cdot\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}\frac{1}{\Gamma(m_1p/2)}t_1^{m_1p/2-1}e^{-t_1}\,. \tag{3.4}$$

Similarly, the density function $g_2(t_2)$ of $\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'$ can be obtained as

$$g_2(t_2)=\frac{d}{dt_2}J_2=|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}\frac{1}{\Gamma(m_2p/2)}t_2^{m_2p/2-1}e^{-t_2}\,. \tag{3.5}$$

Since $\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'$ and $\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'$ are independent, the joint density function is $g_1(t_1)g_2(t_2)$. After making transformations,

$$t=\frac{t_1}{t_2}\,,\qquad t_2=t_2$$

and integrating out t_2, we have the density function $g(\eta)$ of

$$\frac{\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_1\boldsymbol{Z}_1'}{\operatorname{tr}(\Lambda+\boldsymbol{\varepsilon})^{-1}\boldsymbol{Z}_2\boldsymbol{Z}_2'}\,,$$

$$\begin{aligned}g(t)&=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}e^{-\omega^2/2}\\&\quad\cdot\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}\frac{1}{B(m_1p/2,\,m_2p/2)}\frac{t^{m_1p/2-1}}{(1+t)^{m_1p/2+m_2p/2}}\\&=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}e^{-\omega^2/2}\\&\quad\cdot\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}\beta_{\rho_1,\rho_2}(t;0)\,.\end{aligned} \tag{3.6}$$

Hence

$$J=\int_0^\eta g(t)\,dt=|\boldsymbol{I}-\Delta_1\boldsymbol{X}|^{-m_1/2}|\boldsymbol{I}-\Delta_2\boldsymbol{X}|^{-m_2/2}e^{-\omega^2/2}\cdot\exp\left\{\frac{1}{2}E_1\operatorname{tr}(\boldsymbol{I}-\Delta_1\boldsymbol{X})^{-1}\boldsymbol{\Omega}\right\}B_{\rho_1,\rho_2}(\eta;0)\,.$$

The last expression in (3.3) is obtained simply by using formula (3.7) and noting that

$$
\begin{aligned}
e^{-\omega^2/2} &\exp\left\{\frac{1}{2}E_1 \operatorname{tr} \boldsymbol{\Omega}\right\} B_{\rho_1,\rho_2}(\eta\,;0) \\
&= e^{-\omega^2/2} e^{\omega^2 E_1/2} B_{\rho_1,\rho_2}(\eta\,;0) \\
&= e^{-\omega^2/2} \sum_{i=0}^{\infty} \frac{(\omega^2/2)^i E_1^i}{i!} B_{\rho_1,\rho_2}(\eta\,;0) \\
&= \sum_{i=0}^{\infty} \frac{e^{-\omega^2/2}(\omega^2/2)^i}{i!} B_{\rho_1+i,\rho_2}(\eta\,;0) \\
&= B_{\rho_1,\rho_2}(\eta\,;\omega^2)\ .
\end{aligned}
$$

The operators E_1 and E_2 now operate on Beta function and can be defined as

$$
E_1^r B_{\rho_1,\rho_2}(\eta\,;\omega^2) = B_{\rho_1+r,\rho_2}(\eta\,;\omega^2)\ , \qquad E_2^r B_{\rho_1,\rho_2}(\eta\,;\omega^2) = B_{\rho_1,\rho_2+r}(\eta\,;\omega^2)\ , \quad \text{for } r=0,1,2,\cdots .
$$

The following are used to expand J in a power series of ε_{ij}'s starting with the expression of (3.3).

LEMMA 3.4. *Let* $\boldsymbol{A}$ *be a matrix whose characteristic roots are all less than* 1 *in absolute value. Then*

$$
(3.7) \qquad (\boldsymbol{I}-\boldsymbol{A})^{-1} = \sum_{j=0}^{\infty} \boldsymbol{A}^j
$$

$$
\begin{aligned}
(3.8) \qquad |\boldsymbol{I}-\boldsymbol{A}|^{-m/2} &= 1 + \frac{m}{2}s_1 + \frac{m}{8}(2s_2 + ms_1^2) + \frac{m}{48}(8s_3 + 6ms_2s_1 + m^2s_1^3) \\
&\quad + \frac{m}{384}(48s_4 + 32s_3s_1 + 12ms_2^2 + 12m^2s_2s_1^2 + m^3s_1^4) + \cdots
\end{aligned}
$$

where

$$
s_j = \operatorname{tr} \boldsymbol{A}^j\ , \qquad j=1,2,\cdots .
$$

In the course of the expansion, we use the following two kinds of symbols:

(I) $[rs] = \operatorname{tr} \Lambda^{-1}\Lambda_{rs} = \lambda^{rs}$,

$[rs\,|\,tu] = \operatorname{tr} \Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu} = \frac{1}{2}(\lambda^{ur}\lambda^{st} + \lambda^{us}\lambda^{rt})$,   etc.

(II) $(rs) = \operatorname{tr} \Lambda^{-1}\Lambda_{rs}\boldsymbol{\Omega}$, $(rs\,|\,tu) = \operatorname{tr} \Lambda^{-1}\Lambda_{rs}\Lambda^{-1}\Lambda_{tu}\boldsymbol{\Omega}$,   etc.

where $\Lambda_{rs} = \partial_{rs}\Lambda$ and λ^{rs}'s are elements of Λ^{-1}.

4. Derivatives of $B_{\rho_1,\rho_2}(\eta\,;\omega^2)$

It is seen from (2.1) and (2.2) that in order to obtain the expansion of the distribution function of F_0, we need to evaluate the derivatives of $B_{\rho_1,\rho_2}(\eta;\omega^2)$ up to the fourth degree. It turns out that J must be expanded explicitly up to the fourth power of ε_{ij}'s. First of all, we expand (3.3) with respect to $\boldsymbol{X}$ up to the fourth degree with the aid of formulas (3.7) and (3.8). In order to express the resultant in $\boldsymbol{X}$ in terms of ε_{ij}'s, it is convenient to expand $\boldsymbol{X}$ in the form

$$(4.1)\qquad \begin{aligned} \boldsymbol{X}&=(\boldsymbol{\Lambda}+\boldsymbol{\varepsilon})^{-1}\boldsymbol{\Lambda}-\boldsymbol{I}\\ &=(\boldsymbol{I}+\boldsymbol{\Lambda}^{-1}\boldsymbol{\varepsilon})^{-1}-\boldsymbol{I}\\ &=(\boldsymbol{I}+\sum\varepsilon_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs})^{-1}-\boldsymbol{I}\\ &=-\sum\varepsilon_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}+\sum\varepsilon_{rs}\varepsilon_{tu}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{tu}\\ &\quad-\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{rs}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{tu}\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}_{vw}+\cdots, \end{aligned}$$

since $\boldsymbol{\varepsilon}=\sum\varepsilon_{rs}\boldsymbol{\Lambda}_{rs}$.

The result of this computation is

$$(4.2)\qquad \begin{aligned} J=[1&-\sum\varepsilon_{rs}A^{(1)}_{rs}(\eta)+\sum\varepsilon_{rs}\varepsilon_{tu}A^{(2)}_{rs,tu}(\eta)-\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}A^{(3)}_{rs,tu,vw}\\ &+\sum\varepsilon_{rs}\varepsilon_{tu}\varepsilon_{vw}\varepsilon_{xy}A^{(4)}_{rs,tu,vw,xy}(\eta)-\cdots]B_{\rho_1,\rho_2}(\eta;\omega^2), \end{aligned}$$

where

$$(4.3)\qquad A^{(1)}_{rs}(\eta)=\frac{1}{2}(m_1\Delta_1+m_2\Delta_2)[rs]+\frac{1}{2}E_1\Delta_1(rs),$$

$$(4.4)\qquad \begin{aligned} A^{(2)}_{rs,tu}(\eta)=&\frac{1}{4}\{m_1(\Delta_1^2+2\Delta_1)+m_2(\Delta_2^2+2\Delta_2)\}[rs\,|\,tu]\\ &+\frac{1}{8}(m_1\Delta_1+m_2\Delta_2)^2[rs][tu]+\frac{1}{4}(m_1\Delta_1+m_2\Delta_2)E_1\Delta_1[rs](tu)\\ &+\frac{1}{2}(\Delta_1+1)E_1\Delta_1(rs\,|\,tu)+\frac{1}{8}(E_1\Delta_1)^2(rs)(tu), \end{aligned}$$

and the similar but much longer expressions for $A^{(3)}_{rs,tu,vw}(\eta)$ and $A^{(4)}_{rs,tu,vw,xy}(\eta)$ are omitted here to save the space but they are available in [15].

As stated in the end of Section 2, we now need the comparison of (4.2) with (2.4). In doing so, however, we have to take account of the symmetry in subscripts rs, tu, etc. Let us define

$$(4.5)\qquad H^{(j)}_{rs,\ldots}(\eta)=\frac{1}{j!}\sum_{(p_j)}A^{(j)}_{rs,\ldots}(\eta),\qquad j=1,2,\cdots$$

where $\sum_{(p_j)}$ stands for the summation over all the permutations of subscripts $rs, tu,\cdots$ of $A^{(j)}_{rs,tu,\ldots}(\eta)$. Then we have

$$(4.6)\qquad \partial_{rs}B_{\rho_1,\rho_2}(\eta;\omega^2)=-H^{(1)}_{rs}(\eta)B_{\rho_1,\rho_2}(\eta;\omega^2),$$

(4.7) $\partial_{rs}\partial_{tu}B_{\rho_1,\rho_2}(\eta;\omega^2)=2H^{(2)}_{rs,tu}(\eta)B_{\rho_1,\rho_2}(\eta;\omega^2)$,

(4.8) $\partial_{rs}\partial_{tu}\partial_{vw}B_{\rho_1,\rho_2}(\eta;\omega^2)=-6H^{(3)}_{rs,tu,vw}(\eta)B_{\rho_1,\rho_2}(\eta;\omega^2)$,

(4.9) $\partial_{rs}\partial_{tu}\partial_{vw}\partial_{xy}B_{\rho_1,\rho_2}(\eta;\omega^2)=24H^{(4)}_{rs,tu,vw,xy}(\eta)B_{\rho_1,\rho_2}(\eta;\omega^2)$.

Hence from (2.1), the distribution function of F_0 can be written in the following exanded form:

$$\text{(4.10)}\quad \Pr\{F_0\leqq\eta\}=\Big\{1+\frac{2}{n}\sum\lambda_{ur}\lambda_{st}H^{(2)}_{rs,tu}(\eta)+\frac{1}{n^2}[-8\sum\lambda_{wr}\lambda_{st}\lambda_{ur}H^{(3)}_{rs,tu,vw}(\eta)$$
$$+12\sum\lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}H^{(4)}_{rs,tu,vw,xy}(\eta)]+O(n^{-3})\Big\}B_{\rho_1,\rho_2}(\eta;\omega^2)\ .$$

5. The evaluation of the summations in (4.10)

We explain, in this section, the outline of the computation of the summations in (4.10). First of all we simplify the terms in $H^{(j)}_{rs,tu,\dots}(t)$'s using the properties of the trace. For example,

$$\frac{1}{3!}\sum_{(p_3)}[rs|tu](vw)=\frac{1}{3}\{[rs|tu](vw)+[rs|vw](tu)+[tu|vw](rs)\}\ .$$

Next, we evaluate the values of various types of summations like
(a) $\sum\lambda_{ur}\lambda_{st}[rs|tu]$,
(b) $\sum\lambda_{wr}\lambda_{st}\lambda_{uv}(rs|tu)(vw)$, and
(c) $\sum\lambda_{wr}\lambda_{st}\lambda_{uv}[rs|tu](vw)$.
The summation of type (a) can be easily calculated using the concrete displays of symbols of the (I)-type. The summations of the mixed type (c) are obtained by firstly summing with respect to subscripts contained in the brackets and then by using the results for type (b). As an example for the type (b), let us consider

$$K=\sum\lambda_{wr}\lambda_{st}\lambda_{uv}(rs)(tu|vw)\ .$$

Let $\Lambda^{-1}=(\boldsymbol{\lambda}^{(1)},\boldsymbol{\lambda}^{(2)},\cdots,\boldsymbol{\lambda}^{(p)})$, and $\Omega'=(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\cdots,\boldsymbol{\omega}_p)$. Then

$$(rs)(tu|vw)=(\mathrm{tr}\,\Lambda^{-1}\Lambda_{rs}\Omega)(\mathrm{tr}\,\Lambda^{-1}\Lambda_{tu}\Lambda^{-1}\Lambda_{vw}\Omega)$$
$$=\frac{1}{8}\{\boldsymbol{\omega}'_r\boldsymbol{\lambda}^{(s)}(\boldsymbol{\lambda}^{(u)}\lambda^{tw}+\boldsymbol{\lambda}^{(t)}\lambda^{uw})'\boldsymbol{\omega}_v+\boldsymbol{\omega}'_r\boldsymbol{\lambda}^{(s)}(\boldsymbol{\lambda}^{(u)}\lambda^{tv}+\boldsymbol{\lambda}^{(t)}\lambda^{uv})'\boldsymbol{\omega}_w$$
$$+\boldsymbol{\omega}'_s\boldsymbol{\lambda}^{(r)}(\boldsymbol{\lambda}^{(u)}\lambda^{tw}+\boldsymbol{\lambda}^{(t)}\lambda^{uw})'\boldsymbol{\omega}_r+\boldsymbol{\omega}'_s\boldsymbol{\lambda}^{(r)}(\boldsymbol{\lambda}^{(u)}\lambda^{tv}+\boldsymbol{\lambda}^{(t)}\lambda^{uv})'\boldsymbol{\omega}_w\}$$
$$=\frac{1}{8}\{k_1(r,s,t,u,v,w)+k_2(r,s,t,u,v,w)$$
$$+k_3(r,s,t,u,v,w)+k_4(r,s,t,u,v,w)\}\ ,$$

and

$$K_1=\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_1(r, s, t, u, v, w)=\sum_{rv} \boldsymbol{\omega}_r'\boldsymbol{\delta}_r\boldsymbol{\delta}_v'\boldsymbol{\omega}_v+\sum_{rv} \lambda_{vr}\boldsymbol{\omega}_r'\boldsymbol{\Lambda}^{-1}\boldsymbol{\omega}_r$$

where $\boldsymbol{\delta}_r'=(\delta_{1r}, \delta_{2r}, \cdots, \delta_{pr})$ and δ_{jr}'s are Kronecker's symbols. The first term is simply equal to $(\mathrm{tr}\,\boldsymbol{\Omega})^2=\omega^4=s_1^2$ and the second term is equal to

$$\mathrm{tr}\,\boldsymbol{\Lambda}^{-1}\Big(\sum_{rv} \lambda_{vr}\boldsymbol{\omega}_r\boldsymbol{\omega}_r'\Big)=\mathrm{tr}\,\boldsymbol{\Lambda}^{-1}\boldsymbol{\Omega}\boldsymbol{\Lambda}\boldsymbol{\Omega}=\mathrm{tr}\,\boldsymbol{\Lambda}^{-1}(\boldsymbol{\Lambda}^{-1}\boldsymbol{M}\boldsymbol{M}')\boldsymbol{\Lambda}(\boldsymbol{\Lambda}^{-1}\boldsymbol{M}\boldsymbol{M}')=\mathrm{tr}\,\boldsymbol{\Omega}^2=s_2 .$$

Hence, $K_1=s_1^2=s_2$. Similar computations give us

$$\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_2(r, s, t, u, v, w)=(p+1)s_2=\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_4(r, s, t, u, v, w) ,$$

$$\sum \lambda_{wr}\lambda_{st}\lambda_{uv}k_3(r, s, t, u, v, w)=k_1 ,$$

and we have

$$K=\frac{1}{4}\{s_1^2+(p+2)s_2\} .$$

If we put $\boldsymbol{\Omega}=\boldsymbol{I}$, K should be equal to $p(p+1)/2$, which is the value of $\sum \lambda_{wr}\lambda_{st}\lambda_{uv}[rs][tu\,|\,vw]$.

A complete list of values of individual summations of types (a), (b) and (c) is available in [10]. With the aid of these results, we can evaluate each of the summations in (4.10).

$$\begin{aligned}\sum \lambda_{ur}\lambda_{st}H^{(2)}_{rs,tu}(\eta)&=\sum \lambda_{ur}\lambda_{st}\Big\{\frac{1}{2!}\sum_{(p_2)} A^{(2)}_{rs,tu}(\eta)\Big\}\\&=\frac{1}{8}p(p+1)\{m_1(\Delta_1^2+2\Delta_1)+m_2(\Delta_2^2+2\Delta_2)\}\\&\quad+\frac{1}{8}p(m_1\Delta_1+m_2\Delta_2)^2+\frac{1}{4}\Delta^2(m_1\Delta_1+m_2\Delta_2)E_1\Delta_1\\&\quad+\frac{1}{4}(p+1)s_1(\Delta_1+1)E_1\Delta_1+\frac{1}{8}(E_1\Delta_1)^2s_2 .\end{aligned}$$

It is convenient to arrange the above expression with respect to the powers of E_1 and E_2. Then

$$\sum \lambda_{ur}\lambda_{st}H^{(2)}_{rs,tu}(\eta)=\frac{1}{8}\sum_{i=0}^{4}\sum_{j=0}^{2} a_{ij}(m_1, m_2, p; \boldsymbol{\Omega})E_1^iE_2^j \tag{5.1}$$

which gives us the term of order n^{-1} in the expansion of the distribution function of F_0. Two summations in (4.10) which give the term of order n^{-2} can be calculated in the same way as the above. The result can be written in the form

$$\begin{aligned}&-8\sum \lambda_{wr}\lambda_{st}\lambda_{uv}H^{(3)}_{rs,tu,vw}(\eta)+12\sum \lambda_{ur}\lambda_{st}\lambda_{yv}\lambda_{wx}H^{(4)}_{rs,tu,vw,xy}(\eta)\\&=\frac{1}{96}\sum_{i=0}^{8}\sum_{j=0}^{4} b_{ij}(m_1, m_2; p; \boldsymbol{\Omega})E_1^iE_2^j ,\end{aligned} \tag{5.2}$$

where coefficients $a_{ij}(m_1, m_2, p; \boldsymbol{\Omega})$ and $b_{ij}(m_1, m_2, p; \boldsymbol{\Omega})$ will be given in Theorem 6.1.

6. The final result

The desired expanded form of the distribution function of F_0 is now obtained by substituting (5.1) and (5.2) into (4.10). Hence we have immediately the following final result:

THEOREM 6.1. *An asymptotic expansion of the non-null distribution of the ratio of two Hotelling's generalized T_0^2-statistics defined by* (1.1) *is given by*

$$\text{(6.1)} \quad \Pr\{F_0 \leqq \eta\} = B_{\rho_1,\rho_2}(\eta; \omega^2) + \frac{1}{4m}\sum_{i=0}^{4}\sum_{j=0}^{2} a_{ij}(m_1, m_2, p; \boldsymbol{\Omega}) \cdot B_{\rho_1+i,\rho_2+j}(\eta; \omega^2) + \frac{1}{96n^2}\sum_{i=0}^{8}\sum_{j=0}^{4} b_{ij}(m_1, m_2, p; \boldsymbol{\Omega}) \cdot B_{\rho_1+i,\rho_2+j}(\eta; \omega^2) + O(n^{-3}),$$

where with the notations

$$\boldsymbol{\Omega} = \boldsymbol{\Lambda}^{-1}\boldsymbol{M}\boldsymbol{M}', \quad s_j = \operatorname{tr}\boldsymbol{\Omega}^j, \qquad j = 1, 2, \cdots, \ s_1 = \omega^2,$$

coefficients

$$a_{ij}(m_1, m_2, p; \boldsymbol{\Omega}) \equiv a_{ij}, \qquad b_{ij}(m_1, m_2, p; \boldsymbol{\Omega}) \equiv b_{ij}$$

are;

$$a_{00} = (m_1+m_2)p(m_1+m_2-p-1),$$

$$a_{01} = -2(m_1+m_2)m_2p,$$

$$a_{02} = m_2p(m_2+p+1),$$

$$a_{10} = -2(m_1+m_2)(m_1p-s1),$$

$$a_{11} = 2m_2(m_1p-s_1),$$

$$a_{12} = 0,$$

$$a_{20} = m_1p(m_1+p+1) - 2(2m_1+m_2+p+1)s_1 + s_2,$$

$$a_{21} = 2m_2s_1,$$

$$a_{22} = 0,$$

$$a_{30} = 2\{(m_1+p+1)s_1 - s_2\},$$

$$a_{31} = a_{32} = 0,$$

$$a_{40}=s_2\ ,$$

$$a_{41}=a_{42}=0\ ,$$

$$b_{00}=(m_1+m_2)p\{(m_1+m_2)(3(m_1+m_2)p-8)(m_1+m_2-2p-2)$$
$$+(m_1+m_2)(p+1)(3p^2+3p-4)-4(2p^2+3p-1)\}\ ,$$

$$b_{01}=-12(m_1+m_2)^2(m_1+m_2-p-1)m_2p^2\ ,$$

$$b_{02}=6(m_1+m_2)m_2p\{3(m_1+m_2)m_2p+8(m_1+m_2)$$
$$+(m_1-p-1)(p^2+p-4)\}\ ,$$

$$b_{03}=-4m_2p[(m_2+p+1)\{m_2(3(m_1+m_2)p+16)+12m_1\}$$
$$+8m_2(p+1)+4(p^2+3p+4)]\ ,$$

$$b_{04}=3m_2p\{m_2(m_2+2p+2)(m_2p+8)+m_2(p+1)(p^2+p+4)$$
$$+4(2p^2+5p+5)\}\ ,$$

$$b_{10}=-12(m_1+m_2)^2p(m_1+m_2-p-1)(m_1p-s_1)\ ,$$

$$b_{11}=12(m_1+m_2)(m_1p-s_1)m_2\{3(m_1+m_2)p-(p^2+p-4)\}\ ,$$

$$b_{12}=-12m_2(m_1p-s_1)\{(m_1+m_2)(3m_2p+p^2+p+4)+4(m_2+p+1)\}\ ,$$

$$b_{13}=12m_2(m_2+p+1)(m_2p+4)(m_1p-s_1)\ ,$$

$$b_{14}=0\ ,$$

$$b_{20}=6(m_1+m_2)[m_1p\{3(m_1+m_2)m_1p+8(m_1+m_2)$$
$$+(m_2-p-1)(p^2+p-4)\}-2\{((m_1+m_2)p+2)(4(m_1+m_2)$$
$$-p-1)-m_2(3(m_1+m_2)p-p^2-p+4)-(p+1)(p-2)(p+3)\}s_1$$
$$+\{(m_1+m_2)p-(p^2+p-4)\}s_2]\ ,$$

$$b_{21}=-12m_2[m_1p\{(m_1+m_2)(3m_1p+p^2+p+4)+4(m_1+p+1)\}$$
$$-\{3(m_1+m_2)(3m_1+m_2)p+(m_1+m_2)(p^2+p+12)$$
$$+8(m_1+p+1)\}s_1+2(m_1+m_2)s_1^2+((m_1+m_2)p+4)s_2]\ ,$$

$$b_{22}=6m_2[m_1p\{(m_1+m_2+p+1)(p^2+p+4)+3m_1m_2p\}$$
$$-2\{(2m_1+m_2)(3m_2p+p^2+p+4)+(m_2+p+1)(p^2+p+8)\}s_1$$
$$+2m_2s_1^2+(m_2p+p^2+p+4)s_2]\ ,$$

$$b_{23}=12m_2(m_2+p+1)(m_2p+4)s_1\ ,$$

$$b_{24}=0\ ,$$

$$b_{30}=-4[m_1p\{(m_1+p+1)\{m_1(3(m_1+m_2)p+16)+12m_2\}+8m_1(p+1)$$
$$+4(p^2+3p+4)\}-3\{3(m_1+m_2)(m_1p+4)(2m_1+p+1)+(m_1+m_2)$$
$$\cdot\ m_2p(3m_1+p+1)+4m_2(m_2-3p-3)-(m_1+m_2)(p+1)$$

$$\cdot (p^2+p-16)+4(p^2+3p+4)\} s_1+6\{2(m_1+m_2)^2-(m_1+m_2) \cdot (m_2-p-1)+2\} s_1^2+3\{((m_1+m_2)p+8)(2(m_1+m_2)-p-1) - m_2((m_1+m_2)p+4)+4(3p+4)\} s_2-3(m_1+m_2)s_1s_2-4s_3] ,$$

$$b_{31}=12m_2[m_1p(m_1p+4)(m_1+p+1)- \{(5m_1+2m_2)(p^2+p+4) +3m_1((3m_1+2m_2)p-4)+20(m_1+p+1)\} s_1+2(3m_1+2m_2 +p+1)s_1^2-s_1s_2+ \{(3m_1+2m_2)p+16\} s_2] ,$$

$$b_{32}=12m_2[\{(m_1+m_2+p+1)(p^2+p+4)+3m_1m_2p\} s_1-2m_2s_1^2 -(m_2p+p^2+p+4)s_2] ,$$

$$b_{3j}=0 , \qquad j=3, 4 ,$$

$$b_{40}=3[m_1p\{m_1(m_1p+8)(m_1+2p+2)+m_1(p+1)(p^2+p+4) +4(2p^2+5p+5)\} -4\{m_1(m_1p+6)(4m_1+5p+5)+m_1(p+1) \cdot (p^2+p+14)+4(3p^2+8p+9)+3m_2(m_1p+4)(m_1+p+1)\} s_1 +4\{6m_1(m_1+p+1)+(p^2+2p+15)+m_2(6m_1+m_2+4p+4)\} s_1^2 + \{4(3m_1^2p+36m_1+18p+32)+2m_2(6m_1p+m_2p-p^2-p+36)\} s_2 -4(4m_1+3m_2+p+1)s_1s_2-32s_3+s_2^2)] ,$$

$$b_{41}=12m_2[(3m_1p+12)(m_1+p+1)s_1-2(3m_1+m_2+2p+2)s_1^2 - \{(3m_1+m_2)p+20\} s_2+3s_1s_2] ,$$

$$b_{42}=6m_2[2m_2s_1^2+(m_2p+p^2+p+4)s_2] ,$$

$$b_{4j}=0 , \qquad j=3, 4 ,$$

$$b_{50}=12[\{m_1(m_1p+8)(m_1+2p+2)+m_1(p+1)(p^2+p+4) +4(2p^2+5p+5)\} s_1-2\{(2m_1+m_2+p+1)(m_1+p+1)+8\} s_1^2 - \{(m_1p+16)(2m_1+p+1)+8(p+3)+m_2(m_1p+8)\} s_2 +3(2m_1+m_2+p+1)s_1s_2+16s_3-s_2^2] ,$$

$$b_{51}=12m_2[2(m_1+p+1)s_1^2+(m_1p+8)s_2-3s_1s_2] ,$$

$$b_{5j}=0 , \qquad j=2, 3, 4 ,$$

$$b_{60}=2[6\{(m_1+p+1)^2+6\} s_1^2+3\{(m_1p+20)(m_1+p+1)12\} s_2 -6(4m_1+m_2+3p+3)s_1s_2-80s_3+9s_2^2] ,$$

$$b_{61}=12m_2s_1s_2 ,$$

$$b_{6j}=0 , \qquad j=2, 3, 4 ,$$

$$b_{70}=12\{(m_1+p+1)s_1s_2+4s_3-s_2^2\} ,$$

$$b_{7j}=0 , \qquad j=1, 2, 3, 4 ,$$

$$b_{80}=3s_2^2\ ,$$

$$b_{8j}=0\ ,\qquad j=1,2,3,4\ .$$

7. Percentage points of F_0 in the null case

Percentage points of the distribution of F_0 when $H:\boldsymbol{M}=\boldsymbol{0}$ is true has been also obtained by using the result in the previous section, which coincides with the result given by Siotani [3]. Tables of upper 5 and 1 percentage points of F_0 together with the computer program for obtaining these tables are available in [15], [16].

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