NOTE ON THE REDUCTION OF ASSOCIATE CLASSES FOR PBIB DESIGNS

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1. Introduction and summary

When $\lambda_1=\lambda_2$ (an equality among coincidence numbers λ_i, $i=1, 2, \cdots, m$) in a Partially Balanced Incomplete Block (PBIB) design with m associate classes, M. N. Vartak [3] gave a necessary and sufficient condition for the PBIB design to be reducible to a PBIB design with $m-1$ associate classes. Moreover, he stated that repeated applications of the result to any PBIB design will ultimately give a PBIB design whose associate classes are all distinct. However, as there are some reducible PBIB designs to which Vartak's iterative procedure does not apply, a generalization of Vartak's condition is given and furthermore these PBIB designs are shown. By using the generalized Vartak's condition, answers for questions in Author [1], Section 7, are also given. Finally, we consider the reduction of a PBIB design with the hypercubic association scheme.

2. Conditions for the reduction of associate classes

When the parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ of a PBIB design are not all different, the m associate classes of the PBIB design based on a certain association scheme may not be all distinct. The following theorems, which are a generalization of Lemma 4.1 by Vartak [3], give a criterion to determine whether a PBIB design with m associate classes is reducible to a PBIB design with m or fewer distinct associate classes, when its parameters $\lambda_1, \lambda_2, \cdots, \lambda_m$ are not all different. The proof is easily given by considering the association matrices and hence it is omitted here.

THEOREM 2.1. *Let a PBIB design N with m associate classes and with parameters*

$$v, \quad b, \quad r, \quad k, \quad \lambda_i, \quad n_i, \quad p^i_{jk}, \qquad i, j, k=1, 2, \cdots, m$$

be such that $\lambda_1, \lambda_2, \cdots, \lambda_m$ are not all different so that at least l of them *are equal. Without loss of generality we can assume that* $\lambda_1=\lambda_2=\cdots=\lambda_l$. *In this case, the number of associate classes of design N can be reduced from m to* $m-l+1$ *by combining its first l associate classes, if and only if*

$$
(2.1)\qquad \left\|\begin{matrix} \sum_{i,j=1}^{l} p_{ij}^{1}, & \sum_{i=1}^{l} p_{i,l+1}^{1}, & \cdots, & \sum_{i=1}^{l} p_{im}^{1} \\ \sum_{j=1}^{l} p_{l+1,j}^{1}, & p_{l+1,l+1}^{1}, & \cdots, & p_{l+1,m}^{1} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{l} p_{mj}^{1}, & p_{m,l+1}^{1}, & \cdots, & p_{mm}^{1} \end{matrix}\right\|
$$

$$
= \left\|\begin{matrix} \sum_{i,j=1}^{l} p_{ij}^{2}, & \sum_{i=1}^{l} p_{i,l+1}^{2}, & \cdots, & \sum_{i=1}^{l} p_{im}^{2} \\ \sum_{j=1}^{l} p_{l+1,j}^{2}, & p_{l+1,l+1}^{2}, & \cdots, & p_{l+1,m}^{2} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{l} p_{mj}^{2}, & p_{m,l+1}^{2}, & \cdots, & p_{mm}^{2} \end{matrix}\right\| = \cdots
$$

$$
= \left\|\begin{matrix} \sum_{i,j=1}^{l} p_{ij}^{l}, & \sum_{i=1}^{l} p_{i,l+1}^{l}, & \cdots, & \sum_{i=1}^{l} p_{im}^{l} \\ \sum_{j=1}^{l} p_{l+1,j}^{l}, & p_{l+1,l+1}^{l}, & \cdots, & p_{l+1,m}^{l} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{l} p_{mj}^{l}, & p_{m,l+1}^{l}, & \cdots, & p_{mm}^{l} \end{matrix}\right\| .
$$

Further if (2.1) *holds, then the parameters of the reduced PBIB design with* $m-l+1$ *associate classes are as follows:*

$$v'=v\,,\qquad b'=b\,,\qquad r'=r\,,\qquad k'=k\,,$$

$$\lambda_1'=\lambda_1=\lambda_2=\cdots=\lambda_l\,,\qquad \lambda_2'=\lambda_{l+1},\cdots,\lambda_{m-l+1}'=\lambda_m\,,$$

$$n_1'=n_1+n_2+\cdots+n_l\,,\qquad n_2'=n_{l+1},\cdots,n_{m-l+1}'=n_m\,,$$

$$
\|p_{uv}'^{1}\| = \left\|\begin{matrix} \sum_{i,j=1}^{l} p_{ij}^{t}, & \sum_{i=1}^{l} p_{i,l+1}^{t}, & \cdots, & \sum_{i=1}^{l} p_{im}^{t} \\ \sum_{j=1}^{l} p_{l+1,j}^{t}, & p_{l+1,l+1}^{t}, & \cdots, & p_{l+1,m}^{t} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{l} p_{mj}^{t}, & p_{m,l+1}^{t}, & \cdots, & p_{mm}^{t} \end{matrix}\right\| ,
$$

$$\|p'^{w}_{uv}\| = \left\| \begin{matrix} \sum_{i,j=1}^{l} p_{ij}^{w+l-1}, & \sum_{i=1}^{l} p_{i,l+1}^{w+l-1}, & \cdots, & \sum_{i=1}^{l} p_{im}^{w+l-1} \\ \sum_{j=1}^{l} p_{l+1,j}^{w+l-1}, & p_{l+1,l+1}^{w+l-1}, & \cdots, & p_{l+1,m}^{w+l-1} \\ \vdots & \vdots & & \vdots \\ \sum_{j=1}^{l} p_{mj}^{w+l-1}, & p_{m,l+1}^{w+l-1}, & \cdots, & p_{mm}^{w+l-1} \end{matrix} \right\|,$$

where $t=1, 2, \cdots,$ *or* l; $w=2, 3, \cdots, m-l+1$; $u, v=1, 2, \cdots, m-l+1$.

THEOREM 2.2. *Let a PBIB design N with m associate classes and with parameters*

$$v, \quad b, \quad r, \quad k, \quad \lambda_i, \quad n_i, \quad p_{jk}^{i}, \qquad i, j, k=1, 2, \cdots, m$$

be such that $\lambda_1, \lambda_2, \cdots, \lambda_m$ *are not all different so that* l_j λ's $(j=1, 2, \cdots, t)$ *are equal to one another. Without loss of generality we can assume that for* $1 \leqq \theta_1 < \theta_1 + l_1 \leqq \theta_2 < \cdots < \theta_{t-1} + l_{t-1} \leqq \theta_t \leqq m$,

$$\begin{aligned} &\lambda_{\theta_1} = \lambda_{\theta_1+1} = \cdots = \lambda_{\theta_1+l_1-1}, \\ &\lambda_{\theta_2} = \lambda_{\theta_2+1} = \cdots = \lambda_{\theta_2+l_2-1}, \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\lambda_{\theta_t} = \lambda_{\theta_t+1} = \cdots = \lambda_{\theta_t+l_t-1}. \end{aligned} \tag{2.2}$$

In this case, the number of associate classes of design N can be reduced from m to $m - \sum_{i=1}^{t} l_i + t$ *by combining its* l_j *associate classes for each* l_j, *if and only if*

(i) $$\sum_{i=\theta_p}^{\theta_p+l_p-1} \sum_{j=\theta_q}^{\theta_q+l_q-1} p_{ij}^{\theta_u} = \sum_i \sum_j p_{ij}^{\theta_u+1} = \cdots = \sum_i \sum_j p_{ij}^{\theta_u+l_u-1}$$
for $u, p, q=1, 2, \cdots, t$,

(ii) $$\sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u}_{i,q+\sum_{i=1}^{j-1} l_i - j+1} = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+1}_{i,q+\sum_{i=1}^{j-1} l_i - j+1} = \cdots$$
$$= \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+l_u-1}_{i,q+\sum_{i=1}^{j-1} l_i - j+1} \qquad \text{for } j, u=1, 2, \cdots, t;$$

$$\theta_{j-1} - \sum_{i=1}^{j-2} l_i + j - 1 \leqq q \leqq \theta_j - \sum_{i=1}^{j-1} l_i + j - 2, \tag{2.3}$$

(iii) $$\sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u}_{i,q+\sum_{i=1}^{t} l_i - t} = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+1}_{i,q+\sum_{i=1}^{t} l_i - t} = \cdots = \sum_{i=\theta_j}^{\theta_j+l_j-1} p^{\theta_u+l_u-1}_{i,q+\sum_{i=1}^{t} l_i - t}$$
for $j, u=1, 2, \cdots, t$; $\theta_t - \sum_{i=1}^{t-1} l_i + t \leqq q \leqq m - \sum_{i=1}^{t} l_i + t$,

(iv) $p_{\alpha\beta}^{\theta_u} = p_{\alpha\beta}^{\theta_u+1} = \cdots = p_{\alpha\beta}^{\theta_u+l_u-1}$ *for* $u=1, 2, \cdots, t$;

$$1 \leqq \alpha, \beta \notin \bigcup_{i=1}^{t} \{\theta_i, \theta_i+1, \cdots, \theta_i+l_i-1\} \leqq m,$$

(v) *the above conditions* (i), (ii), (iii) *and* (iv) *remain true under any permutation of two subscripts of* p^i_{jk}.

Further if (2.3) *holds, then the parameters of the reduced PBIB design with* $m-\sum_{i=1}^{t} l_i+t$ *associate classes are as follows*:

$$v'=v\,,\qquad b'=b\,,\qquad r'=r\,,\qquad k'=k\,,$$

for $j=1, 2, \cdots, t$,

$$\lambda'_{\theta_j-\sum_{i=1}^{j-1} l_i+j-1}=\lambda_{\theta_j}=\lambda_{\theta_j+1}=\cdots=\lambda_{\theta_j+l_j-1}\,,$$

$$\lambda'_i=\begin{cases}\lambda_{i+\sum_{i=1}^{j-1} l_i-j+1} & \text{if } \theta_{j-1}-\sum_{i=1}^{j-2} l_i+j-1\leqq i\leqq\theta_j-\sum_{i=1}^{j-1} l_i+j-2\,,\\ \lambda_{i+\sum_{i=1}^{t} l_i-t} & \text{if } \theta_t-\sum_{i=1}^{t-1} l_i+t\leqq i\leqq m-\sum_{i=1}^{t} l_i+t\,,\end{cases}$$

$$n'_{\theta_j-\sum_{i=1}^{j-1} l_i+j-1}=n_{\theta_j}+n_{\theta_j+1}+\cdots+n_{\theta_j+l_j-1}\,,$$

$$n'_i=\begin{cases}n_{i+\sum_{i=1}^{j-1} l_i-j+1} & \text{if } \theta_{j-1}-\sum_{i=1}^{j-2} l_i+j-1\leqq i\leqq\theta_j-\sum_{i=1}^{j-1} l_i+j-2\,,\\ n_{i+\sum_{i=1}^{t} l_i-t} & \text{if } \theta_t-\sum_{i=1}^{t-1} l_i+t\leqq i\leqq m-\sum_{i=1}^{t} l_i+t\,.\end{cases}$$

$||p'^x_{yz}||$ can be written in a form similar to that of Theorem 2.1 and hence omitted here.

Though Theorem 2.1 is a special case of Theorem 2.2, it has been written especially because of its frequent use. The matrix representation of (2.3) is omitted, as it will make this note unduly lengthy.

3. PBIB designs showing the validity of the above theorems

Design (I). We consider a 5-associate PBIB design with the hypercubic association scheme [2] having the following parameters:

$$v=s^5,\quad b,\quad r,\quad k,\quad \lambda_i\ (i=1, 2, \cdots, 5)\,,$$

$$n_1=5(s-1),\quad n_2=10(s-1)^2,\quad n_3=10(s-1)^3,\quad n_4=5(s-1)^4,$$
$$n_5=(s-1)^5\,,$$

$$||p^1_{ij}||=\left\|\begin{matrix} s-2 & 4(s-1) & 0 & 0 & 0\\ & 4(s-1)(s-2) & 6(s-1)^2 & 0 & 0\\ & & 6(s-1)^2(s-2) & 4(s-1)^3 & 0\\ & \text{Sym.} & & 4(s-1)^3(s-2) & (s-1)^4\\ & & & & (s-1)^4(s-2)\end{matrix}\right\|,$$

$$\|p_{ij}^2\|=\left\|\begin{matrix} 2 & 2(s-2) & 3(s-1) & 0 & 0 \\ & (s-2)^2+6(s-1) & 6(s-1)(s-2) & 3(s-1)^2 & 0 \\ & & 3(s-1)(s-2)^2+6(s-1)^2 & 6(s-1)^2(s-2) & (s-1)^3 \\ & \text{Sym.} & & 3(s-1)^2(s-2)^2+2(s-1)^3 & 2(s-1)^3(s-2) \\ & & & & (s-1)^3(s-2)^2 \end{matrix}\right\|,$$

$$\|p_{ij}^3\|=\left\|\begin{matrix} 0 & 3 & 3(s-2) & 2(s-1) & 0 \\ & 6(s-2) & 3(s-2)^2+6(s-1) & 6(s-1)(s-2) & (s-1)^2 \\ & & (s-2)^3+12(s-1)(s-2) & 6(s-1)(s-2)^2+3(s-1)^2 & 3(s-1)^2(s-2) \\ & \text{Sym.} & & 2(s-1)(s-2)^3+6(s-1)^2(s-2) & 3(s-1)^2(s-2)^2 \\ & & & & (s-1)^2(s-2)^3 \end{matrix}\right\|,$$

$$\|p_{ij}^4\|=\left\|\begin{matrix} 0 & 0 & 4 & 4(s-2) & s-1 \\ & 6 & 12(s-2) & 6(s-2)^2+4(s-1) & 4(s-1)(s-2) \\ & & 12(s-2)^2+6(s-1) & 4(s-2)^3+12(s-1)(s-2) & 6(s-1)(s-2)^2 \\ & \text{Sym.} & & (s-2)^4+12(s-1)(s-2)^2 & 4(s-1)(s-2)^3 \\ & & & & (s-1)(s-2)^4 \end{matrix}\right\|,$$

$$\|p_{ij}^5\|=\left\|\begin{matrix} 0 & 0 & 0 & 5 & 5(s-2) \\ & 0 & 10 & 20(s-2) & 10(s-2)^2 \\ & & 30(s-2) & 30(s-2)^2 & 10(s-2)^3 \\ & \text{Sym.} & & 20(s-2)^3 & 5(s-2)^4 \\ & & & & (s-2)^5 \end{matrix}\right\|.$$

In this case, it is clear that every combination of two associate classes in the sense of Vartak does not give a new reduced PBIB design, so that Vartak's iterative procedure is impossible. Some combinations of other types of associate classes, however, give new reduced PBIB designs under certain conditions. For example, from Theorems 2.1 and 2.2, when (i) $\lambda_1=\lambda_3=\lambda_5$ or (ii) $\lambda_1=\lambda_2$, $\lambda_3=\lambda_4$ or (iii) $\lambda_1=\lambda_3$, $\lambda_2=\lambda_4$, if $s=2$, then the design reduces to 3-associate PBIB designs. When $\lambda_1=\lambda_4$ and $\lambda_2=\lambda_5$, if $s=3$, then the design reduces to a 3-associate PBIB design. Further when (i) $\lambda_1=\lambda_2=\lambda_3=\lambda_4$, (if $s=2$), or (ii) $\lambda_2=\lambda_4$, $\lambda_1=\lambda_3=\lambda_5$, (if $s=4$), then the design reduces to 2-associate PBIB designs.

Design (II). We consider a 7-associate PBIB design with the F_3 type association scheme described in Section 4 of [1], i.e., $N=N_1\otimes N_2\otimes N_3$, (where N_i is a BIB design with parameters v_i, b_i, r_i, k_i, λ_i), having the following parameters:

$$v'=v_1v_2v_3\,,\quad b'=b_1b_2b_3\,,\quad r'=r_1r_2r_3\,,\quad k'=k_1k_2k_3\,,$$
$$\lambda_1'=r_1\lambda_2r_3\,,\quad \lambda_2'=\lambda_1r_2r_3\,,\quad \lambda_3'=\lambda_1\lambda_2r_3\,,\quad \lambda_4'=r_1r_2\lambda_3\,,$$
$$\lambda_5'=r_1\lambda_2\lambda_3\,,\quad \lambda_6'=\lambda_1r_2\lambda_3\,,\quad \lambda_7'=\lambda_1\lambda_2\lambda_3\,,$$

$$n_1=v_2-1\,,\quad n_2=v_1-1\,,\quad n_3=(v_1-1)(v_2-1)\,,\quad n_4=v_3-1\,,$$
$$n_5=(v_2-1)(v_3-1)\,,\quad n_6=(v_3-1)(v_1-1)\,,$$
$$n_7=(v_1-1)(v_2-1)(v_3-1)\,,$$

$$
\|p^1_{ij}\|=\begin{Vmatrix}
v_2-2 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & v_1-1 & 0 & 0 & 0 & 0 \\
 & & (v_1-1)(v_2-2) & 0 & 0 & 0 & 0 \\
 & \text{Sym.} & & 0 & v_3-1 & 0 & 0 \\
 & & & & (v_3-1)(v_2-2) & 0 & 0 \\
 & & & & & 0 & (v_1-1)(v_3-1) \\
 & & & & & & (v_1-1)(v_2-2)(v_3-1)
\end{Vmatrix},
$$

$$
\|p^2_{ij}\|=\begin{Vmatrix}
0 & 0 & v_2-1 & 0 & 0 & 0 & 0 \\
 & v_1-2 & 0 & 0 & 0 & 0 & 0 \\
 & & (v_1-2)(v_2-1) & 0 & 0 & 0 & 0 \\
 & \text{Sym.} & & 0 & 0 & v_3-1 & 0 \\
 & & & & 0 & 0 & (v_2-1)(v_3-1) \\
 & & & & & (v_1-2)(v_3-1) & 0 \\
 & & & & & & (v_1-2)(v_2-1)(v_3-1)
\end{Vmatrix},
$$

$$
\|p^3_{ij}\|=\begin{Vmatrix}
0 & 1 & v_2-2 & 0 & 0 & 0 & 0 \\
 & 0 & v_1-2 & 0 & 0 & 0 & 0 \\
 & & (v_1-2)(v_2-2) & 0 & 0 & 0 & 0 \\
 & \text{Sym.} & & 0 & 0 & 0 & v_3-1 \\
 & & & & 0 & v_3-1 & (v_2-2)(v_3-1) \\
 & & & & & 0 & (v_1-2)(v_3-1) \\
 & & & & & & (v_1-2)(v_2-2)(v_3-1)
\end{Vmatrix},
$$

$$
\|p^4_{ij}\|=\begin{Vmatrix}
0 & 0 & 0 & 0 & v_2-1 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & v_1-1 & 0 \\
 & & 0 & 0 & 0 & 0 & (v_1-1)(v_2-1) \\
 & & & v_3-2 & 0 & 0 & 0 \\
 & & & & (v_2-1)(v_3-2) & 0 & 0 \\
 & & \text{Sym.} & & & (v_1-1)(v_3-2) & 0 \\
 & & & & & & (v_1-1)(v_2-1)(v_3-2)
\end{Vmatrix},
$$

$$
\|p^5_{ij}\|=\begin{Vmatrix}
0 & 0 & 0 & 1 & v_2-2 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & v_1-1 \\
 & & 0 & 0 & 0 & v_1-1 & (v_1-1)(v_2-2) \\
 & & & 0 & v_3-2 & 0 & 0 \\
 & & & & (v_2-2)(v_3-2) & 0 & 0 \\
 & & \text{Sym.} & & & 0 & (v_1-1)(v_3-2) \\
 & & & & & & (v_1-1)(v_2-2)(v_3-2)
\end{Vmatrix},
$$

$$
\|p^6_{ij}\|=\begin{Vmatrix}
0 & 0 & 0 & 0 & 0 & 0 & v_2-1 \\
 & 0 & 0 & 1 & 0 & v_1-2 & 0 \\
 & & 0 & 0 & v_2-1 & 0 & (v_1-2)(v_2-1) \\
 & & & 0 & 0 & v_3-2 & 0 \\
 & & & & 0 & 0 & (v_2-1)(v_3-2) \\
 & \text{Sym.} & & & & (v_1-2)(v_3-2) & 0 \\
 & & & & & & (v_1-2)(v_2-1)(v_3-2)
\end{Vmatrix},
$$

$$\|p_{ij}^{7}\|=\left\|\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 0 & 1 & v_2-2 \\ & 0 & 0 & 0 & 1 & 0 & v_1-2 \\ & & 0 & 1 & v_2-2 & v_1-2 & (v_1-2)(v_2-2) \\ & & & 0 & 0 & 0 & v_3-2 \\ & & & & 0 & v_3-2 & (v_2-2)(v_3-2) \\ & & \text{Sym.} & & & 0 & (v_1-2)(v_3-2) \\ & & & & & & (v_1-2)(v_2-2)(v_3-2) \end{array}\right\|.$$

In this case, it is clear that every combination of two associate classes in the sense of Vartak does not give a new reduced PBIB design, so that Vartak's iterative procedure is impossible. Some combinations of other types of associate classes, however, give new reduced PBIB designs under certain conditions. It follows from Theorems 2.1 and 2.2 that all cases of reduction are as follows:

(i) When $\lambda_1'=\lambda_2'$ and $\lambda_5'=\lambda_6'$, if $v_1=v_2$, or (ii) when $\lambda_1'=\lambda_4'$ and $\lambda_3'=\lambda_6'$, if $v_2=v_3$, or (iii) when $\lambda_2'=\lambda_4'$ and $\lambda_3'=\lambda_5'$, if $v_3=v_1$, then the design reduces to 5-associate PBIB designs and vice versa. (iv) When $\lambda_1'=\lambda_2'=\lambda_4'$ and $\lambda_3'=\lambda_5'=\lambda_6'$, if $v_1=v_2=v_3$, then the design reduces to a 3-associate PBIB design with the cubic association scheme [1] and vice versa.

It follows clearly from the form of λ_i' ($i=1, 2, \cdots, 7$) and Theorem 2.1 that by assuming a relation $\lambda_i'=\lambda_j'$ ($i\neq j$; $=1, 2, 4$ or $3, 5, 6$) only, the design N does not reduce to a 6-associate PBIB design. It should be noted that these results give answers for Section 7 in [1], i.e. under conditions (7.1) and (7.2) in [1], the design $N=N_1\otimes N_2\otimes N_3$ cannot be reduced to a 6-associate PBIB design, and moreover the above case (i) (i.e. the case of Section 7 in [1]) may be called the Singular Reduced F_3 type association scheme.

To this effect, a hypercubic association scheme of m associate classes described in the next section may be called the Regular Reduced F_m type association scheme.

4. Reduction of the hypercubic association scheme

Since it is well known [4] that an F_m type association scheme of $v=s_1 s_2\cdots s_m$ treatments is reducible to a hypercubic association scheme of m associate classes (or C_m type association scheme) provided $s_1=s_2=\cdots=s_m=s$, we now further consider the possibility of reduction of a PBIB design with the C_m type association scheme of $v=s^m$ treatments, except in the case of $\lambda_1=\lambda_2=\cdots=\lambda_m$ under which the design reduces to a balanced incomplete block design. That is, we consider the case in which $m\geqq 3$.

In a C_3 type association scheme, from Theorems 2.1 and 2.2, we clearly obtain $s=2$ and 4 as a necessary and sufficient condition for a PBIB design with its association scheme to be reducible.

It follows from the exhaustive investigation by using Theorems 2.1 and 2.2 that a necessary and sufficient condition for a PBIB design with the C_m type association scheme of $v=s^m$ treatments to be reducible for $m=4$ and 5, is that $s=2$, 3 and 4.

Since this is also almost true for $m=6$, in general it will be conjectured for $m\geqq 6$. This will be shown after the strenuous attempt of exhaustive investigation about the parameters of the C_m type association scheme by using Theorems 2.1 and 2.2.

Thus from Theorems 2.1 and 2.2, we can make a study of the reduction of associate classes for many PBIB designs with the known association scheme.

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REFERENCES

[1] Kageyama, S. (1972). On the reduction of associate classes for certain PBIB designs, *Ann. Math. Statist.*, **43**, 1528-1540.
[2] Kusumoto, K. (1965). Hypercubic designs, *Wakayama Medical Reports*, **9**, 123-132.
[3] Vartak, M. N. (1955). On an application of Kronecker product of matrices to statistical designs, *Ann. Math. Statist.*, **26**, 420-438.
[4] Yamamoto, S., Fujii, Y. and Hamada, N. (1965). Composition of some series of association algebras, *J. Sci. Hiroshima Univ.*, Ser. A-I, **29**, 181-215.

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