ON MULTIVARIATE MODIFIED POLYA AND INVERSE POLYA DISTRIBUTIONS AND THEIR PROPERTIES

K. G. JANARDAN AND G. P. PATIL

(Received Oct. 13, 1969; revised June 21, 1973)

1. Introduction and summary

In connection with the negative multinomial distribution, Sibuya, Yoshimura, and Shimizu [5] discuss a slight modification of the Eggenberger Polya Urn Scheme and obtain a multivariate distribution which tends to the negative multinomial under certain limiting conditions. In this paper we treat this distribution to be called “multivariate modified Polya (MMP) distribution” and discuss some of its interesting structural properties, which we hope are at least of pedagogical interest. If the random vector \((rv) (x_1, x_2, \cdots, x_r)\) has \(s\)-variate modified Polya distribution, then it is shown that the marginal distribution of the subvector \((x_1, x_2, \cdots, x_r)\), and the conditional distribution of \((x_{r+1}, x_{r+2}, \cdots, x_s)\) given \((x_1, x_2, \cdots, x_r)\) do not belong to the family of MMP distributions unlike in the case of multivariate Polya and inverse Polya families (see [2], [3]), these are shown to be products of Gaussian hypergeometric function (see [1]) and a \(r\)-variate modified Polya distribution. We also introduce “multivariate inverse Polya (MMIP) distribution,” and give similar properties for MMIP distribution.

2. Genesis of the MMP and MMIP distributions

(a) Suppose an urn contains \(N_1, N_2, \cdots, N_s, \left(\sum_{i=1}^{s} N_i=M\right)\) balls of \(s\)-different colors as well as \(N_0\) white balls. We draw a ball and if it is colored, it is replaced with additional \(c_1, c_2, \cdots, c_s\) balls of \(s\)-colors and if it is white, it is replaced with additional \(d=\sum_{i=1}^{s} c_i\) white balls. This is repeated \(n\) times. If \(x_i\) denotes the number of balls of \(i\)th color, and \(x_0\) the number of white balls such that \(\sum_{i=8}^{r} x_i=n\), then the \(rv\) \(x=(x_1, x_2, \cdots, x_r)\) has been shown (see [5]) to have a probability distribution whose probability function (pf) is given by
\( \text{mmp} (x; d, n, N) = \binom{n}{x} \left( \frac{N_0^{(x, d)} M^{(x, d)}}{N^{(x, d)}} \right) \prod_{i=1}^{s} \left( \frac{N_i}{M} \right)^{x_i} \)

where \( x_i = 0, 1, 2, \ldots, n \) for \( i = 1, 2, \ldots, s \) so that \( x = \sum_{i=1}^{s} x_i \), \( N = N_0 + M \);

\( a^{(k,d)} = a(a+d)(a+2d) \cdots (a+(k-1)d) \)

and \( \binom{n}{x} \) denotes the multinomial coefficient.

We note that when \( s = 1 \) the pf (2.1) reduces to the univariate Polya distribution (see [4]). This is also evident from the urn scheme. For, in the case when \( s = 1 \) there are only "white" and "black" balls in the urn and the scheme of sampling would reduce to Eggenberger-Polya sampling scheme for univariable case.

(b) In this paragraph, we give a different chance mechanism which generates the MMP distribution. Put \( x = (x_1, x_2, \ldots, x_n) \) and \( x = \sum x_i \).

If the conditional distribution of \( x \) given \( x \) is singular multinomial (SM) distribution (see [4]) with parameters \( x \) and \( \phi = (\phi_1, \phi_2, \ldots, \phi_s) \) and the distribution of \( x \) is univariate Polya distribution with parameters \( n, \nu \) and \( p_0 \), then the distribution of \((x, x)\) is MMP distribution as in (2.1).

**Remark.** It may be noted, as the referee pointed out, that (1) marginal distribution of the component \( x \) of \((x, x)\) follows a univariate Polya distribution and that (2) the sum \( x_1 + x_2 + \cdots + x_s + x = 2(x_1 + x_2 + \cdots + x_s) \) takes even numbers with probability one. For,

\[
P(x, x) = P(x|x) P(x) = \binom{n}{x} \prod_{i=1}^{s} \phi_i^{x_i} \left( \frac{-q_0/\nu}{x} \right)^{x_i} \left( \frac{-p_0/\nu}{n-x} \right)^{(n-x)/n}
\]

where \( q_0 = 1 - p_0 \). The pf (2.2) can be easily shown to be equivalent to the pf (2.1), by taking \( \phi_i = p_i/(1-p_0) \), \( p_i = N_i/N \) and \( \nu = d/N \).

(c) Now in the above urn sampling scheme of (a) suppose we draw balls until we observe exactly \( k \) white balls following the same procedure. Let \( x_i \) denote the number of balls of \( i \)th color drawn in the process of getting \( k \) white balls, then it is easy to show that the pf of the rv \( x \) is

\[
\text{mmip} (x; d, k, N) = \binom{k+x-1}{x} \left( \frac{N_0^{(k,d)} M^{(x, d)}}{N^{(k,x,d)}} \right) \prod_{i=1}^{s} \left( \frac{N_i}{M} \right)^{x_i}
\]

where \( x_i = 0, 1, 2, \ldots, \infty \) for \( i = 1, 2, \ldots, s \); \( x = \sum_{i=1}^{s} x_i \), \( M = \sum_{i=1}^{s} N_i \) and \( N = N_0 + M \).

We note when \( s = 1 \) the pf (2.3) reduces to the pf of univariate inverse Polya distribution (see [4]).

(d) Here we give a different chance mechanism which generates the MMIP distribution. Again, as in (b), put \( x = (x_1, x_2, \ldots, x_s) \) and
$x = \sum x_i$. If the conditional distribution of $x$ given $x$ is SM distribution and the distribution of $x$ is univariate inverse Polya distribution with parameters $k$, $\nu$ and $p_0$, then the distribution of $(x, x)$ is MMIP as in (2.3). A similar remark as in (b) applies here also. For,

$$P(x, x) = P(x | x) P(x)$$

$$= \left( x! \prod_{i=1}^{x} x_i! \right) \prod_{i=1}^{x} \phi_t^{i-1} \left( \frac{k}{k+x} \right) \left( -\frac{p_0/\nu}{k} \right) \left( -\frac{q_0/\nu}{x} \right) \left( -\frac{1}{\nu} \right)$$

where $q_0 = 1 - p_0$. The pf (2.4) can be easily seen to be equivalent to the pf (2.3) after little algebra with $\phi_t = p_i/(1-p_0)$, $p_i = N_i/N$ and $\nu = d/N$.

3. Moments of the distributions

The probability generating functions (pgfs) of MMP and MMIP distributions are respectively,

$$G_x(t) = C_1 \; _2F_1(-n; q_0^{-1}, -n+1; Z)$$

and

$$G_x(t) = C_2 \; _2F_1(k; q_0^{-1}, \nu^{-1}+k; Z)$$

where

$$C_1 = \frac{(-p_0/\nu)^{(n)}}{(-1/\nu)^{(n)}} \quad , \quad C_2 = \frac{(-p_0/\nu)^{(k)}}{(-1/\nu)^{(k)}}$$

$$Z = \phi_1 t_1 + \phi_2 t_2 + \cdots + \phi_s t_s \quad , \quad a^{(b)} = a(a-1) \cdots (a-b+1)$$

and $\phi_i$, $p_0$ and $\nu$ are as defined earlier. For the definition of $_2F_1$ see [1].

Remark. It may be noted that the pgfs of MMP and MMIP are equivalent to the pgfs of univariate Polya (with parameters $n$, $\nu$ and $p_0$), and univariate inverse Polya (with parameters $k$, $\nu$ and $p_0$), evaluated at $Z = \phi_1 t_1 + \phi_2 t_2 + \cdots + \phi_s t_s$.

The factorial moment generating functions of MMP and MMIP, obtained by substituting $t_i = 1 + \alpha_i$ ($i = 1, 2, \cdots, s$) in (3.1) and (3.2) are clearly,

$$M_x(\alpha) = C_1 \; _2F_1(-n; q_0^{-1}, -p_0/\nu - n+1; W)$$

and

$$M_x(\alpha) = C_2 \; _2F_1(k; q_0^{-1}, \nu^{-1}+k; W)$$

where $C_1$ and $C_2$ are as given above and $W = 1 + \phi_1 \alpha_1 + \phi_2 \alpha_2 + \cdots + \phi_s \alpha_s$. 
Hence the factorial moments $\mu'[r_1, r_2, \ldots, r_s]$ of MMP and MMIP are respectively given by,

$$(n^xM^{(r,d)}) \prod_{i=1}^{s} \left( \frac{N_i}{M} \right)^{r_i} \text{ and } \frac{(k_rM^{(r,d)})}{(N_0-rd)^{k_r}} \prod_{i=1}^{s} \left( \frac{N_i}{M} \right)^{r_i}$$

where $r = r_1 + r_2 + \ldots + r_s$ and $(k_r) = k(k+1)(k+2)\ldots(k+r-1)$.

4. Structural properties of MMP and MMIP

The MMP and MMIP distributions possess several remarkable and interesting properties which are not similar to the properties of multivariate Pólya and inverse Pólya distributions. The more important of these properties refer to any group $r<s$ variables out of the $s$ considered. These properties, which can be proved using the pgf techniques or otherwise, are stated as theorems without proofs in this section.

4.1. Properties of MMP

If the rv $x$ has MMP as in (2.1), then

**Theorem 1.** The sum of the components of $x$ follows univariate Pólya distribution with parameters $n$, $d$, and $M$.

**Theorem 2.** The conditional distribution of $x$ given $\sum x_i = m$ follows SM distribution with parameters $m$ and $(\phi_1, \phi_2, \ldots, \phi_s)$ where $\phi_i = p_i/(1-p_0)$ and $p_i = N_i/N$.

**Theorem 3.** The rv $(Z_1, Z_2, \ldots, Z_s)$ also follows MMP with parameters $n$, $d$, and $M$, where $Z_j = \Sigma x_i$, $M_j = \Sigma N_i$ with $\Sigma$ standing for the summation taken over $i \in G_j$ for $j = 1, 2, \ldots, s$ such that $\bigcup_{j=1}^{r} G_j = \{1, 2, \ldots, s\}$ and $G_j \cap G_{j'} = \phi$ for $j \neq j' = 1, 2, \ldots, r$.

**Theorem 4.** The conditional distribution of the rv $x$ given the rv $(Z_1, Z_2, \ldots, Z_s)$ is the product of $r$ independent SM distributions of the type SM$(\{p_{ij}\}, Z_j)$ where $P_{ij} = N_i/M_j$ for $i \in G_j$ ($j = 1, 2, \ldots, r$) and $Z_j$ and $G_j$ are as defined in the above theorem.

**Theorem 5.** The marginal distribution of $x_i = (x_1, x_2, \ldots, x_r)$ where $1 \leq r \leq s$ is available as

$$\binom{n}{x_i} \left( \frac{N_0^{(n-y,d)}M^{(y,d)}}{N^{(n,d)}} \right) \prod_{i=1}^{s} \left( \frac{N_i}{M} \right)^{x_i}.$$

$$\frac{2F_1}{\binom{-n+y; M+yd}{d} + \frac{N_0}{d} + n-y-1; \sum_{i=r+1}^{s} \frac{N_i}{M}}$$
where \( y = \sum_{i=1}^{r} x_i \) and \( _2F_1 \) is Gaussian hypergeometric function (see [1]).

**Theorem 6.** The conditional distribution of \( x_2 \) given \( x_1 \), where \( x_2 = (x_{r+1}, x_{r+2}, \ldots, x_r) \) is given by

\[
\binom{n-y}{x_2} \left( \frac{N_0^{(n-x,d)}M^{(x,d)}}{N_0^{(n-y,d)}M^{(y,d)} _2F_1(\cdots)} \right) \prod_{i=r+1}^{r} \left( \frac{N_i}{M} \right) ^{x_i}
\]

where \( _2F_1(\cdots) \) is as in Theorem 5.

### 4.2. Properties of MMIP

Let the rv \( x \) have MMIP as in (2.3), then

**Theorem 7.** The sum \( \sum_{i=1}^{s} x_i \) has the univariate inverse Polya distribution with parameters \( k, d \) and \( M \).

**Theorem 8.** The conditional distribution of \( x \) given \( \sum_{i=1}^{s} x_i = m \) follows SM distribution with parameters \( m \) and \( \phi_1, \phi_2, \ldots, \phi_s \).

**Theorem 9.** The distribution of the rv \( (Z_1, Z_2, \ldots, Z_r) \) is also MMIP with parameters \( k, d \) and \( M \), where \( Z_j, M_j (j = 1, 2, \ldots, r) \) are as defined in Theorem 4.

**Theorem 10.** The conditional distribution of \( x \) given \( (Z_1, Z_2, \ldots, Z_r) \) is the product of \( r \) independent SM distributions of the type, SM(\( \{P_{ij}\} \), \( Z_i \)) where \( P_{ij} = N_i/M_i \) for \( i \in G_j \).

**Theorem 11.** The marginal distribution of \( x = (x_1, x_2, \ldots, x_r) \) is available as

\[
\binom{k+y-1}{x_1, x_2, \ldots, x_r} \left( \frac{N_{i}^{(k,d)}M^{(d,y)}}{N^{(k+y,d)}} \right) \prod_{i=1}^{r} \left( \frac{N_i}{M} \right) ^{x_i} \cdot _2F_1(k+y; M+yd; N, \frac{N}{d}+k+y; \sum_{i=1}^{r} \frac{N_i}{M})
\]

where \( y = \sum_{i=1}^{r} x_i \).

**Theorem 12.** The conditional distribution of \( x_2 \) given \( x_1 \) is given by

\[
\binom{k^*+z-1}{x_{r+1}, \ldots, x_r} \left( \frac{(M+yd)^{(d,z)}}{(N+k^*d)^{(d,z)} _2F_1(\cdots)} \right) \prod_{i=r+1}^{r} \left( \frac{N_i}{M} \right) ^{x_i}
\]

where \( k^* = k+y \), \( x_2 = (x_{r+1}, \ldots, x_r) \) and \( _2F_1(\cdots) \) is as in Theorem 11.
Acknowledgements

The authors wish to thank the referee for his suggestions which greatly improved this paper. In particular, the referee's comments led to results (b) and (d) of Section 2.

SANGAMON STATE UNIVERSITY
PENNSYLVANIA STATE UNIVERSITY

REFERENCES


