ON SEQUENTIAL DISTINGUISHABILITY FOR THE EXPONENTIAL FAMILY*

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1. Introduction

Let $\{X_n,\ n\geqq 1\}$ be an iid (independent and identically distributed) stochastic sequence assumed to be governed by a member of a countable family of probability measures $\Phi=\{P_\theta:\ \theta\in\Omega\}$ where P_θ are defined on an appropriate probability space and Ω is countable. Observing sequentially the stochastic sequence $\{X_n,\ n\geqq 1\}$ we want to stop at some finite stage and decide in favour of a member of the family Φ with a uniformly small probability of error. The family Φ is said to be "Sequentially Distinguishable" if for any given ε $(0<\varepsilon<1)$ there exists a stopping time t and a terminal decision function $\delta(X_1,\cdots,X_t)$ such that $\mathrm{P}_\theta(t<\infty)=1\ \forall\,\theta\in\Omega$ and $\sup_{\theta\in\Omega}\mathrm{P}_\theta(\delta(X_1,\cdots,X_t)\neq\theta)\leqq\varepsilon$.

The family Φ being countable, there is no loss of generality in assuming the existence of probability densities $\{f_\theta:\ \theta\in\Omega\}$ with respect to some σ-finite measure μ. To simplify notations we write $f_i=f_{\theta_i}$, and $P_i=P_{\theta_i}$, etc. Further, let $f_{i,n}$ be the joint probability density function of $(X_1, X_2,\cdots,X_n)$ with respect to μ_n (the μ-measure in n-dimensions). In what follows we shall take a doubly indexed sequence of constants $\{a_{ij}\}$ such that $a_{ij}>1$ and $\sum_{i\neq j}a_{ij}^{-1}\leqq\varepsilon$ for a given ε $(0<\varepsilon<1)$ and $\forall\,j$. Motivated by Wald's sequential probability ratio test (SPRT) Robbins [7] defined a general stopping time for the sequential distinguishability problem as follows:

$$(1.1)\qquad \begin{aligned} N&=\inf\{n\geqq 1:\ f_{i,n}\geqq\sup_{j\neq i}a_{ij}f_{j,n}\ \text{for some}\ i\}\\ &=\infty\qquad \text{if no such}\ n\,, \end{aligned}$$

and assert P_i $(\Longleftrightarrow\theta_i)$ if N stops with i. Let $a_i=\{\text{accept}\ \theta_i\}$. By assuming N terminates, it follows that

$$\begin{aligned}&\mathrm{P}_j(\delta(X_1,\cdots,X_N)\neq\theta_j)\\ &\quad=\sum_{i\neq j}\mathrm{P}_j(a_i)=\sum_{i\neq j}\sum_{n=1}^{\infty}\int_{\{N=n,a_i\}}f_{j,n}d\mu_n\leqq\sum_{i\neq j}a_{ij}^{-1}\,\mathrm{P}_i(a_i)\leqq\varepsilon\qquad\forall\,j\,.\end{aligned}$$

* Part of this paper was written while the author was at Columbia University.

This stopping time has been studied by the author ([5], [6]) and a number of results have been obtained. The object of this paper is to generalize some of the specific examples of [6] to the case where f_θ is the one parameter exponential probability density function. In Section 2 we give some definitions and results from [6] which are essential for the succeeding main sections.

2. Preliminaries

Let $I(f_i:f_j)$ denote the Kullback-Leibler information measure between P_i and P_j, and is defined as

$$I(f_i:f_j)=\mathrm{E}_i \log (f_i/f_j)=\int f_i \log (f_i/f_j)d\mu \ .$$

We assume that $0<\inf_{j\neq i} I(f_i:f_j)\leqq\infty \ \forall\, i$. It has been shown in [6] that

$$\mathrm{E}_i\, N \geqq \sup_{j\neq i} \{(\log a_{ij})/I(f_i:f_j)\} \ . \tag{2.1}$$

This inequality is universal for the stopping time (1.1).

We note that the problem of sequential distinguishability can be done (at least in principle) by a sequence of SPRT's. Consequently, let t_{ij} denote the stopping time of an SPRT for testing $H_i: f=f_i$ against $H_j: f=f_j$, when the error probabilities are $\alpha=\beta=\varepsilon$. The following definition is due to Robbins [7].

DEFINITION. The stopping rule N is said to be asymptotically optimal if

$$\lim_{\varepsilon\to 0} \frac{\mathrm{E}_i\, N}{\sup_{j\neq i} \mathrm{E}_i\, t_{ij}}=1 \ .$$

The definition is interesting because of the known optimality of SPRT. In [6] we proved the following elementary lemma.

LEMMA 2.1. *The stopping rule N is asymptotically optimal if*

$$\mathrm{E}_i\, N \sim (-\log \varepsilon)/\inf_{j\neq i} I(f_i:f_j) \qquad \text{as } \varepsilon\to 0 \ .$$

We now proceed with the main problem of this paper.

3. The problem and the stopping rule

Let $f(x,\theta)=\exp(\theta x-b(\theta))$ be the p.d.f. of a random variable X with respect to some σ-finite measure μ. It is known that the natural parameter set $\Theta=\left\{\theta: \int \exp(\theta x-b(\theta))d\mu(x)<\infty\right\}$ is a convex set. Further, it is well known that $b(\theta)$ is differentiable on the interior of Θ, and $\mathrm{E}_\theta X=b'(\theta)$, $\sigma_X^2(\theta)=b''(\theta)>0$, so that $b'(\theta)$ is strictly increasing and $b(\theta)$ is a strict convex function. Let $\Omega=\{\theta_i: i\in Z\}$, where θ_i's are the interior points of Θ, and Z is the set of integers. With no loss of generality we can assume that Ω is an ordered set in the usual direction, i.e. $\cdots<\theta_{i-1}<\theta_i<\theta_{i+1}<\cdots$. Let P_θ denote the probability measure pertaining to $f(x,\theta)$, and set $\Phi=\{P_\theta: \theta\in\Omega\}$. Let $X_1, X_2, \cdots$ be an iid sequence assumed to be governed by a member of Φ. We want to find a sequential procedure (N, δ) where N is a stopping rule and δ a terminal decision function which accepts a member $\theta_i\in\Omega$ such that (i) $\mathrm{P}_{\theta_i}(N<\infty)=1\ \forall\, \theta_i\in\Omega$, and (ii) $\sup_{\theta_i\in\Omega}\mathrm{P}_{\theta_i}(\text{error})\leqq\varepsilon$ for a given ε $(0<\varepsilon<1)$.

First we make some useful observations. We write $P_i=P_{\theta_i}$, $\mathrm{E}_i=\mathrm{E}_{\theta_i}$, $f_i=f(X,\theta_i)$, and $I(\theta_i:\theta_j)=I(f_i:f_j)$, etc. without any further comment. The Kullback-Leibler information measure is

$$\begin{aligned}(3.1)\qquad I(\theta_i:\theta_j)&=\mathrm{E}_i\log(f_i/f_j)\\&=\{(\theta_i-\theta_j)b'(\theta_i)-(b(\theta_i)-b(\theta_j))\}>0\ .\end{aligned}$$

If we set $F(\theta_j)=(b(\theta_j)-b(\theta_i))/(\theta_j-\theta_i)$, $j\neq i$, (θ_i fixed), then $\partial F/\partial\theta_j=(I(\theta_i:\theta_j))/(\theta_i-\theta_j)^2>0$. Hence $F(\theta_j)$ is increasing in θ_j for each fixed θ_i. It follows from (3.1) that $\partial I/\partial\theta_j\gtrless 0$ according as $\theta_j>\theta_i$ or $\theta_j<\theta_i$ (since $b'(\cdot)$ is increasing). Thus $I(\theta_i:\theta_j)$ is increasing (decreasing) in θ_j according as $\theta_j>\theta_i$ or $\theta_j<\theta_i$. Before choosing a_{ij} for the proposed stopping time (1.1) we make the following assumptions.

Let $g(x)$ be a real function such that

(i) $g'(x)>0$, $g''(x)\geqq 0$, so that $g(\cdot)$ is a strictly increasing convex function and $g'(\cdot)$ is non-decreasing.

(ii) $g(\theta_{i+1})-g(\theta_i)\geqq 1\ \forall\, i\in Z$.

(iii) $\psi(\theta_j)=\{(b(\theta_j)-b(\theta_i))/(\theta_j-\theta_i)-(g(\theta_j)-g(\theta_i))(\log\alpha)/n(\theta_j-\theta_i)\}$ $(\alpha>1)$, is increasing in $\theta_j>\theta_i$ for some $n\geqq m<\infty$.

(iv) $d(\theta_i,\theta_j)=(I(\theta_i:\theta_j))/(g(\theta_j)-g(\theta_i))$ is increasing in $\theta_j>\theta_i$.

We now choose $a_{ij}=\alpha^{|g(\theta_i)-g(\theta_j)|}$, $\alpha>1$. Assumption (ii) $\Rightarrow|g(\theta_i)-g(\theta_j)|\geqq|j-i|$, and hence

$$(3.2)\qquad \sum_{i\neq j}a_{ij}^{-1}\leqq\sum_{i\neq j}\alpha^{-|i-j|}\leqq 2/(\alpha-1)\ .$$

Recalling the stopping time (1.1) we have

$$\begin{aligned}(3.3)\qquad N&=\inf\{n>1:\ f_{i,n}\geqq\sup_{j\neq i}a_{ij}f_{j,n}\ \text{for some}\ i\}\\&=\inf\{n>1:\ 0\geqq\max[\sup_{j>i}R_n(i,j),\ \sup_{j<i}R_n(i,j)]\ \text{for some}\ i\}\ ,\end{aligned}$$

where $R_n(i,j)=|g(\theta_i)-g(\theta_j)|(\log\alpha)/n+(\theta_j-\theta_i)S_n/n-(b(\theta_j)-b(\theta_i))$, and where $S_n = X_1 + X_2 + \cdots + X_n$. Using assumptions (i) and (iii), simple computations simplify (3.3) to as follows:

$$(3.4) \qquad N = \inf\left\{n \geqq m : \frac{b(\theta_i)-b(\theta_{i-1})}{\theta_i-\theta_{i-1}} + \frac{g(\theta_i)-g(\theta_{i-1})}{\theta_i-\theta_{i-1}} \frac{\log\alpha}{n} \leqq \frac{S_n}{n} \leqq \frac{b(\theta_{i+1})-b(\theta_i)}{\theta_{i+1}-\theta_i} - \frac{g(\theta_{i+1})-g(\theta_i)}{\theta_{i+1}-\theta_i} \frac{\log\alpha}{n} \text{ for some } i\right\},$$

and accept θ_i (a_i) if N stops with i. First we have the following.

LEMMA 3.1. (i) $P_i(N<\infty)=1 \ \forall i \in Z$, *and* (ii) $P_i(\text{error}) \leqq 2/(\alpha-1)$ $\forall i \in Z$.

PROOF. Since by the strong law of large numbers $S_n/n \to b'(\theta_i)$ a.s. P_i (as $n\to\infty$), (i) follows from the fact that $b'(\cdot)$ is increasing. To prove the second part, we have

$$\begin{aligned} P_j(\text{error}) &= \sum_{i\neq j} P_j(a_i) \\ &= \sum_{i\neq j}\sum_{n=1}^{\infty}\int_{\{N=n,a_i\}} f_{j,n}d\mu_n \\ &= \sum_{i<j}\sum_{n=1}^{\infty}\int_{\{N=n,a_i\}} f_{j,n}d\mu_n + \sum_{i>j}\sum_{n=1}^{\infty}\int_{\{N=n,a_i\}} f_{j,n}d\mu_n . \end{aligned}$$

Note that $f_{j,n}/f_{i,n} = \exp((\theta_j-\theta_i)S_n - n(b(\theta_j)-b(\theta_i)))$. Consider $i<j \Longleftrightarrow \theta_i < \theta_j$. On $\{N=n, a_i\}$, $S_n/n \leqq (b(\theta_{i+1})-b(\theta_i))/(\theta_{i+1}-\theta_i) - (g(\theta_{i+1})-g(\theta_i))(\log\alpha)/n(\theta_{i+1}-\theta_i)$, and note that the right-hand side of this inequality is the infinum (assumption (iii)) of $\phi(\theta_j) = (b(\theta_j)-b(\theta_i))/(\theta_j-\theta_i) - (g(\theta_j)-g(\theta_i))\cdot(\log\alpha)/n(\theta_j-\theta_i)$ over $\theta_j > \theta_i$. Hence on the set $\{N=n, a_i\}$ we have

$$f_{j,n}/f_{i,n} \leqq \exp(-(g(\theta_j)-g(\theta_i))\log\alpha) \leqq \alpha^{-(g(\theta_j)-g(\theta_i))} .$$

The case $i>j$ can be considered in a similar fashion. At any rate

$$P_j(\text{error}) \leqq \sum_{i\neq j} \alpha^{-|g(\theta_i)-g(\theta_j)|} \leqq \sum_{i\neq j} \alpha^{-|i-j|} \leqq 2/(\alpha-1) .$$

4. Bounds for $E_i N$ and an asymptotic expression

From (2.1) we have

$$E_i N \geqq \sup_{j\neq i} [(\log a_{ij})/I(\theta_i:\theta_j)]$$

where $I(\theta_i:\theta_j)$ is the Kullback-Leibler information measure. It follows from the choice of a_{ij} that

$$(4.1) \qquad E_i N \geqq (\log\alpha)/\min\,[\inf_{j>i} d(\theta_i,\theta_j),\ \inf_{j<i} d(\theta_i,\theta_j)]$$

where $d(\theta_i,\theta_j) = (I(\theta_i:\theta_j))/|g(\theta_i)-g(\theta_j)|$, $i\neq j$. It follows from assumption (iv) that

$$\inf_{j>i} d(\theta_i, \theta_j) = d(\theta_i, \theta_{i+1}) = I(\theta_i : \theta_{i+1})(g(\theta_{i+1}) - g(\theta_i))^{-1} .$$

For $j<i$, $\partial d/\partial\theta_j = -[(b(\theta_i)-b(\theta_j))(g(\theta_i)-g(\theta_j)) - Ig(\theta_j)](g(\theta_i)-g(\theta_j))^{-2}$. Since $g'(\cdot)$ is increasing, hence

$$\partial d/\partial\theta_j \leqq -[(b(\theta_i)-b(\theta_j))(g(\theta_i)-g(\theta_j)) - Ig(\theta_i)](g(\theta_i)-g(\theta_j))^{-2} .$$

By assumption (iv) it follows that $\partial d/\partial\theta_j \leqq 0$ for $j<i$, so that $\inf_{j<i} d(\theta_i, \theta_j) = d(\theta_i, \theta_{i-1})$. Thus from (4.1) we have

$$\mathrm{E}_i N \geqq (\log\alpha)/\min\{d(\theta_i, \theta_{i+1}),\ d(\theta_i, \theta_{i-1})\} ,$$

i.e.

$$\mathrm{E}_i N \geqq K_i^{-1}\log\alpha \tag{4.2}$$

where

$$K_i = \min[I(\theta_i : \theta_{i+1})/(g(\theta_{i+1})-g(\theta_i)),\ I(\theta_i : \theta_{i-1})/(g(\theta_i)-g(\theta_{i-1})] . \tag{4.3}$$

Hence

$$\liminf_{\alpha\to\infty} (K_i \mathrm{E}_i N)/(\log\alpha) \geqq 1 . \tag{4.4}$$

Now we want to find an upper bound and an asymptotic expression for $\mathrm{E}_i N$. To this end, we set $r = m\log\alpha$, where $m > K_i^{-1}$ and assume for simplicity that r is an integer. From the stopping rule (3.4) we have

$$\mathrm{P}_i(N>r) \leqq \mathrm{P}_i(S_r/r>a) + \mathrm{P}_i(S_r/r<c) \tag{4.5}$$

where $a = (b(\theta_{i+1})-b(\theta_i))/(\theta_{i+1}-\theta_i) - (g(\theta_{i+1})-g(\theta_i))m^{-1}/(\theta_{i+1}-\theta_i)$, and $c = (b(\theta_i) - b(\theta_{i-1}))/(\theta_i-\theta_{i-1}) + (g(\theta_i)-g(\theta_{i-1}))m^{-1}/(\theta_i-\theta_{i-1})$. It follows from the Definition (4.3) that $m > K_i^{-1} > (g(\theta_{i+1})-g(\theta_i))/I(\theta_i : \theta_{i+1}) \Rightarrow a > (b(\theta_{i+1})-b(\theta_i))/(\theta_{i+1}-\theta_i) - I(\theta_i : \theta_{i+1})/(\theta_{i+1}-\theta_i)$. On substituting the value of I we see that $a > b'(\theta_i) = \mathrm{E}_i X$, and hence a theorem of Chernoff [1] implies

$$\mathrm{P}_i(S_r/r>a) \leqq \rho_1^r$$

where $\rho_1 = \rho_1(a) = \inf_t e^{-at}M(t)$, $M(t) = \mathrm{E}_i e^{tX} = e^{b(\theta_i+t)-b(\theta_i)}$. One can show that $0<\rho_1<1$. A similar argument shows that

$$\mathrm{P}_i(S_r/r<c) \leqq \rho_2^r , \qquad 0<\rho_2<1$$

where $\rho_2 = \rho_2(c) = \inf_t e^{-ct}M(t)$. It follows from (4.5) that

$$\mathrm{P}_i(N>r) \leqq \begin{cases} \rho_1^r + \rho_2^r , & 0<\rho_1,\ \rho_2<1 \\ 2\rho^r , & \rho = \max(\rho_1, \rho_2) . \end{cases} \tag{4.6}$$

Note that (4.6) is sufficient to imply: (i) $\mathrm{E}_i N<\infty\ \forall i$, and (ii) $\mathrm{E}_i e^{tN}<\infty$ for some $t>0$ ($\forall i$). Moreover, $N\sim K_i^{-1}\log\alpha$ in probability P_i (as $\alpha\to\infty$). To get the bound for $\mathrm{E}_i N$ we have

$$\begin{aligned}\mathrm{E}_i N=\sum_{n=1}^{\infty} n\,\mathrm{P}_i(N=n)&=\sum_{n<r} n\,\mathrm{P}_i(N=n)+\sum_{n>r} n\,\mathrm{P}_i(N=n)\\ &\leqq r+(r+1)\,\mathrm{P}_i(N>r)+\sum_{n>r}\mathrm{P}_i(N>n)\\ &\leqq r+(r+1)\{\rho_1^r+\rho_2^r\}+\sum_{n>r}\{\rho_1^n+\rho_2^n\}\ .\end{aligned}$$

Since $r=m\log\alpha$, the last series $\to 0$ as $\alpha\to\infty$. Thus we have

$$\mathrm{E}_i N\leqq m\log\alpha+(m\log\alpha+1)\{\rho_1^{m\log\alpha}+\rho_2^{m\log\alpha}\}+o(1)\ , \tag{4.7}$$

so that

$$\limsup_{\alpha\to\infty}\,(K_i\,\mathrm{E}_i N)/(\log\alpha)\leqq 1\ . \tag{4.8}$$

From (4.4) and (4.8) we have

$$\mathrm{E}_i N\sim K_i\log\alpha \qquad \text{as } \alpha\to\infty\ . \tag{4.9}$$

5. Asymptotic optimality and special cases

Regarding the asymptotic optimality we have the following theorem.

THEOREM 5.1. *If* $g(\theta_{k+1})-g(\theta_k)=1\ \forall k\in Z$, *then the stopping rule defined by* (3.4) *is asymptotically optimal.*

PROOF. Recall that the uniform bound on the error probability is $\varepsilon=2/(\alpha-1)$. By using the condition $g(\theta_{k+1})-g(\theta_k)=1\ \forall k\in Z$, it follows from (4.3) and (4.9) that

$$\mathrm{E}_i N\sim(-\log\alpha)/\min\,[I(\theta_i:\theta_{i+1}),\ I(\theta_i:\theta_{i-1})] \qquad \text{as } \alpha\to\infty\ . \tag{5.1}$$

But since $I(\theta_i:\theta_j)$ is increasing in $\theta_j>\theta_i$ and decreasing in $\theta_j<\theta_i$, hence we have

$$\mathrm{E}_i N\sim(-\log\varepsilon)/\inf_{j\neq i} I(\theta_i:\theta_j) \qquad \text{as } \alpha\to\infty\ . \tag{5.2}$$

Hence the conclusion follows from Lemma 2.1.

COROLLARY 1. *Let* $\Omega=\{\theta_i:\ i\in Z\}$ *be an ordered set in the usual direction with positive minimum spacing* δ ($\delta>0$). *Then a possible choice of* $g(x)$ *is* $g(x)=x/\delta$. *Moreover, if there is uniform spacing, then the corresponding rule* N *is asymptotically optimal.*

COROLLARY 2. *If* $\theta_k=ak+c$, $a>0$, *then we can take* $g(x)=(x-c)/a$, *and the rule would be assymptotically optimal.*

Remark. Theorem 5.1 gives asymptotically optimal rules for the sequences which are uniformly spaced relative to $g(\cdot)$. However, we can always modify our choice of a_{ij} to overcome this restriction. Assume for simplicity that $\Omega=(\theta_1<\theta_2<\cdots)$, and modify a_{ij} as follows:

$$a_{ij}=\begin{cases} \alpha^{(g(\theta_j)-g(\theta_i))/(g(\theta_{i+1})-g(\theta_i))}, & j>i \\ \alpha^{(g(\theta_i)-g(\theta_j))/(g(\theta_i)-g(\theta_{i-1}))}, & j<i\,. \end{cases}$$

Then it is easy to show that the modified rule (3.4) is asymptotically optimal under mild conditions ensuring $\sum_{i\neq j} a_{ij}^{-1}\leqq 2/\alpha+o(\alpha^{-1})$. For example, let $g(\cdot)$ satisfy the previous conditions and the following:

(1) $\inf_i \inf_{j\neq i} |g(\theta_i)-g(\theta_j)|=\delta>0$

(2) $g(\theta_{i+1})-g(\theta_i)<\Delta,\ \Delta>0,\ i>1.$

Then, $\sum_{i\neq j} a_{ij}^{-1}\leqq 2/\alpha+2/\alpha(\alpha^{\delta/\Delta}-1)=2/\alpha+o(\alpha^{-1})$. That the associated stopping rule for the modified choice of a_{ij} is asymptotically optimal follows from the fact that $\mathrm{E}_i\, N\sim(-\log \varepsilon)/\inf_{j\neq i} I(\theta_i:\theta_j)$, as $\alpha\to\infty$, where $\varepsilon=2\alpha^{-1}$.

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