ON A CLASS OF SIMULTANEOUS RANK ORDER TESTS IN MANOCOVA*

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Summary

For the one-criterion multivariate analysis of covariance (MANOCOVA) model, the rank order tests for the overall hypothesis of no treatment effect considered by Quade [9], Puri and Sen [7] and Sen and Puri [11] are extended here to some simultaneous tests for various component hypotheses. The theory is based on an extension of rank order estimates of contrasts in multivariate analysis of variance (MANOVA) developed by Puri and Sen [6] to the MANOCOVA problem, and is formulated in the set up of Gabriel and Sen [1] and Krishnaiah [3], [4].

1. Introduction

Let $\boldsymbol{Z}_\alpha^{(k)}=(\boldsymbol{Y}_\alpha^{(k)}, \boldsymbol{X}_\alpha^{(k)})'=(Y_{1\alpha}^{(k)},\cdots,Y_{p\alpha}^{(k)}, X_{1\alpha}^{(k)},\cdots,X_{q\alpha}^{(k)})'$, $\alpha=1,\cdots,n_k$ be n_k independent and identically distributed random vectors (iidrv) with a continuous cumulative distribution function (cdf) $\Pi_k(\boldsymbol{z})$, $\boldsymbol{z}\in R^{p+q}$, the $(p+q)$-dimensional Euclidean space, where $p\geqq 1$, $q\geqq 1$, and $k=1,\cdots,c$ $(\geqq 2)$; all these $N\left(=\sum_{k=1}^{c} n_k\right)$ stochastic vectors are assumed to be independent. The q-variate marginal cdf of $\boldsymbol{X}_\alpha^{(k)}$ is denoted by $G_k(\boldsymbol{x})$, $\boldsymbol{x}\in R^q$, and the p-variate conditional cdf of $\boldsymbol{Y}_\alpha^{(k)}$, given $\boldsymbol{X}_\alpha^{(k)}=\boldsymbol{x}$, is denoted by $F_k(\boldsymbol{y}|\boldsymbol{x})$, $\boldsymbol{y}\in R^p$, $\boldsymbol{x}\in R^q$, $k=1,\cdots,c$. Our basic model is the following:

$$(1.1)\quad G_k(\boldsymbol{x})=G(\boldsymbol{x}) \quad\text{and}\quad F_k(\boldsymbol{y}|\boldsymbol{x})=F(\boldsymbol{y}-\boldsymbol{\tau}_k|\boldsymbol{x})\,,\qquad \text{for } k=1,\cdots,c\,,$$

where F and G are unknown continuous cdfs, and $\boldsymbol{\tau}_1,\cdots,\boldsymbol{\tau}_c$ are the unknown treatment effects (p-vectors). For justification of this model, we may refer to Scheffé ([10], Ch. 6) and Sen and Puri [11]. Tests for the overall hypothesis

$$(1.2)\quad H_0:\boldsymbol{\tau}_1=\cdots=\boldsymbol{\tau}_c=\boldsymbol{0} \text{ vs. } \boldsymbol{\tau}_k\neq\boldsymbol{0} \qquad \text{for at least one } k\ (1\leqq k\leqq c)\,,$$

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based on suitable rank order statistics, were considered in an increasing order of generality by Quade [9], Puri and Sen [7], and Sen and Puri [11]. We are interested here in testing simultaneously for contrasts among $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_c$, both in the set up of Gabriel and Sen [1] and Krishnaiah [3], [4]. These simultaneous procedures involve the concomitant variates adjusted rank order estimates of contrasts among $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_c$, which are considered first in Section 2. Section 3 deals with the proposed simultaneous tests, and their asymptotic relative efficiency (ARE) results are presented in the last section.

2. Covariate adjusted rank order estimates of contrasts in MANOCOVA

The statistics considered by Quade [9], Puri and Sen [7] and Sen and Puri [11], being based on the combined sample ranking, are not suitable for our simultaneous inference procedures to be considered in the next section. For this reason, we extend the results of Puri and Sen [6] on rank order estimates in MANOVA to the MANOCOVA problem, which provides the access to our proposed procedures.

Consider the pair (k, l) of samples, and let $n_{kl}=n_k+n_l$, for $1 \leqq k < l \leqq c$. Let $R_{i,\alpha}^{(k,l)}$ be the rank of $Y_{i\alpha}^{(k)}$ among the n_{kl} observations $Y_{i1}^{(k)}, \cdots, Y_{in_k}^{(k)}, Y_{i1}^{(l)}, \cdots, Y_{in}^{(l)}$, for $\alpha=1, \cdots, n_k$, $1 \leqq i \leqq p$, and let $S_{i,\alpha}^{(k,l)}$ be the rank of $X_{i\alpha}^{(k)}$ among the n_{kl} observations $X_{i1}^{(k)}, \cdots, X_{in_k}^{(k)}, X_{i1}^{(l)}, \cdots, X_{in}^{(l)}$, $\alpha=1, \cdots, n_k$, $i=1, \cdots, q$, for $1 \leqq k < l \leqq c$. Consider now $(p+q)$ sets of rank scores

$$\text{(2.1)} \qquad J_{n_{kl}}^{(i)}(\alpha/(n_{kl}+1)) , \qquad \alpha=1, \cdots, n_{kl},\ i=1, \cdots, p \text{ (primary variate scores)}$$

$$\text{(2.2)} \qquad J_{n_{kl}}^{*(j)}(\alpha/(n_{kl}+1)) , \qquad \alpha=1, \cdots, n_{kl},\ j=1, \cdots, q \text{ (covariate scores)},$$

defined for every (k, l): $1 \leqq k < l \leqq c$, in the same fashion as in Puri and Sen ([8], Section 3.6). Define then the statistics

$$\text{(2.3)} \qquad T_{kl}^{(i)}=n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_{kl}}^{(i)}(R_{i,\alpha}^{(k,l)}/(n_{kl}+1)) , \qquad i=1, \cdots, p;\ 1 \leqq k < l \leqq c ,$$

$$\text{(2.4)} \qquad T_{kl}^{*(j)}=n_k^{-1} \sum_{\alpha=1}^{n_k} J_{n_{kl}}^{*(j)}(S_{j,\alpha}^{(k,l)}/(n_{kl}+1)) , \qquad j=1, \cdots, q;\ 1 \leqq k < l \leqq c .$$

If instead of $Y_{i\alpha}^{(k)}$ and $Y_{i\alpha}^{(l)}$, one works with $Y_{i\alpha}^{(k)}-a$, $\alpha=1, \cdots, n_k$ and $Y_{i\alpha}^{(l)}$, $\alpha=1, \cdots, n_l$, the corresponding rank order statistic, defined by (2.3) is denoted by $T_{kl}^{(i)}(a)$, $1 \leqq k < l \leqq c$, $i=1, \cdots, p$, and $-\infty < a < \infty$. We now assume that $J_{n_{kl}}^{(i)}(u)$ is $\uparrow$ (monotone increasing) in $u: 0<u<1$, so that

$$\text{(2.5)} \qquad T_{kl}^{(i)}(a) \text{ is } \downarrow \text{ in } a: -\infty < a < \infty .$$

Next, we rank the observations coordinatewise within each sample separately. Let $R^{*(k)}_{i,\alpha}$ (or $S^{*(k)}_{i,\alpha}$) be the rank of $Y^{(k)}_{i\alpha}$ among $Y^{(k)}_{i1},\cdots,Y^{(k)}_{in_k}$ (or $X^{(k)}_{i\alpha}$ among $X^{(k)}_{i1},\cdots,X^{(k)}_{in_k}$), for $\alpha=1,\cdots,n_k$, $i=1,\cdots,p$ (or $i=1,\cdots,q$), and $k=1,\cdots,c$. Define $J^{(i)}_{n_k}$, $i=1,\cdots,p$, and $J^{*(j)}_{n_k}$, $j=1,\cdots,q$ as in (2.1) and (2.2) with the n_{kl} being replaced by n_k, and let

$$v^{(k)}_{ij}=n_k^{-1}\sum_{\alpha=1}^{n_k} J^{(i)}_{n_k}(R^{*(k)}_{i,\alpha}/(n_k+1))J^{(j)}_{n_k}(R^{*(k)}_{j,\alpha}/(n_k+1))-\bar{J}^{(i)}_{n_k}\bar{J}^{(j)}_{n_k}\,, \tag{2.6}$$

for $i, j=1,\cdots,p$, where

$$\bar{J}^{(i)}_{n_k}=n_k^{-1}\sum_{\alpha=1}^{n_k} J^{(i)}_{n_k}(\alpha/(n_k+1))\,, \qquad i=1,\cdots,p\,. \tag{2.7}$$

Similarly, define $v^{(k)}_{ij}$ for $p+1\leqq i,\ j\leqq p+q$ by replacing $J^{(i)}_{n_k}$ and $R^{*(k)}_{i,\alpha}$ by $J^{*(i)}_{n_k}$ and $S^{*(k)}_{i,\alpha}$ respectively (and also for j) in the definitions in (2.6) and (2.7). Finally, keeping $J^{(i)}_{n_k}$ and $R^{*(k)}_{i,\alpha}$ as they are while replacing $J^{(j)}_{n_k}$ and $R^{*(k)}_{j,\alpha}$ by $J^{*(j)}_{n_k}$ and $S^{*(k)}_{j,\alpha}$, respectively, in (2.6) and (2.7), we define in the same way $v^{(k)}_{ij}$ for $i=1,\cdots,p$; $j=p+1,\cdots,p+q$. Let then

$$\boldsymbol{V}^{(k)}=((v^{(k)}_{ij}))_{i,j=1,\cdots,p+q}\,, \qquad k=1,\cdots,c\,, \tag{2.8}$$

where $v^{(k)}_{ji}=v^{(k)}_{ij}$ for all i, j. Further, let

$$\bar{\boldsymbol{V}}_N=\sum_{k=1}^{c}(n_k/N)\boldsymbol{V}^{(k)}=\begin{pmatrix}\bar{\boldsymbol{V}}_{N,11} & \bar{\boldsymbol{V}}_{N,12}\\ \bar{\boldsymbol{V}}_{N,21} & \bar{\boldsymbol{V}}_{N,22}\end{pmatrix}, \tag{2.9}$$

where $\bar{\boldsymbol{V}}_{N,11}$, $\bar{\boldsymbol{V}}_{N,12}=\bar{\boldsymbol{V}}'_{N,21}$ and $\bar{\boldsymbol{V}}_{N,22}$ are respectively of the order $p\times p$, $p\times q$ and $q\times q$. Finally, let

$$\bar{\boldsymbol{V}}^*_N=\bar{\boldsymbol{V}}_{N,11}-\bar{\boldsymbol{V}}_{N,12}\bar{\boldsymbol{V}}^-_{N,22}\bar{\boldsymbol{V}}_{N,21}\,, \tag{2.10}$$

where $\bar{\boldsymbol{V}}^-_{N,22}$ is a generalized inverse of $\bar{\boldsymbol{V}}_{N,22}$. Define then $\bar{J}^{(i)}_{n_{kl}}$, $i=1,\cdots,p$, and $\bar{J}^{*(j)}_{n_{kl}}$, $j=1,\cdots,q$, for $1\leqq k<l\leqq c$ as in (2.7) with n_k being replaced by n_{kl}, and let

$$\boldsymbol{T}_{kl}(\boldsymbol{a})=(T^{(1)}_{kl}(a_1),\cdots,T^{(p)}_{kl}(a_p))'\,, \qquad 1\leqq k<l\leqq c\,; \tag{2.11}$$

$$\boldsymbol{T}^*_{kl}=(T^{*(1)}_{k},\cdots,T^{*(q)}_{k})'\,, \qquad 1\leqq k<l\leqq c\,; \tag{2.12}$$

$$\bar{\boldsymbol{J}}_{kl}=(\bar{J}^{(1)}_{kl},\cdots,\bar{J}^{(p)}_{kl})'\,,\quad \bar{\boldsymbol{J}}^*_{k}=(\bar{J}^{*(1)}_{kl},\cdots,\bar{J}^{*(q)}_{kl})'\,, \qquad 1\leqq k<l\leqq c\,; \tag{2.13}$$

$$\begin{aligned}\boldsymbol{T}^0_{kl}(\boldsymbol{a})&=(\boldsymbol{T}_{kl}(\boldsymbol{a})-\bar{\boldsymbol{J}}_{n_{kl}})-\bar{\boldsymbol{V}}_{N,12}\bar{\boldsymbol{V}}^-_{N,22}(\boldsymbol{T}^*_{kl}-\bar{\boldsymbol{J}}^*_{kl})\\ &=(T^{0(1)}_{kl}(a_1),\cdots,T^{0(p)}_{kl}(a_p))'\,, \qquad \text{for } 1\leqq k<l\leqq c\,.\end{aligned} \tag{2.14}$$

The statistics in (2.14) are the covariate adjusted rank order statistics as defined in Sen and Puri [11] with the only difference that their combined sample permutation covariance matrix is replaced here by $\bar{\boldsymbol{V}}_N$, the average within sample rank order covariance matrix. This replacement is made with the primary objective of making the adjustments for the covariates in (2.14) unaffected by $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_c$; $\bar{\boldsymbol{V}}_N$ is translation invariant, so that the unknown $\boldsymbol{\tau}_1, \cdots, \boldsymbol{\tau}_c$ have no effect on it, while the combined sample permutation covariance matrix rests on the assumption that $\boldsymbol{\tau}_1 = \cdots = \boldsymbol{\tau}_c$, which is not necessarily true.

Now, precisely by the same alignment logic as in Puri and Sen [6], we define

$$(2.15) \qquad \hat{\Delta}_{kl,1}^{(i)} = \inf\{a:\ T_{kl}^{0(i)}(a) < 0\}, \qquad \hat{\Delta}_{kl,2}^{(i)} = \sup\{a:\ T_{kl}^{0(i)}(a) > 0\},$$

for $1 \leq k < l \leq c$, $i = 1, \cdots, p$. Then, our proposed estimator of $\boldsymbol{\Delta}_{kl} = \boldsymbol{\tau}_k - \boldsymbol{\tau}_l$ is

$$(2.16) \qquad \hat{\boldsymbol{\Delta}}_{kl} = (\hat{\Delta}_{kl}^{(1)}, \cdots, \hat{\Delta}_{kl}^{(p)})'; \quad \hat{\Delta}_{kl}^{(i)} = \frac{1}{2}(\hat{\Delta}_{kl,1}^{(i)} + \hat{\Delta}_{kl,2}^{(i)}), \qquad 1 \leq i \leq p.$$

Conventionally, we let $\Delta_{kk} = \hat{\Delta}_{kk} = 0$, $k = 1, \cdots, c$ and $\Delta_{kl} = -\Delta_{lk}$ for $1 \leq k < l \leq c$. Then, as in Puri and Sen ([6]; [8], Ch. 6), we define the *compatible estimators*

$$(2.17) \qquad \hat{\boldsymbol{\Delta}}_{kl}^{*} = \hat{\boldsymbol{\Delta}}_{k\cdot} - \hat{\boldsymbol{\Delta}}_{l\cdot}; \quad \hat{\boldsymbol{\Delta}}_{k\cdot} = c^{-1}\sum_{l=1}^{c} \hat{\boldsymbol{\Delta}}_{kl}, \qquad 1 \leq k < l \leq c.$$

Let $H_{(i)}(x)$ (and $G_{(j)}(x)$) be the marginal cdf of $Y_{i\alpha}^{(k)} - \tau_k^{(i)}$ (and $X_{i\alpha}^{(k)}$), $i = 1, \cdots, p$ (and $j = 1, \cdots, q$), and let $H_{(i,i')}^{(1)}(x, y)$, $H_{(i,j)}^{(2)}(x, y)$ and $G_{(j,j')}(x, y)$ be respectively the joint cdf of $(Y_{i\alpha}^{(k)} - \tau_k^{(i)}, Y_{i'\alpha}^{(k)} - \tau_k^{(i')})$, $(Y_{i\alpha}^{(k)} - \tau_k^{(i)}, X_{j\alpha}^{(k)})$ and $(X_{j\alpha}^{(k)}, X_{j'\alpha}^{(k)})$, for $i(\neq i') = 1, \cdots, p$ and $j(\neq j') = 1, \cdots, q$. Define then

$$(2.18) \qquad J^{(i)}(u) = \lim_{n\to\infty} J_n^{(i)}(u), \quad J^{*(j)}(u) = \lim_{n\to\infty} J_n^{*(j)}(u), \qquad 0 < u < 1,$$

for $i = 1, \cdots, p$, $j = 1, \cdots, q$, and denote by

$$(2.19) \quad \mu_i = \int_0^1 J^{(i)}(u)du, \quad i = 1, \cdots, p; \qquad \mu_j^* = \int_0^1 J^{*(j)}(u)du, \quad j = 1, \cdots, q;$$

$$(2.20) \quad \nu_{ij,11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{(i)}(H_{(i)}(x))J^{(j)}(H_{(j)}(y))dH_{(i,j)}^{(1)}(x, y) - \mu_i\mu_j, \qquad i, j = 1, \cdots, p;$$

$$(2.21) \quad \nu_{ij,12} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{(i)}(H_{(i)}(x))J^{*(j)}(G_{(j)}(y))dH_{(i,j)}^{(2)}(x, y) - \mu_i\mu_j^*, \qquad i = 1, \cdots, p;\ j = 1, \cdots, q;$$

$$(2.22) \quad \nu_{ij,22} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} J^{*(i)}(G_{(i)}(x))J^{*(j)}(G_{(j)}(y))dG_{(i,j)}(x, y) - \mu_i^*\mu_j^*, \qquad i, j = 1, \cdots, q;$$

$$(2.23) \quad \boldsymbol{\nu} = \begin{pmatrix} \boldsymbol{\nu}_{11} & \boldsymbol{\nu}_{12} \\ \boldsymbol{\nu}_{21} & \boldsymbol{\nu}_{22} \end{pmatrix} \qquad \boldsymbol{\nu}_{11} = ((\nu_{ij,11})), \ \boldsymbol{\nu}_{22} = ((\nu_{ij,22})),$$

and $\boldsymbol{\nu}_{12} = \boldsymbol{\nu}_{21}' = ((\nu_{ij,12}))$. Further, let

(2.24) $\boldsymbol{\nu}^* = \boldsymbol{\nu}_{11} - \boldsymbol{\nu}_{12}\boldsymbol{\nu}_{22}^{-}\boldsymbol{\nu}_{21} = ((\nu_{ij}^*))_{i,j=1,\cdots,p}$;

(2.25) $\boldsymbol{\Gamma}^* = ((\gamma_{ij}^*)) = ((\nu_{ij}^*/B_iB_j))_{i,j=1,\cdots,p}$,

where

$$B_i = \int_{-\infty}^{\infty} (d/dx)J^{(i)}(H_{(i)}(x))dH_{(i)}(x) , \qquad i=1,\cdots,p . \tag{2.26}$$

Finally, consider a contrast

$$\boldsymbol{\phi}(\boldsymbol{l}) = l_1\boldsymbol{\tau}_1 + \cdots + l_c\boldsymbol{\tau}_c , \quad \boldsymbol{l} = (l_1, \cdots, l_c)' \perp \text{(orthogonal to) } \mathbf{1} = (1, \cdots, 1)', \tag{2.27}$$

for which we have the compatible estimator

$$\boldsymbol{\phi}^*(\boldsymbol{l}) = l_1\hat{\boldsymbol{\Delta}}_{1\cdot} + \cdots + l_c\hat{\boldsymbol{\Delta}}_{c\cdot} , \tag{2.28}$$

where the $\hat{\boldsymbol{\Delta}}_{k\cdot}$ are defined by (2.17). For any $\boldsymbol{l} \perp \mathbf{1}$, (2.28) provides a robust, translation invariant and concomitant variate adjusted estimator.

For the study of the asymptotic properties of $\boldsymbol{\phi}^*(\boldsymbol{l})$, we assume that as $N \to \infty$,

$$n_k/N \to \lambda_k : 0 < \lambda_k < 1 , \qquad \text{for } k=1,\cdots,c . \tag{2.29}$$

Then, by the same technique as in Theorems 2.1 and 2.2 of Puri and Sen [6] (with direct adaptations from Theorem 5.1 of Puri and Sen [5], yielding the joint asymptotic normality of the covariate adjusted rank order statistics in (2.14)), we obtain that

$$\mathcal{L}(N^{1/2}[\boldsymbol{\phi}^*(\boldsymbol{l}) - \boldsymbol{\phi}(\boldsymbol{l})]) \to \mathcal{N}\left(\mathbf{0}, \left(\sum_{k=1}^{c} l_k^2/\lambda_k\right)\boldsymbol{\Gamma}^*\right) . \tag{2.30}$$

Similarly, if we have $r(\geqq 1)$ linearly independent contrasts

$$\boldsymbol{\phi}(\boldsymbol{l}_s) = \sum_{k=1}^{c} l_{sk}\boldsymbol{\tau}_k , \qquad \boldsymbol{l}_s \perp \mathbf{1},\ s=1,\cdots,r , \tag{2.31}$$

and we define $\boldsymbol{C} = ((c_{ss'}))_{s,s'=1,\cdots,r}$ by $c_{ss'} = \sum_{k=1}^{c} l_{sk}l_{s'k}/\lambda_k$, $s, s'=1,\cdots,r$, we have

$$\mathcal{L}(N^{1/2}[\boldsymbol{\phi}^*(\boldsymbol{l}_s) - \boldsymbol{\phi}(\boldsymbol{l}_s)],\ 1 \leqq s \leqq r) \to \mathcal{N}(\mathbf{0}, \boldsymbol{C} \otimes \boldsymbol{\Gamma}^*) , \tag{2.32}$$

where $\otimes$ stands for the Kronecker product of two matrices.

We shall make use of (2.30) and (2.32) for providing suitable confidence intervals to $\boldsymbol{\phi}(\boldsymbol{l})$, $\boldsymbol{l} \perp \mathbf{1}$. For this, we require to estimate the unknown $\boldsymbol{\Gamma}^*$. Now, defining t_α as the upper $100\alpha\%$ point of the standard normal distribution, and letting

$$\hat{\Delta}_{kl,U}^{(i)} = \sup\{a :\ T_{kl}^{0(i)}(a) > -t_{\alpha/2}[\bar{v}_{N,ii}^*(n_{kl}/n_kn_l)]^{1/2}\} , \tag{2.33}$$

$$\hat{\Delta}^{(i)}_{kl,L}=\inf\{a:\ T^{0(i)}_k(a)<t_{\alpha/2}[\bar{v}^*_{N,ii}(n_{kl}/n_k n_l)]^{1/2}\}\ , \tag{2.34}$$

and following the same method of proof as in Theorem 1 of Puri and Sen [6], it follows that

$$\hat{B}^{(i)}_{kl}=(2t_{\alpha/2}(\bar{v}^*_{N,ii})^{1/2})/[(n_{kl}/n_k n_l)^{1/2}(\hat{\Delta}^{(i)}_{kl,U}-\hat{\Delta}^{(i)}_{kl,L})]\ ,\qquad 1\leqq k<l\leqq c \tag{2.35}$$

are all translation invariant consistent estimators of B_i, defined by (2.26), for $i=1,\cdots,p$. Hence,

$$\hat{B}^{(i)}=\binom{c}{2}^{-1}\sum_{1\leqq k<l\leqq c}\hat{B}^{(i)}_{kl}\ ,\qquad i=1,\cdots,p \tag{2.36}$$

are translation invariant robust and consistent estimators of $B_1,\cdots,B_p$, respectively. Also, by Theorem 4.2 of Puri and Sen [5] (see also Theorem 3.2 of Ghosh and Sen [2]) and some standard manipulations, it follows that whatever be $\boldsymbol{\tau}_1,\cdots,\boldsymbol{\tau}_c$,

$$\bar{\boldsymbol{V}}_N\xrightarrow{P}\boldsymbol{\nu}\qquad \text{as } N\to\infty\ . \tag{2.37}$$

Thus, from (2.10), (2.24), (2.25), (2.36) and (2.37), we obtain on defining

$$\hat{\boldsymbol{\Gamma}}^*_N=((\hat{\gamma}^*_{ij}))\ ;\quad \hat{\gamma}^*_{ij}=\bar{v}^*_{N,ij}/[\hat{B}^{(i)}\hat{B}^{(j)}]\ ,\qquad i,j=1,\cdots,p\ . \tag{2.38}$$

that

$$\hat{\boldsymbol{\Gamma}}^*_N\xrightarrow{P}\boldsymbol{\Gamma}^*\qquad \text{as } N\to\infty\ . \tag{2.39}$$

Now, the space of all possible contrasts $\boldsymbol{\phi}(\boldsymbol{l})$, $\boldsymbol{l}\perp\mathbf{1}$, is spanned by a set of $c-1$ linearly independent contrasts. Hence, using (2.31) and (2.32) with $r=c-1$, and (2.39), we obtain by the classical Scheffé method that

$$\lim_{N\to\infty}\mathrm{P}\left\{N^{1/2}|\boldsymbol{t}'(\boldsymbol{\phi}^*(\boldsymbol{l})-\boldsymbol{\phi}(\boldsymbol{l}))|\left[(\boldsymbol{t}'\hat{\boldsymbol{\Gamma}}^*_N\boldsymbol{t})\left(\sum_{k=1}^{c}l_k^2/\lambda_k\right)\right]^{-1/2}\right.$$
$$\left.\leqq\chi_{p(c-1),\alpha}\ \text{for every } \boldsymbol{l}\perp\mathbf{1},\ \boldsymbol{t}\neq\mathbf{0}\right\}=1-\alpha\ , \tag{2.40}$$

which provide a simultaneous confidence region to the set of all possible contrasts where $\chi^2_{p(c-1),\alpha}$ is the upper 100% point of the chi-square distribution with $p(c-1)$ degrees of freedom. ARE results will be considered in Section 4.

3. Simultaneous test procedures

Along the lines of Krishnaiah [3], [4], we consider here some simultaneous test procedures (STP) for contrasts in MANOCOVA based on the robust estimators derived in the earlier section.

Procedure I. Consider a set of $r(\geqq 1)$ linearly independent contrasts (vectors) $\boldsymbol{\phi}(\boldsymbol{l}_s)$, $s=1,\cdots,r$, defined as in (2.31), and let

$$H_s:\boldsymbol{\phi}(\boldsymbol{l}_s)=\mathbf{0} \text{ vs. } A_s:\boldsymbol{\phi}(\boldsymbol{l}_s)\neq\mathbf{0}\ , \qquad s=1,\cdots,r\ ; \tag{3.1}$$

$$H=H_1\cap\cdots\cap H_r \qquad \text{and} \qquad A=A_1\cup\cdots\cup A_r\ . \tag{3.2}$$

Let us define the compatible estimators $\boldsymbol{\phi}^*(\boldsymbol{l}_s)$, $s=1,\cdots,r$, as in (2.28), and let

$$Q_{N,s}=(N/c_{ss})[\boldsymbol{\phi}^*(\boldsymbol{l}_s)]'(\hat{\boldsymbol{\Gamma}}_N^*)^{-1}[\boldsymbol{\phi}^*(\boldsymbol{l}_s)]\ , \qquad s=1,\cdots,r\ , \tag{3.3}$$

where $\hat{\boldsymbol{\Gamma}}_N^*$ is defined by (2.38), and c_{ss}, in between (2.31) and (2.32). By using (2.32), (2.39) and a basic result of Krishnomoorthy and Parthasarathy (1951), we conclude that under H in (3.2), $\boldsymbol{Q}_N=(Q_{N,1},\cdots,Q_{N,r})'$ has asymptotically a (r-) multivariate chi-square distribution. Hence, there exists a Q_α such that

$$\lim_{N\to\infty} \mathrm{P}\{Q_{N,s}\leqq Q_\alpha,\ \forall s=1,\cdots,r\,|\,H\}=1-\alpha\ . \tag{3.4}$$

Our proposed STP consists in rejecting H_s in favor of A_s only for those s for which $Q_{N,s}>Q_\alpha$; otherwise, accept H_s, $s=1,\cdots,r$. The total hypothesis H is accepted iff all the component hypotheses are accepted. The associated simultaneous confidence intervals for $\boldsymbol{t}'\boldsymbol{\phi}(\boldsymbol{l}_s)$, $\boldsymbol{t}\neq\mathbf{0}$, are

$$|\boldsymbol{t}'(\boldsymbol{\phi}^*(\boldsymbol{l}_s)-\boldsymbol{\phi}(\boldsymbol{l}_s))|\leqq\{Q_\alpha c_{ss}(\boldsymbol{t}'\hat{\boldsymbol{\Gamma}}_N^*\boldsymbol{t})/N\}^{1/2}\ , \qquad s=1,\cdots,r,\ \boldsymbol{t}\neq\mathbf{0}\ , \tag{3.5}$$

with an overall confidence coefficient, asymptotically, equal to $1-\alpha$.

Procedure II. This is really a stepdown procedure. We denote by $\boldsymbol{C}_j$ the principal minor of $\boldsymbol{C}$ comprising of the first j rows and columns, and write for $2\leqq j\leqq r$, $\boldsymbol{C}_j=\begin{pmatrix}\boldsymbol{C}_{j-1} & \boldsymbol{c}_{j-1}\\ \boldsymbol{c}'_{j-1} & c_{jj}\end{pmatrix}$, $c^*_{jj}=c_{jj}-\boldsymbol{c}'_{j-1}\boldsymbol{C}^{-1}_{j-1}\boldsymbol{c}_{j-1}$, $j=2,\cdots,r$, and let $c^*_{11}=c_{11}$. Then, by (2.32), on denoting by $\boldsymbol{Z}_1=N^{1/2}[\boldsymbol{\phi}^*(\boldsymbol{l}_1)-\boldsymbol{\phi}(\boldsymbol{l}_1)]$, and for $j\geqq 2$,

$$\boldsymbol{Z}_j=N^{1/2}[(\boldsymbol{\phi}^*(\boldsymbol{l}_1)-\boldsymbol{\phi}(\boldsymbol{l}_1))',\cdots,(\boldsymbol{\phi}^*(\boldsymbol{l}_{j-1})-\boldsymbol{\phi}(\boldsymbol{l}_{j-1}))']'\ , \tag{3.6}$$

we conclude that $\boldsymbol{Z}_1$ has asymptotically a multinormal distribution with mean $\mathbf{0}$ and dispersion matrix $c_{11}.\boldsymbol{\Gamma}_N^*$, while given $\boldsymbol{Z}_j$, the conditional distribution of $N^{1/2}[\boldsymbol{\phi}^*(\boldsymbol{l}_j)-\boldsymbol{\phi}(\boldsymbol{l}_j)]$ is asymptotically multinormal with mean vector

$$(\boldsymbol{c}'_{j-1}\boldsymbol{C}^{-1}_{j-1}\otimes\boldsymbol{I})\boldsymbol{Z}_j \tag{3.7}$$

and dispersion matrix

$$c^*_{jj}\boldsymbol{\Gamma}_N^*\ , \qquad \text{for } j=2,\cdots,r\ . \tag{3.8}$$

Let us then write $\boldsymbol{\eta}_1=\boldsymbol{\phi}(\boldsymbol{l}_1)$ and for $2\leqq j\leqq r$,

$$\boldsymbol{\eta}_j = \boldsymbol{\phi}(\boldsymbol{l}_j) - (\boldsymbol{c}'_{j-1}\boldsymbol{C}^{-1}_{j-1} \otimes \boldsymbol{I})[(\boldsymbol{\phi}(\boldsymbol{l}_1))', \cdots, (\boldsymbol{\phi}(\boldsymbol{l}_{j-1}))'] \tag{3.9}$$

Then, the total hypothesis H in (3.2) may be written as $H = H_1^* \cap \cdots \cap H_r^*$, where

$$H_j^* : \boldsymbol{\eta}_j = \boldsymbol{0}, \quad \text{for } j = 1, \cdots, r . \tag{3.10}$$

We cesire to provide an STP for H_j^*, $j=1, \cdots, r$. For this, let

$$\boldsymbol{\eta}_j^* = \boldsymbol{\phi}^*(\boldsymbol{l}_j) - (\boldsymbol{c}'_{j-1}\boldsymbol{C}_{j-1} \otimes \boldsymbol{I})[(\boldsymbol{\phi}^*(\boldsymbol{l}_1))', \cdots, (\boldsymbol{\phi}^*(\boldsymbol{l}_{j-1}))'] , \tag{3.11}$$

for $j=2, \cdots, r$ and $\boldsymbol{\eta}_1^* = \boldsymbol{\phi}^*(\boldsymbol{l}_1)$, and let

$$Q^*_{N,j} = (N/c^*_{jj})[(\boldsymbol{\eta}_j^*)'(\hat{\boldsymbol{\Gamma}}^*_N)^{-1}(\boldsymbol{\eta}_j^*)] , \quad j=1, \cdots, r . \tag{3.12}$$

By (2.32), (2.39), (3.7) and (3.8), we conclude that under H, $Q^*_{N,1}, \cdots, Q^*_{N,r}$ are asymptotically independent, each having a central chi-square distribution with p degrees of freedom. Hence, there exists a Q^*_α such that

$$\lim_{N\to\infty} \mathrm{P}\{Q^*_{N,s} \leqq Q^*_\alpha, \ s=1, \cdots, r \mid H\} = 1-\alpha . \tag{3.13}$$

Our proposed STP consists in testing for $H_1^*, \cdots, H_r^*$ sequentially. That is, if $Q^*_{N,1} > Q^*_\alpha$, reject H_1^* and hence H; otherwise proceed to test for H_2^*. If $Q^*_{N,2} > Q^*_\alpha$, reject H_2^* and hence H; otherwise proceed to H_3^*, and so on. The total hypothesis H is accepted iff $Q^*_{N,j} \leqq Q^*_\alpha$ for all $j = 1, \cdots, r$. By virtue of (3.7) and (3.8), we obtain the following associated simultaneous confidence intervals

$$|N^{1/2}(\boldsymbol{\eta}_j^* - \boldsymbol{\eta}_j)'\boldsymbol{t}| \leqq [Q^*_\alpha(\boldsymbol{t}'(\hat{\boldsymbol{\Gamma}}^*_N)\boldsymbol{t})c_{jj}]^{1/2} , \quad j=1, \cdots, r, \ \boldsymbol{t} \neq \boldsymbol{0} , \tag{3.14}$$

with an asymptotic confidence coefficient equal to $1-\alpha$.

Procedure III. Let $\boldsymbol{\tau}' = (\boldsymbol{\tau}'_1, \cdots, \boldsymbol{\tau}'_c)$, and let

$$\boldsymbol{a}' = (a_{11}, a_{12}, \cdots, a_{1p}, \cdots, a_{c1}, a_{c2}, \cdots, a_{cp}) \neq \boldsymbol{0}' \tag{3.15}$$

be a pc-vector such that $a_{1i}, \cdots, a_{ci}$ are not all equal for $i=1, \cdots, p$; the totality of all possible $\boldsymbol{a}$ satisfying the above condition is denoted by $\mathcal{A}$. Then, the hypothesis H in (3.2) may be written as $H = \bigcap_{\boldsymbol{a}} H_{\boldsymbol{a}}$, where

$$H_{\boldsymbol{a}} : \boldsymbol{a}'\boldsymbol{\tau} = \boldsymbol{0} \quad \text{for } \boldsymbol{a} \in \mathcal{A} . \tag{3.16}$$

We want to consider an STP for all $H_{\boldsymbol{a}}$, $\boldsymbol{a} \in \mathcal{A}$.

Let us denote by $\hat{\boldsymbol{\Delta}}' = (\hat{\boldsymbol{\Delta}}'_{1\cdot}, \cdots, \hat{\boldsymbol{\Delta}}'_{c\cdot})$, where the $\hat{\boldsymbol{\Delta}}_{k\cdot}$ are defined by (2.17), and let $\boldsymbol{n} = \mathrm{Diag}(n_1, \cdots, n_c)$. Define then

$$Q_{N,0} = \hat{\boldsymbol{\Delta}}'(\boldsymbol{n} \otimes (\hat{\boldsymbol{\Gamma}}^*_N)^{-1})\hat{\boldsymbol{\Delta}} = \sum_{k=1}^{c} n_k \hat{\boldsymbol{\Delta}}'_{k\cdot}(\hat{\boldsymbol{\Gamma}}^*_N)^{-1}\hat{\boldsymbol{\Delta}}_{k\cdot} . \tag{3.17}$$

By (2.32) and (2.39), under H, $Q_{N,0}$ has asymptotically chi-square distribution with $p(c-1)$ degrees of freedom. Hence,

$$\lim_{N\to\infty} \mathrm{P}\{Q_{N,0} \leqq \chi^2_{p(c-1),\alpha} | H\} = 1-\alpha . \tag{3.18}$$

Replacing $\hat{\boldsymbol{\Delta}}$ by $\hat{\boldsymbol{\Delta}}-\boldsymbol{\tau}$ in (3.17), and thereby eliminating H in (3.18), we obtain by the Schwarz inequality that asymptotically with a confidence coefficient $1-\alpha$,

$$|\boldsymbol{a}'(\hat{\boldsymbol{\Delta}}-\boldsymbol{\tau})| \leqq \chi_{p(c-1),\alpha}[\boldsymbol{a}(\boldsymbol{n}^{-1}\otimes\hat{\boldsymbol{\Gamma}}^*_N)\boldsymbol{a}]^{1/2} , \qquad \forall\, \boldsymbol{a}\in\mathcal{A} , \tag{3.19}$$

which provides a simultaneous confidence region to all possible $\boldsymbol{a}'\boldsymbol{\tau}$, $\boldsymbol{a}\in\mathcal{A}$. The STP consists in rejecting those $H_{\boldsymbol{a}}$ ($\boldsymbol{a}\in\mathcal{A}$) for which $|\boldsymbol{a}'\hat{\boldsymbol{\Delta}}|>\chi_{p(c-1),\alpha}[\boldsymbol{a}'(\boldsymbol{n}^{-1}\otimes\hat{\boldsymbol{\Gamma}}^*_N)\boldsymbol{a}]^{1/2}$. A similar procedure for MANOVA (but involving a loss of A.R.E. for contrasts not involving all the c samples) is due to Gabriel and Sen [1].

Procedure III has the maximum flexibility to all $\boldsymbol{a}\in\mathcal{A}$, but, in practice, when one may be interested in a set of specified contrasts, it is usually less efficient than the two other procedures. The simplicity of the computation of Q^*_α over Q_α (derived from the asymptotic independence of $Q^*_{N,1},\cdots,Q^*_{N,r}$) has to be weighed against the arbitrariness and complications involved in the choice of the sequence of the suffixes $1,\cdots,r$ and the formulation of $H^*_1,\cdots,H^*_r$. One may also consider some other procedures along the lines of Krishnaiah [4].

4. ARE results

One could have ignored the covariates totally, and proceeding as in Puri and Sen [6], obtained rank order estimates of the contrasts derived solely from the $T^{(i)}_{kl}$, $1\leqq k<l\leqq c$, $i=1,\cdots,p$. In this case, the asymptotic covariance matrix in (2.30) or (2.32) would have involved $\boldsymbol{\Gamma}$ instead of $\boldsymbol{\Gamma}^*$, where

$$\boldsymbol{\Gamma}=((\gamma_{ij})) , \qquad \gamma_{ij}=\nu_{ij,11}/B_iB_j ,\ i,j=1,\cdots,p , \tag{4.1}$$

and $\nu_{ij,11}$ and B_i are defined by (2.20) and (2.26). Thus, as in Puri and Sen [6], defining the ARE as the reciprocal of the ratio of the generalized variances raised to the power p^{-1}, the ARE of the covariate adjusted rank order estimates with respect to the unadjusted ones is

$$e_1=\{|\boldsymbol{\Gamma}|/|\boldsymbol{\Gamma}^*|\}^{1/p}=\{|\boldsymbol{\nu}_{11}|/|\boldsymbol{\nu}^*|\}^{1/p}=\{|\boldsymbol{\nu}_{11}|/|\boldsymbol{\nu}_{11}-\boldsymbol{\nu}_{12}\boldsymbol{\nu}^-_{22}\boldsymbol{\nu}_{21}|\}^{1/p} , \tag{4.2}$$

where $\boldsymbol{\nu}_{12}\boldsymbol{\nu}^-_{22}\boldsymbol{\nu}_{21}$ is positive semi-definite. Hence, we have

$$e_1\geqq 1 , \qquad \text{where the equality sign holds when } \boldsymbol{\nu}_{12}=\boldsymbol{0} . \tag{4.3}$$

This explains the asymptotic supremacy of the covariate adjusted estimates. It also follows similarly that the ARE is a non-decreasing function of the number of concomitant variates, i.e., additional covariates induce more information on the estimates.

For the classical normal theory estimates of contrasts based on the sample mean vectors, let $\boldsymbol{\Sigma}=\begin{pmatrix}\boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12}\\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22}\end{pmatrix}$ be the covariance matrix of $(\boldsymbol{Y}_\alpha^{(k)\prime}, \boldsymbol{X}_\alpha^{(k)\prime})'$, and let

$$\boldsymbol{\Sigma}^*=\boldsymbol{\Sigma}_{11}-\boldsymbol{\Sigma}_{12}\boldsymbol{\Sigma}_{22}^{-}\boldsymbol{\Sigma}_{21}\ . \tag{4.4}$$

Then, from the results of Sen and Puri [11], we may conclude (on omitting the details) that the ARE of $\boldsymbol{\phi}^*(\boldsymbol{l})$ with respect to the classical normal theory estimator of $\boldsymbol{\phi}(\boldsymbol{l})$ is

$$e_2=\{|\boldsymbol{\Sigma}^*|/|\boldsymbol{\Gamma}^*|\}^{1/p}\ , \tag{4.5}$$

which depends on $\boldsymbol{\Sigma}_{11}$, $\boldsymbol{\Sigma}_{22}$, $\boldsymbol{\Sigma}_{12}$ as well as $\boldsymbol{\nu}_{11}$, $\boldsymbol{\nu}_{22}$, $\boldsymbol{\nu}_{12}$ and $B_1,\cdots,B_p$. In general, it is not possible to attach suitable bounds to e_2. However,

$$\{|\boldsymbol{\Sigma}^*|/|\boldsymbol{\Gamma}|\}^{1/p}\leqq e_2\leqq\{|\boldsymbol{\Sigma}_{11}|/|\boldsymbol{\Gamma}^*|\}^{1/p}\ . \tag{4.6}$$

In particular, if $\boldsymbol{\Sigma}_{12}=\boldsymbol{0}$, $e_2\geqq\{|\boldsymbol{\Sigma}|\,|\boldsymbol{\Gamma}|\}^{1/p}$, where the equality sign holds if $\boldsymbol{\nu}_{12}=\boldsymbol{0}$. Thus, if the original variates are uncorrelated but not necessarily independent, we expect to gain in efficiency by considering the robust estimates $\boldsymbol{\phi}^*(\boldsymbol{l})$. The same ARE results holds for the simultaneous confidence intervals in (2.40) when compared with the parallel procedure for the MANOVA model and the parametric Scheffé type procedure.

For the simultaneous tests in Section 3, the ARE with respect to their parametric counterparts considered in detail in Krishnaiah [4] depends on the particular set of variates or samples included in the set of contrasts under test. The overall ARE agrees with e_1 or e_2 in the respective cases, while the minimum and maximum (over all possible choice of variates and samples) ARE are respectively the minimum and the maximum characteristic roots of $\boldsymbol{\Sigma}^*(\boldsymbol{\Gamma}^*)^{-1}$. Since, these bounds are similar to the ones discussed in Sen and Puri [11], we omit the details here. In passing, we may remark that on using the normal scores statistics for deriving the robust estimates in Section 2, we are able to achieve asymptotic optimality when the underlying distribution is normal.

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