A BAYES RULE FOR THE SYMMETRIC MULTIPLE COMPARISONS PROBLEM II[1]

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1. Statement of problem

This paper presents the mathematical derivation of a Bayes solution for the symmetric multiple comparisons problem presented earlier by Waller and Duncan [4]. Because the earlier paper details the model and assumptions on which the solution is based, only the functions and results needed to complete the development are stated here.

Typically the data for a multiple comparisons problem consists of n independent normally distibuted treatment means $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ which estimate n true means $(\mu_1, \mu_2, \cdots, \mu_n)$ and an independent estimate, say s^2_{eE} with f_E degrees of freedom, of the error variance σ^2_e. Further, we suppose that each mean $\bar{x}_i$ is based on r replications. The problem is then to choose for each of the $h=n(n-1)$ ordered pairs of means either decision $d^+_{ij}: \mu_i>\mu_j$ or decision $d^0_{ij}: \mu_i$ not ranked relative to μ_j. However, Duncan [2] shows that under the assumptions of symmetry and an additive loss model, the Bayes rule for the full multiple comparisons problem is the simultaneous application of the Bayes rule to select either d^+_{12} or d^0_{12} for the means (μ_1, μ_2) to all h pairs of means. Thus our major concern is to obtain the Bayes rule for selecting one of the decisions

$$(1.1)\qquad \begin{aligned} &d^+_{12}: \mu_1>\mu_2\,, \\ &d^0_{12}: \mu_1 \text{ not ranked relative to } \mu_2, \end{aligned}$$

for the pair of means (μ_1, μ_2). Following that development we state an extension of the result as the Bayes rule for the full problem.

To begin the derivation of a Bayes rule to select one of the decisions (1.1) for the pair of means $(\mu_1, \mu_2,)$, we summarize the probability

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model, the prior distribution, and the loss function. For a more complete presentation of the model the reader should refer to Waller and Duncan [4].

For reasons of location invariance we consider only solutions that depend on the means $(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n)$ through the $(n-1)$ Helmert transformations

$$y_1=\frac{\bar{x}_1-\bar{x}_2}{\sqrt{2}}, \quad y_2=\frac{\bar{x}_1+\bar{x}_2-2\bar{x}_3}{\sqrt{6}}, \quad \cdots,$$

$$y_{n-1}=\frac{\bar{x}_1+\bar{x}_2+\cdots+\bar{x}_{n-1}-(n-1)\bar{x}_n}{\sqrt{n(n-1)}}.$$

Thus the joint probability density of our data is

$$f(\boldsymbol{z}|\boldsymbol{\theta})=\left[\prod_1^{n-1} P_1\left(y_i \,\middle|\, \eta_i, \frac{\sigma_e^2}{r}\right)\right] P_2(s_{eE}^2|\sigma_e^2, f_E) \tag{1.2}$$

where $P_1(x|\mu, \sigma^2)$ is the density of a normal random variable with mean μ and variance σ^2, $P_2(s^2|\sigma^2, f)$ is the chi-square density of the random variable (fs^2/σ^2) with f degrees of freedom, $\boldsymbol{z}=(y_1, y_2, \cdots, y_{n-1}, s_{eE}^2)$, $\eta_i =\mathrm{E}\,(y_i)$, and $\boldsymbol{\theta}=(\eta_1, \eta_2, \cdots, \eta_{n-1}, \sigma_e^2)$.

The prior density is the product of $(n-1)$ independent normal densities for $(\eta_1, \eta_2, \cdots, \eta_{n-1})$, each with mean zero and variance σ_μ^2, and an independent joint density for σ_e^2 and $\sigma_T^2=r\sigma_\mu^2+\sigma_e^2$ (by definition) deduced from two independent χ^2 densities. Thus, the full prior density is

$$\lambda(\boldsymbol{\theta})=K^{-1}\int_{\sigma_e^2}^{\infty}\left[\prod_1^{n-1} P_1\left(\eta_i \,\middle|\, 0, \frac{\sigma_T^2-\sigma_e^2}{r}\right)\right] P_3(\sigma_T^2|s_{TP}^2, q_p) P_3(\sigma_e^2|s_{eP}^2, f_P)\, d\sigma_T^2 \tag{1.3}$$

where P_1 is defined for (1.2), $P_3(\sigma^2|s^2, f)$ is the fiducial distribution of σ^2 deduced by assuming s^2 is distributed so that $(fs^2)/\sigma^2$ has a χ^2 density with f degrees of freedom, and K is the normalizing constant

$$K=\int_0^{\infty}\int_{\sigma_e^2}^{\infty} P_3(\sigma_T^2|s_{TP}^2, q_P) P_3(\sigma_e^2|s_{eP}^2, f_P)\, d\sigma_T^2\, d\sigma_e^2 .$$

The loss model is defined as the sum of the losses associated with the decision selected for each pair of means. Then for reasons of symmetry the same loss function is assumed for every pair of means. The common loss function written for means (μ_1, μ_2) is

$$L(d_{12}^{+}: \boldsymbol{\theta})=\begin{cases} k_1|\delta|, & \delta\leqq 0 \\ 0, & \delta>0, \end{cases} \qquad L(d_{12}^{0}: \boldsymbol{\theta})=\begin{cases} 0, & \delta\leqq 0 \\ k_2\delta, & \delta>0 \end{cases} \tag{1.4}$$

in which $\delta=\mu_1-\mu_2$ and k_1 and k_2 are positive constants such that $k_1\gg k_2$.

2. Simplification of the risk equation

This section has two purposes. The first objective is to define the Bayes solution (with respect to the foregoing model) for selecting either decision d_{12}^{+} or decision d_{12}^{0} for means (μ_1, μ_2). The second objective is to simplify the risk equation that is used to define the Bayes solution. To meet those aims our first step is to prove the following lemma:

LEMMA 2.1. *The Bayes rule for selecting one of the decisions* (1.1) *for means* (μ_1, μ_2) *is*

$$\begin{aligned} &\textit{make decision } d_{12}^{+} \textit{ if } r(z)>k, \\ &\textit{make decision } d_{12}^{0} \textit{ if } r(z)\leqq k, \end{aligned} \tag{2.1}$$

where $k=k_1/k_2$ *and*

$$r(z)=\frac{I_1^{+}(z)}{I_1^{-}(z)}, \tag{2.2}$$

in which I_1^{*} *is the integral*

$$I_1^{*}(z)=\int_{\Omega^{*}}|\delta| f(z|\boldsymbol{\theta})\lambda(\boldsymbol{\theta})d\boldsymbol{\theta}, \qquad *=+,-, \tag{2.3}$$

$\Omega^{+}=\{\boldsymbol{\theta}:\ \delta>0\}$, $\Omega^{-}=\{\boldsymbol{\theta}:\ \delta\leqq 0\}$, *and* $f(z|\boldsymbol{\theta})$ *and* $\lambda(\boldsymbol{\theta})$ *are defined by* (1.2) *and* (1.3), *respectively.*

PROOF. Using (1.2), (1.3) and (1.4), the Bayes risk is minimized by making decision d_{12}^{+} when $g(z)<0$ and by making decision d_{12}^{0} when $g(z)\geqq 0$ where

$$g(z)=\int [L(d_{12}^{+}:\ \boldsymbol{\theta})-L(d_{12}^{0}:\ \boldsymbol{\theta})] f(z|\boldsymbol{\theta})\lambda(\boldsymbol{\theta})d\boldsymbol{\theta}.$$

Substituting for $L(d_{12}^{+}:\ \boldsymbol{\theta})$ and $L(d_{12}^{0}:\ \boldsymbol{\theta})$ from (1.4), we find $g(z)<0$, if, and only if, $r(z)>k$. That completes the proof of the lemma.

To begin simplifying the *critical equation*

$$r(z)=k \tag{2.4}$$

apply Fubini's theorem to interchange the order of integration in I_1^{+} and I_1^{-} from $\sigma_T^2, \eta_1, \cdots, \eta_{n-1}, \sigma_e^2$ to $\eta_2, \cdots, \eta_{n-1}, \eta_1, \sigma_T^2, \sigma_e^2$. Then the integrations in (2.4) with respect to $\eta_2, \eta_3, \cdots, \eta_{n-1}$ are performed by evaluating the product of the integrals

$$W(y_i)=\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\frac{r}{\sigma_e^2}(y_i-\eta_i)^2+\frac{\eta_i^2}{\sigma_\mu^2}\right]\right\}d\eta_i, \qquad i=2, 3, \cdots, n-1.$$

By completing the square and letting $\sigma_T^2 = r\sigma_\mu^2 + \sigma_e^2$, we have

$$\prod_{i=2}^{n-1} W(y_i) = \left[\frac{2\pi r\sigma_T^2}{\sigma_e^2(\sigma_T^2 - \sigma_e^2)}\right]^{-(n+2)/2} \exp\left\{-r\sum_{i=2}^{n-1} y_i^2 \Big/ 2\sigma_T^2\right\}.$$

At this point in the evaluation of the integrals involved in (2.4), the solution comes to depend on the data through the difference $\sqrt{2}\,y_1$ $(=\bar{x}_1 - \bar{x}_2)$, the among treatment variance estimator

$$s_{TE}^2 = r\sum_{i=1}^{n-1} y_i^2 \Big/ (n-1),$$

and the error variance estimator s_{eE}^2. Then by transforming from η_1, σ_T^2, and σ_e^2 to

$$r = \frac{\sqrt{r}\,\eta_1}{\sigma_e}, \quad \Phi = \frac{\sigma_T^2}{\sigma_e^2}, \quad u^2 = \frac{s_e^2}{\sigma_e^2} \tag{2.5}$$

and defining the "pooled" estimates of σ_T^2 and σ_e^2

$$\begin{aligned} s_T^2 &= \frac{q_P s_{TP}^2 + (n-1)s_{TE}^2}{q}, \\ s_e^2 &= \frac{f_P s_{eP}^2 + f_E s_{eE}^2}{f}, \end{aligned} \tag{2.6}$$

where $q = q_P + (n-1)$ and $f = f_P + f_E$, the solution of the critical equation (2.4) depends on the data, z, through the two statistics

$$\begin{aligned} t &= \frac{\bar{x}_1 - \bar{x}_2}{s_d}, \\ F &= \frac{s_T^2}{s_e^2}, \end{aligned} \tag{2.7}$$

in which $s_d^2 = 2s_e^2/r$.

After the foregoing indicated integrations and transformations are performed in both I_1^+ and I_1^-, the critical equation, (2.4), reduces to

$$\frac{I_2^+(t|F, q, f)}{I_2^-(t|F, q, f)} = k \tag{2.8}$$

where

$$\begin{aligned} I_2^*(t|F, q, f) = \int_1^\infty \int_{\tau^*} \int_0^\infty & |\tau|(\Phi - 1)^{-1/2}\Phi^{-(q+1)/2} u^{q+f-3} \\ & \times \exp\left\{-\frac{1}{2}u^2\left[A\left(\frac{\tau}{u}, \Phi, t, F\right)\right]\right\} du^2 d\tau\, d\Phi \end{aligned}$$

in which $* = +, -$, $\tau^+ = \{\tau | 0 < \tau < \infty\}$, $\tau^- = \{\tau | -\infty < \tau \leq 0\}$, and

$$A(w, x, y, z)=\left\{y\left[\frac{x-1}{x}\right]^{1/2}-w\left[\frac{x}{x-1}\right]^{1/2}\right\}^2+f+\frac{qz}{x}.$$

Since I_2^+ and I_2^- are further reduced by identical steps, we indicate in detail only the reduction steps for I_2^+ and state the analogous results in I_2^-.

Before integrating with respect to u^2, a change of variable from τ to

(2.9) $$\gamma=\left\{\frac{f+q}{f+qF/\Phi}\right\}^{1/2}\left\{\frac{\tau}{u}\left[\frac{\Phi}{\Phi-1}\right]^{1/2}-t\left[\frac{\Phi-1}{\Phi}\right]^{1/2}\right\}$$

is made. The integration with respect to u^2 then gives

(2.10) $$I_3^+(t|F, q, f)=\int_1^\infty(\Phi-1)^{1/2}\Phi^{-(q+3)/2}\left\{f+\frac{qF}{\Phi}\right\}^{-(f'-1)/2}\times\int_{-t\cdot\beta(\Phi)}^\infty[\gamma+t\cdot\beta(\Phi)]\left\{1+\frac{\gamma^2}{f'}\right\}^{-(f'+1)/2}d\gamma\,d\Phi$$

where $I_3^+(t|F, q, f)\propto I_2^+(t|F, q, f)$ and

(2.11) $$f'=f+q,\qquad \beta(\Phi)=\left\{\frac{f'(\Phi-1)}{f\Phi+qF}\right\}^{1/2}.$$

Let $h(z|f')$ denote the density of a t variable with f' degrees of freedom, and let $H(z|f')$ denote the cumulative distribution of the same t variable. Then the integration with respect to γ produces

(2.12) $$I_4^+(t|F, q, f)=\int_1^\infty(\Phi-1)^{1/2}G(\Phi)M^+(t\beta(\Phi)|f')d\Phi$$

where $I_4^+(t|F, q, f)\propto I_3^+(t|F, q, f)$, $G(\Phi)$ is similar to an F density with q and f degrees of freedom and defined by

$$G(\Phi)=\Phi^{-(q+3)/2}\left\{f+\frac{qF}{\Phi}\right\}^{-(f'-1)/2},$$

and M^+ is defined by

(2.13) $$M^+(z|f')=\left\{\frac{f'}{f'-1}\right\}\left\{1+\frac{z^2}{f'}\right\}h(z|f')+zH(z|f').$$

The same sequence of steps will yield $I_4^-(t|F, q, f)\propto I_2^-(t|F, q, f)$ where the constants of proportionality are the same for I_4^- and I_4^+. Thus, the critical equation is reduced to the ratio of two single integrals and expressed by

$$\frac{I_4^+(t|F, q, f)}{I_4^-(t|F, q, f)}=k \tag{2.14}$$

in which $M^-(z|f')=M^+(-z|f')$.

Differentiation of the left member of equation (2.14) with respect to t shows that the ratio is a monotonically increasing function of t. Thus, if t_c is the solution of the critical equation (2.14) for fixed values of k, F, q, and f, then $I_4^+(t_c|F, q, f)/I_4^-(t_c|F, q, f)=k$ and $I_4^+(t|F, q, f)/I_4^-(t|F, q, f)>k$ if, and only if, $t>t_c$. Therefore, the Bayes rule can be expressed as

$$\begin{aligned} &\text{make decision } d_{12}^+ \text{ if } t>t_c,\\ &\text{make decision } d_{12}^0 \text{ if } t\leqq t_c, \end{aligned} \tag{2.15}$$

where t is defined by (2.7) and t_c is the solution of (2.14) for given values of k, F, q, and f.

Now that the Bayes rule can be expressed in terms of the solution to Equation (2.14), the next step is to solve that equation.

3. General recursive solution of the critical equation

This section is used to derive a basic recursive method to generate integrals that are proportional to the integrals I_4^+ and I_4^- in the critical equation (2.14). A specialization of this recursion formula is used in Section 4 to solve for t in the equation $I_4^+(t|F, q, f)/I_4^-(t|F, q, f)=k$ for each fixed set of values F, q, f and k.

The first step in the development is to define a transformation from Φ to ϕ by

$$\cos^2\phi=\frac{qF}{f\Phi+qF} \tag{3.1}$$

and put (for notational convenience only)

$$\sin\theta=\frac{[1-\cos^2\phi/\cos^2\phi_0]^{1/2}t}{[f+(1-\cos^2\phi/\cos^2\phi_0)t^2]^{1/2}} \tag{3.2}$$

in which

$$\cos\phi_0=\left[\frac{qF}{f+qF}\right]^{1/2}.$$

The critical equation is then given by

$$\frac{I^+(q-1, f-3, -2, 1, f'-2)}{I^-(q-1, f-3, -2, 1, f'-2)}=k \tag{3.3}$$

where, for $*=+$ or $-$

$$I^*(m, n, p, w, r)=\int_{\phi_0}^{\pi/2} \cos^m \phi \sin^n \phi \cos^p \theta \sin^w g^*(\theta|r)d\phi , \tag{3.4}$$

$$g^*(\theta|r)=\frac{2\pi^{1/2}\Gamma[(r+2)/2]}{(r+2)^{1/2}\Gamma[(r+1)/2]} \cos \theta M^*[(r+2)^{1/2} \tan \theta|r+2] , \tag{3.5}$$

in which M^* is the function defined by (2.13).

To develop a recursive formula to generate $I^*(m, n, p, w, r)$, we use the familiar recursive formula to evaluate $\int \cos^n x\, dx$ to obtain

$$g^*(\theta|r)=\frac{r}{r-1} g^*(\theta|r-2)-\frac{\cos^r \theta}{r-1} , \tag{3.6}$$

for $r=2, 3, \cdots$ and for $*=+$ or $-$. The initial values for the recursion given in (3.6) are

$$\begin{aligned} &g^{\pm}(\theta|0)=1\pm\sin \theta , \\ &g^{\pm}(\theta|1)=\cos \theta \pm\left(\frac{\pi}{2}\pm\theta\right) \sin \theta . \end{aligned} \tag{3.7}$$

Using the result (3.6) along with a method patterned after the standard recursive formula for evaluation $\int \sin^n x \cos^m x\, dx$, we prove the following theorem.

THEOREM 3.1 (*Basic Recursion Theorem*). *If* I^*, *for* $*=+$ *or* $-$, *is the integral defined in* (3.4), *then*

$$\begin{aligned} &I^*(m, n, -2, 1, r) \\ &\quad =A_0B^*(m-1, n-1, -4, 3, r)+A_1I^*(m, n-2, -2, 1, r) \\ &\quad\quad +A_2I^*(m-2, n, -2, 1, r)+\frac{1}{r}J(m, n, r, 1) , \end{aligned} \tag{3.8}$$

where $S^2=t^2/(f+t^2)$, $C^2=1-S^2$,

$$D=\frac{f+qF}{qF} , \quad A_0=\frac{C^2}{rS^2D} , \quad A_1=(n-1)\frac{D-1}{rD} , \quad A_2=\frac{m-1}{rD} ,$$

$$J(m, n, p, w)=\int_{\phi_0}^{\pi/2} \cos^m \phi \sin^n \phi \cos^p \theta \sin^w \theta\, d\phi ,$$

and

$$B^*(m, n, p, w, r)=\cos^m \phi \sin^n \phi \cos^p \theta \sin^w \theta g^*(\theta|r)\Big|_{\phi=\phi_0}^{\pi/2} .$$

PROOF. To establish (3.8) we put

$$I^*(m, n, p, w, r)=\int_{\phi_0}^{\pi/2} u\,dv$$

where

$$(3.9)\quad \begin{aligned} u&=\frac{C^2}{S^2D}(\cos\phi)^{m-1}(\sin\phi)^{n-1}(\cos\theta)^{p-3}(\sin\theta)^w g^*(\theta|r)\,,\\ dv&=\frac{S^2D}{C^2}\cos\phi\sin\phi\cos^3\theta\,d\phi\,. \end{aligned}$$

From (3.2) we have

$$(3.10)\quad \frac{d\theta}{d\phi}=\frac{S^2D}{C^2}\left\{\frac{\cos\phi\sin\phi\cos^3\theta}{\sin\theta}\right\}.$$

We then use (3.10) and the chain rule of differentiation to determine du and $v=\int\sin\theta\,d\theta=-\cos\theta$. Thus, for all choices of m, n, p, w, and r, the integration by parts formula gives

$$\begin{aligned}(3.11)\quad &(p-2)I^*(m, n, p, w, r)\\ &=-\frac{C^2}{S^2D}B^*(m-1, n-1, p-2, w, r)\\ &\quad+(1+w)I^*(m, n, p+2, w-2, r)-J(m, n, p+r+2, w-2)\\ &\quad+\frac{C^2(n-1)}{S^2D}I^*(m, n-2, p-2, w, r)\\ &\quad+\frac{C^2(m-1)}{S^2D}I^*(m-2, n, p-2, w, r)\,.\end{aligned}$$

To simplify (3.11), we first use the identity $\sin^2 x=1-\cos^2 x$ to write

$$(3.12)\quad I^*(a, b, c, d, e)=I^*(a, b, c, d-2, e)-I^*(a, b, c+2, d-2, e)\,.$$

We then use (3.2) to deduce

$$(3.13)\quad 1-S^2D\cos^2\phi=\frac{C^2}{\cos^2\theta}\,.$$

Then we use (3.13) to write

$$(3.14)\quad \begin{aligned} I^*(a, b, c, d, e)&=\frac{I^*(a, b, c+2, d, e)-S^2DI^*(a+2, b, c+2, d, e)}{C^2}\,,\\ I^*(a, b, c, d, e)&=\frac{1-S^2D}{C^2}I^*(a, b, c+2, d, e)\\ &\quad-\frac{S^2D}{C^2}I^*(a, b+2, c+2, d, e)\,. \end{aligned}$$

Using (3.12) and (3.14) and putting $w=3$ and $p=-2$ reduces (3.11) to (3.8). That completes the proof.

The next objective is to specialize the basic recursive relationship (3.8) to generate the integrals $I^*(q-1, f-3, -2, -1, f'-2)$ for $*=+$ or $-$ and fixed values of q, f, and $f'=f+q$.

4. Specific recursive solutions of the critical equation

This section gives a method to determine the value of t, say t_c, which satisfies the critical equation (2.14) for any specified set of values for (k, F, q, f) in these cases:

Case 1: $F<\infty$, q is an even integer, and f is an even integer.
Case 2: $F<\infty$, $q=\infty$, and f is an even integer.
Case 3: $F=\infty$, q is an even integer, and f is an even integer.

Those three cases are discussed in Sections 4.2, 4.3, and 4.4, respectively.

4.1 *Refining the basic recursion formula*

In our recursive scheme with Equation (3.8), m is an index to be terminated at $q-1$, n is an index to be terminated at $f-3$, and r is an index to be terminated at $f'-2$. But to satisfy the constraint $f'=f+q$, we require that $r=m+n+2$. Thus, we simplify the notation of the previous section as follows:

$$\begin{aligned} &J(m, n)=J(m, n, r, 1), && \text{for } r=m+n+2, \\ (4.1)\qquad &I^{\pm}(m, n)=I^{\pm}(m, n, -2, 1, r), && \text{for } r=m+n+2, \\ &B_n^{\pm}=B^{\pm}(0, n, -4, 3, n+4). && \end{aligned}$$

Further, from the definition of $B^{\pm}(\ )$ in Theorem 3.1, it follows that

$$(4.2)\qquad B^{\pm}(m, n, -4, 3, m+n+4)=\begin{cases} \dfrac{S^3}{C^4}g_{n+4}^{\pm}, & \text{for } m=0 \\ 0, & \text{for } m>0, \end{cases}$$

where

$$(4.3)\qquad g_r^{\pm}=g^{\pm}(\theta|r)\Big|_{\theta=\sin^{-1}(S)}$$

and C and S are defined for Theorem 3.1. Also, it follows directly from (3.6) and (3.7) that

$$g_r^{\pm}=\frac{rg_{r-2}^{\pm}-C^r}{r-1},$$

$$(4.4)\qquad g_0^{\pm}=1\pm S,$$

$$g_1^{\pm}=C+S\left[\tan^{-1}\left(\frac{t^2}{f}\right)\pm\left(\frac{\pi}{2}\right)\right].$$

Then by arguments similar to those used in proving Theorem 3.1, it can be shown that

$$(4.5)\qquad J(m,n)=\begin{cases}\dfrac{C^{r-2}S-(n-1)(D-1)J(1,n-2)}{(n+2)D}, & \text{for } m=1,\\[2ex] \dfrac{(n-1)(D-1)J(m,n-2)+(m-1)J(m-2,n)}{(r-1)D} & \text{for } m>1,\end{cases}$$

where C, S, and D are as defined for Theorem 3.1.

By using the definitions in (4.1) and the results in (4.2), (4.3), (4.4) and (4.5) in the recursion (3.8), we can write

$$(4.6)\qquad I^*(m,n)=\begin{cases}\dfrac{(S/C^2)g_{n+1}^*+(n-1)(D-1)I^*(1,n-2)}{(n+2)D}, & \text{for } m=1,\\[2ex] \dfrac{(n-1)(D-1)I^*(m,n-2)+(m-1)I^*(m-2,n)}{(r-1)D} & \text{for } m>1,\end{cases}$$

for $*=+$ or $-$, $r=n+m+2$, and where S, C and D are previously defined.

The recursions (4.6) are not precise enough to avoid difficulties in round off error on a computer. Therefore, one final refinement is made in the recursive method to generate the ratio of the integrals in Equation (3.3). The result is stated in the following theorem:

THEOREM 4.1. *Let*

$$S^2=\frac{t^2}{f+t^2},\qquad C^2=1-S^2,$$

$$D=\frac{f+qF}{qF},\qquad E=\frac{f}{f+qF},$$

$$(4.7)\qquad G_n^{\pm}=\frac{2[(n-1)/2]!\,g_n^{\pm}}{\pi^{1/2}[(n-2)/2]!},$$

$$L^{\pm}(m,n)=\frac{2(m+n+1)C^2D^{(m+1)/2}[(m+n-1)/2]!}{\pi^{1/2}S[(m-1)/2]!\,[(n-1)/2]!}I^{\pm}(m,n).$$

Then

$$(4.8)\qquad L^{\pm}(m, n)=\begin{cases} G_2^{\pm}, & \text{for } n=1 \text{ and } m\geqq 1, \\ G_{n+1}^{\pm}+E\cdot L^{\pm}(1, n-2), & \text{for } m=1 \text{ and } n>1, \\ L^{\pm}(m-2, n)+E\cdot L^{\pm}(m, n-2), & \text{for } m>1 \text{ and } n>1, \end{cases}$$

where

$$(4.9)\qquad \begin{aligned} G_1^{\pm}&=\frac{2\{C+S[\tan^{-1}(t^2/f)\pm(\pi/2)]\}}{\pi}, \qquad G_2^{\pm}=(1\pm S)^2, \\ G_n^{\pm}&=n\frac{G_{n-2}^{\pm}}{n-2}-\frac{C^n[(n-3)/2]!}{\pi^{1/2}[(n-2)/2]!}, \qquad \text{for } n>2. \end{aligned}$$

PROOF. The results in (4.8) are immediate by using the definition (4.7) and the recursion (4.6) while (4.9) follows directly from the definition of $G_n^{\pm}$ and (4.4).

4.2 *Solution for Case* 1 ($F<\infty$, *q even, and f even*)

In this section we outline an iterative method that uses recursions (4.8) and (4.9) to determine the value of t, say t_c, which satisfies Equation (3.3) for a specific set of values (k, F, q, f) where $F<\infty$, q is an even positive integer, and f is an even positive integer.

We note that finding t_c, which satisfies the critical equation in forms (2.4), (2.14), and (3.3) for given values of (k, F, q, f) is equivalent to finding t_c, which satisfies

$$(4.10)\qquad \frac{L^{+}(q-1, f-3)}{L^{-}(q-1, f-3)}=k$$

for the same values of (k, F, q, f). Thus we use the recursions in (4.8) for odd values of both m and n with termination points $(q-1)$ and $(f-3)$, respectively. By simultaneously using Equation (4.9) to generate the necessary $G_n^{\pm}$ values and Equation (4.8) to generate $L^{\pm}(q-1, f-3)$ we obtain an iterative method to solve (4.10). To illustrate the method, we suppose

$$(4.11)\qquad k=100, \quad F=4, \quad q=6, \quad f=10,$$

and list the iteration steps.

1. Let t_1 be a first guess at the solution to (4.10) for the arguments given in (4.11).
2. Use (4.7) to obtain S, C, D, and E from t_1, F, q, and f.
3. Use (4.9) to generate G_r^{*}, for $r=2, 4, 6, 8$ and $*=+, -$.
4. Use (4.8) to generate

(i) $L^*(m, 1)$, $m=1, 3, 5$, for $*=+,-$.
(ii) $L^*(1, n)$, $L^*(3, n)$, $L^*(5, n)$, $n=3, 5, 7$, for $*=+,-$.

5. Calculate
$$k(t_1)=L^+(5, 7)/L^-(5, 7)\,.$$
6. Compute the derivatives $L^{*\prime}(5, 7)$ of $L^*(5, 7)$ with respect to t for $*=+,-$. Those calculations can be completely in conjunction with the corresponding calculations of $L^*(5, 7)$ by differentiating the recursions which generate $L^*(m, n)$ and G_n^* to obtain recursions which generate $L^{*\prime}(m, n)$ and $G_n^{*\prime}$.
7. Compute the derivative $k'(t_1)$ of the ratio $k(t_1)$ where
$$k'(t_1)=[L^-(5, 7)L^{+\prime}(5, 7)-L^+(5, 7)L^{-\prime}(5, 7)]/[L^-(5, 7)]^2.$$
8. Obtain a second approximation of t, using
$$t_2=t_1+(k-k(t_1))/k'(t_1)\,.$$
9. Repeat steps 2 through 8 replacing t_1 by t_2 and obtain a third approximation t_3 for t_c. Iterate to convergence.

The foregoing method was used along with appropriate interpolation formulae to compute a set of critical t value tables for selected values of (k, F, q, f). The tables in Waller and Duncan [4] contained an error, but a corrected set of tables are available in Waller and Duncan [5]. An algorithm (ASA FORTRAN) for calculating the critical t-values, t_c, is available on request.

4.3 *Solution for Case* 2 ($F<\infty$, $q\to\infty$, f *even*)

Case 2 is identical to Case 1 except that q is no longer finite. Rather than proceeding directly to the limit of (4.8) as $q\to\infty$, it is convenient to prove this theorem:

THEOREM 4.2. *If q is an even integer* ($\geqq 2$) *and if f is an even integer* ($\geqq 4$), *then*

$$L^*(q-1, f-3)=\sum_{i=0}^{(f-4)/2} E^i\binom{(q-2)/2+i}{i}G^*_{f-2-2i}\,, \qquad *=+,-, \tag{4.12}$$

where E, L^*, and *G^* are defined in* (4.7), (4.8) *and* (4.9).

PROOF. (i) If $f=4$, then by (4.8) $L^*(q-1, 1)=G_2^*$ for all values of q and (4.12) holds.

(ii) If $q=2$, a proof by induction yields (4.12). First, part (i) shows that $L^*(1, 1)=G_2^*$. Second, from the induction assumption that

$$L^*(1, f-5)=\sum_{i=0}^{(f-6)/2} E^iG^*_{f-4-2i}$$

and (4.8) it follows that (4.12) holds.

(iii) Finally, that (4.12) holds for $q>2$ and $f>4$ follows by substituting the inductive hypotheses

$$L^*(q-3,\, f-3)=\sum_{i=0}^{(f-4)/2} E^i\binom{(q-4)/2+i}{i} G^*_{f-2+2i}\,,$$

$$L^*(q-1,\, f-5)=\sum_{i=0}^{(f-6)/2} E^i\binom{(q-2)/2+i}{i} G^*_{f-4+2i}\,,$$

into (4.8).

An immediate consequence of Theorem 4.2 is as follows:

THEOREM 4.3. *If f is an even integer, then*

$$L^*(\infty,\, f-3)=\sum_{i=0}^{(f-4)/2}\left(\frac{f}{2F}\right)^i \frac{1}{i!} G^*_{f-2-2i}\,, \qquad \textit{for } *=+,\,-. \tag{4.13}$$

PROOF. In view of (4.12), we need only show that

$$\lim_{q\to\infty}\binom{(q-2)/2+i}{i}E^i=\left(\frac{f}{2F}\right)^i\frac{1}{i!}$$

to establish (4.13). From the definition of E in (4.7) and $\binom{a}{b}$ we have

$$\lim_{q\to\infty}\binom{(q-2)/2+i}{i}E^i=\left(\frac{f}{2F}\right)^i\frac{1}{i!}\lim_{q\to\infty}\prod_{j=1}^{i}\frac{1+2(j-1)/q}{1+1/q}=\left(\frac{f}{2F}\right)^i\frac{1}{i!}\,.$$

The critical t values for Case 2 can now be found by using (4.13) in an iterative method similar to that given in Section 4.2 for Case 1.

4.4 *Solution for Case* 3 ($F\to\infty$, *q even, f even*)

To obtain the desired integrals for $F=\infty$, q an even integer, and f an even integer, we use (4.12) to prove this theorem:

THEOREM 4.4. *If q is an even integer* ($\geqq 2$) *and f is an even integer* ($\geqq 4$), *then*

$$\lim_{F\to\infty} L^*(q-1,\, f-3)=G^*_{f-2}\,, \qquad \textit{for } *=+,\,-. \tag{4.14}$$

Remark. When $F=\infty$, we obtain the same critical value for all values of q.

PROOF. Since G^*_n is not a function of F, it follows from (4.12) that

$$\lim_{F\to\infty} L^*(q-1,\, f-3)=G^*_{f-2}+\sum_{i=1}^{(f-4)/2} G^*_{f-2-2i}\binom{(q-2)/2+i}{i}\lim_{F\to\infty}E^i\,.$$

Now by (4.7) $E=f/(f+qF)$. Therefore $\lim_{F\to\infty}E^i=0$, for all finite values of i, which establishes (4.14).

Here again an iterative method similar to that outlined for Case 1 can be used to calculate the critical t value for any given pair of values for k and f.

That concludes the solutions for the three cases defined at the beginning of this section. The somewhat more cumbersome recursive techniques to obtain solutions to some remaining cases are discussed in the next two sections.

5. Case 4: The two treatment experiment

The solution for the two-treatment experiment is developed in this section. In this special case ($q=1$), the quantities F and t^2 (see (2.7)) are identical. Setting $q=1$ and $F=t^2$ simplifies many of the quantities defined in Theorem 3.1. The following lemmas summarize the results for $q=1$ and lead to the recursion formula for the integrals $I^{\pm}(0, n)$ given in Theorem 5.1.

LEMMA 5.1. *If* $q=1$, *then*

(5.1)
$$\text{(i)}\quad D=\frac{1}{S^2},$$
$$\text{(ii)}\quad E=C^2,$$
$$\text{(iii)}\quad \sin\phi=\frac{C}{\cos\theta},$$
$$\text{(iv)}\quad \cos\phi=\frac{(\cos^2\theta-C^2)^{1/2}}{\cos\theta},$$
$$\text{(v)}\quad d\phi=\frac{C\sin\theta\, d\theta}{(\cos^2\theta-C^2)^{1/2}\cos\theta}.$$

PROOF. Using $F=t^2$ and $q=1$ in the definitions of D and E as given (4.7) establishes (i) and (ii). Parts (iii), (iv), and (v) follow directly from (3.12) when D is replaced by $1/S^2$.

LEMMA 5.2. *If* $q=1$ *and* $J(0, n)$ *is as defined for* (4.1), *then*

$$J(0, n)=\frac{C^{n+1}S^2\pi}{4}. \tag{5.2}$$

PROOF. By definition (see (4.1) and Theorem 3.8)

$$J(0, n)=\int_{\phi_0}^{\pi/2}\sin^n\phi\cos^{n+2}\theta\sin\theta\, d\phi.$$

Using parts (iii) and (v) of (5.1) to replace $\sin\phi$ and $d\phi$ gives

$$J(0, n)=C^{n+1}\int_{\theta}^{\theta_0}\frac{\cos\theta\sin^2\theta}{(\cos^2\theta-C^2)^{1/2}}d\theta, \tag{5.3}$$

where $\sin^2\theta_0 = S^2$. Evaluation of (5.3) gives (5.2).

LEMMA 5.3. *If* E, D, $I^{\pm}(m, n)$ *and* $J(m, n)$ (*see* (4.1) *and* (5.1)), *are as previously defined, then*

$$(5.4)\qquad I^{\pm}(0, n) = \left[1 + \left(\frac{n-2}{n+1}\right)C^2\right] I^{\pm}(0, n-2) - \frac{(n-3)nC^2}{n^2-1} I^{\pm}(0, n-4) - \frac{C^{n-1}S^2\pi}{2(n^2-1)} .$$

PROOF. Using $m=2$ in (4.6), we obtain

$$(5.5)\qquad I^{\pm}(2, n) = \left(\frac{n-1}{n+3}\right) E I^{\pm}(2, n-2) + \frac{1}{(n+3)D} I^{\pm}(0, n) .$$

Definition (3.6) and (4.1) imply that

$$(5.6)\qquad I^{\pm}(2, n) = \left(\frac{n+4}{n+3}\right) I^{\pm}(0, n) - \frac{1}{n+3} J(0, n) - I^{\pm}(0, n+2) .$$

Likewise,

$$(5.7)\qquad I^{\pm}(2, n-2) = \left(\frac{n+2}{n+1}\right) I^{\pm}(0, n-2) - \left(\frac{1}{n+1}\right) J(0, n-2) - I^{\pm}(0, n) .$$

By substituting (5.6) and (5.7) into (5.5) and by replacing $(n+2)$ by n we obtain

$$(5.8)\qquad I^{\pm}(0, n) = \left[\left(\frac{n-3}{n+1}\right)E + \frac{1}{(n+1)D} + \left(\frac{n+2}{n+1}\right)\right] I^{\pm}(0, n-2) - \frac{(n-3)n}{n^2-1} E I^{\pm}(0, n-4) - \left(\frac{1}{n+1}\right) J(0, n-2) + \frac{(n-3)E}{n^2-1} J(0, n-4) .$$

The use of (5.2) and parts (i) and (ii) of (5.1) establishes (5.4).

The foregoing lemmas and definitions are used to state and prove the following theorem which can be used to obtain critical t-values for a two-treatment experiment.

THEOREM 5.1. *Let*

$$(5.9)\qquad L^{\pm}(0, n) = \frac{2C^2(n+1)}{\pi S^2} I^{\pm}(0, n) .$$

If $q=1$, *then*

$$(5.10)\qquad L^{\pm}(0, n) = \frac{(n+1)+(n-2)C^2}{n-1} L^{\pm}(0, n-2) - \frac{nC^2}{n-1} L^{\pm}(0, n-4)$$

$$-\frac{C^{n+1}}{n-1}, \qquad \textit{for } n \geqq 4 .$$

PROOF. Multiplying (5.4) by $2C^2(n+1)/\pi S^2$ and replacing $I^{\pm}(0, n-2)$ and $I^{\pm}(0, n-4)$ by $[\pi S^2/2C^2(n-1)]L^{\pm}(0, n-2)$ and $[\pi S^2/2C^2(n-3)]L^{\pm}(0, n-4)$, respectively, gives (5.10).

The following quantities are needed to initialize the recursion (5.10) and can be obtained by integration:

$$\text{(5.11)} \qquad \begin{aligned} L^{\pm}(0, 0) &= \frac{1-2C^2+C^3}{S^2} \pm \frac{2}{\pi S^2}\left[CS+(S^2-C^2)\tan^{-1}\left(\frac{S}{C}\right)\right], \\ L^{\pm}(0, 1) &= (1 \pm S)^2, \\ L^{\pm}(0, 2) &= \frac{1}{S^2}[3-4C^2+C^5] \\ &\quad \pm \frac{2}{\pi S^2}\left[CS(1+2S^2)+(3-4C^2)\tan^{-1}\left(\frac{S}{C}\right)\right], \\ L^{\pm}(0, 3) &= \frac{2[2+2S^2+S^2C^2] \pm 2S(4+C^2)}{2} . \end{aligned}$$

The recursion in (5.10) and the starting values in (5.11) can be used in an iterative scheme (similar to that presented in Section 4.2) to determine that value of t, say t_c, which satisfies the equation

$$\text{(5.12)} \qquad \frac{L^{+}(0, f-3)}{L^{-}(0, f-3)} = k$$

for each given pair of values (k, f) such that $f \geqq 3$.

That concludes the development of the solution for the two-treatment experiment.

6. Discussion of other cases

Three remaining cases of interest for $F < \infty$ can be identified according to the possible values of q and f as follows:

Case 5: q and f are both odd integers $\geqq 4$.
Case 6: q is an even integer $\geqq 2$, and f is an odd integer $\geqq 3$.
Case 7: q is an odd integer $\geqq 3$, and f is an odd integer $\geqq 3$.

To use the recursive techniques in Sections 3 and 4 to solve the three cases above, recursions that generate the quantities $L^{\pm}(0, n)$ and $L^{\pm}(m, 0)$ are needed to initiate the recursive system given in (4.8). Those recursions can be derived with the method used to prove Theorem 3.1 along the results and definitions in Theorem 4.1. The derivations are similar to those used in Section 5 to derive a recursive method for generating $L^{\pm}(0, n)$ when $q=1$. However, the recursions and their initial values require evaluation of certain elliptic integrals, which tends to make the development tedious and bulky. Therefore, a detailed treatment of the recursive method for the cases identified above will be presented after further research directed to simplifying the technique is completed. In the meantime access to the development in its current state is available on microfilm in the thesis by Waller [3].

7. The Bayes rule

The foregoing development completes the derivation of the Bayes rule for the symmetric multiple comparisons problem presented and discussed by Waller and Duncan [4]. In this section we state that Bayes rule. First, we state the rule for those cases where $q>1$. Second, the rule for the case when $q=1$ is given.

Bayes rule ($q>1$): Let $t_c=t(k, F, q, f)$ be that value of t that satisfies Equation (2.14) (or equivalently (4.10)) for a specified vector (k, F, q, f). Let $t_{ij}=(\bar{x}_i-\bar{x}_j)/s_d$ where s_d is defined for (2.7). Then the Bayes rule for the symmetric multiple comparisons problem is the simultaneous application of this rule to all $n(n-1)/2$ pairs of treatment means:

$$\begin{aligned} &\text{If } t_{ij}>t_c, \text{ conclude } \mu_i>\mu_j. \\ (6.1)\qquad &\text{If } |t_{ij}|<t_c, \text{ conclude } \mu_i \text{ is unranked relative to } \mu_j. \\ &\text{If } t_{ij}<-t_c, \text{ conclude } \mu_i<\mu_j. \end{aligned}$$

Bayes rule ($q=1$): Let t_0 be that value of t that satisfies Equation (5.1) for a specified vector (k, f). Let $t_{12}=(\bar{x}_1-\bar{x}_2)/s_d$ where s_d is defined for (2.7). The Bayes rule for the symmetric multiple comparisons problem for means μ_1 and μ_2 is the application of (6.1) with $t_c=t_0$, $i=1$, and $j=2$.

That completes the statement of the Bayes rule for the symmetric multiple comparisons problem.

8. Conclusion

The mathematical derivation of a Bayes rule for the symmetric multiple comparisons problem has been presented in the foregoing sections of this paper. An application method for that Bayes rule and tables of the critical t-values for selected values of k, F, q and f were presented in Waller and Duncan [4], [5].

There are, however, some open problems remaining that need to be solved to complete the solution. One open problem is the asymptotic solution for the case when $f \to \infty$ and q is finite. Another result of interest is refining the recursive technique to obtain a set of recursions for Cases 5, 6, and 7, which are as precise and efficient for computer use as those used in Cases 1, 2, 3, and 4. That result would eliminate the necessity of using interpolation methods to determine the critical t-values for Cases 5, 6, and 7. We note that even though the available tables values for those cases were determined by interpolation the numbers are accurate to two decimal places. Thus, the approximations involved are of no practical importance.

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