ASYMPTOTIC FORMULAS FOR THE HYPERGEOMETRIC FUNCTION ${}_2F_1$ OF MATRIX ARGUMENT, USEFUL IN MULTIVARIATE ANALYSIS

NARIAKI SUGIURA

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Summary

Let S_i have the Wishart distribution $W_p(\Sigma_i, n_i)$ for $i=1, 2$. An asymptotic expansion of the distribution of $-2\log\lambda=-2\log\left[\prod_{\alpha=1}^{2}|S_\alpha/n_\alpha|^{n_\alpha/2}\cdot|(S_1+S_2)/n|^{-n/2}\right]$ for large $n=n_1+n_2$ is derived, when $\Sigma_1\Sigma_2^{-1}=I+n^{-1/2}\theta$, based on an asymptotic solution of the system of partial differential equations for the hypergeometric function ${}_2F_1$, obtained recently by Muirhead [2]. Another asymptotic formula is also applied to the distributions of $-2\log\lambda$ and $-\log|S_2(S_1+S_2)^{-1}|$ under fixed $\Sigma_1\Sigma_2^{-1}$, which gives the earlier results by Nagao [4]. Some useful asymptotic formulas for ${}_1F_1$ were investigated by Sugiura [7].

1. Preliminaries

Based on a random sample of size n_i+1 from p-variate normal population $N(\mu_i, \Sigma_i)$ for $i=1$ and 2, the modified likelihood ratio statistic for testing $\Sigma_1=\Sigma_2$ against $\Sigma_1\neq\Sigma_2$ can be expressed as

$$(1.1)\qquad \lambda=|S_1/n_1|^{n_1/2}|S_2/n_2|^{n_2/2}/|(S_1+S_2)/n|^{n/2},$$

where S_1 has the Wishart distribution $W_p(\Gamma, n_1)$ and S_2 has $W_p(I, n_2)$ independently with $n=n_1+n_2$. The diagonal matrix Γ has the characteristic roots of $\Sigma_1\Sigma_2^{-1}$ on the diagonal. The moments of λ was given by Sugiura [5] as

$$(1.2)\qquad \mathrm{E}\,[\lambda^h]=\left(\frac{n^n}{n_1^{n_1}n_2^{n_2}}\right)^{ph/2}\frac{\Gamma_p(n/2)}{\Gamma_p(n_1/2)\Gamma_p(n_2/2)}\,\frac{\Gamma_p(n_1(1+h)/2)\Gamma_p(n_2(1+h)/2)}{\Gamma_p(n(1+h)/2)}\cdot|\Gamma|^{n_1h/2}\cdot{}_2F_1(nh/2,\, n_1(1+h)/2;\, n(1+h)/2;\, I-\Gamma)\ ,$$

which can be derived by starting the joint distribution of (S_1, S_2) and using the formula

$$\operatorname{etr}\left[\frac{1}{2}(I-\Gamma^{-1})S_1\right]=\sum_{k=0}^{\infty}\Sigma_{(\kappa)}C_\kappa\left(\frac{1}{2}(I-\Gamma^{-1})S_1\right)\Big/k!\ ,$$

together with (12) and (22) in Constantine [1]. Kummer transformation formula ${}_2F_1(a_1, a_2; b; Z)=|I-Z|^{-a_2}{}_2F_1(b-a_1, a_2; b; -Z(I-Z)^{-1})$ is also used in the final expression.

Put $m_\alpha=\rho n_\alpha$ for $\alpha=1, 2$ and the correction factor $\rho=1-(2p^2+3p-1)(n_1^{-1}+n_2^{-1}-n^{-1})/\{6(p+1)\}$ with $m=m_1+m_2$. Asymptotic expansion of the distribution of $-2\rho\log\lambda$ for large m and fixed $\rho_\alpha=m_\alpha/m>0$, when $\Gamma\doteq I+m^{-1}\theta$, was obtained by Sugiura [5] in terms of χ^2-distributions and when Γ is fixed, by Nagao [4] in terms of normal distribution function and its derivatives, in a more general case, namely for k-sample problem. When $\Gamma=I+m^{-1/2}\theta$, however, both methods are not available and the present approach by system of partial differential equations for ${}_2F_1$ by Muirhead [3] is essentially useful.

Another asymptotic formula for ${}_2F_1$, based on the differential equations gives the asymptotic expansions of the distributions of $-2\rho\log\lambda$ and $-\log|S_2(S_1+S_2)^{-1}|$ under fixed Γ obtained in Nagao [4] by different technique. The characteristic function of $-\sqrt{n}\log|S_2(S_1+S_2)^{-1}|$ can be written as

$$(1.3)\qquad \Gamma_p\left(\frac{1}{2}n_2-\sqrt{n}\,it\right)\Gamma_p\left(\frac{1}{2}n\right)\Big/\left[\Gamma_p\left(\frac{1}{2}n_2\right)\Gamma_p\left(\frac{1}{2}n-\sqrt{n}\,it\right)\right]$$
$$\cdot\ {}_2F_1\left(-\sqrt{n}\,it, \frac{1}{2}n_1; \frac{1}{2}n-\sqrt{n}\,it; I-\Gamma\right)\ ,$$

by the similar argument as for (1.2), which was remarked by Sugiura [6]. This statistic can be used to test $\Sigma_1=\Sigma_2$ against $\Sigma_1\Sigma_2^{-1}\geqq I$ for two normal populations.

2. General asymptotic formulas for ${}_2F_1$

Muirhead [2] has proved that the hypergeometric function ${}_2F_1(a, b; c; R)$ for $R=\operatorname{diag}(r_1, \cdots, r_p)$ can be characterized as the unique solution of the partial differential equation

$$(2.1)\qquad r_1(1-r_1)\frac{\partial^2F}{\partial r_1^2}+\left[c-\frac{1}{2}(p-1)-\left\{a+b+1-\frac{1}{2}(p-1)\right\}r_1\right.$$
$$\left.+\frac{1}{2}\sum_{j=2}^{p}\frac{r_1(1-r_1)}{r_1-r_j}\right]\frac{\partial F}{\partial r_1}-\frac{1}{2}\sum_{j=2}^{p}\frac{r_j(1-r_j)}{r_1-r_j}\frac{\partial F}{\partial r_j}=abF\ ,$$

subject to the condition that F is symmetric with respect to r_i and F is analytic at $R=0$ with $F(0)=1$.

Generalizing the characteristic function of $-2\rho\log\lambda$ obtained by putting $h=-2\rho it$ in (1.2) under $\Gamma=I+m^{-1/2}\theta$, we shall consider $G(R)=$

$\log[{}_2F_1(\alpha n;\beta_1 n+\beta_0;\gamma_1 n+\gamma_0;n^{-1/2}R)]$ for $\gamma_1\neq 0$, which is the unique solution for

$$(2.2)\quad r_1(1-r_1/\sqrt{n})\left\{\frac{\partial^2 G}{\partial r_1^2}+\left(\frac{\partial G}{\partial r_1}\right)^2\right\}+\Big[\gamma_1 n-(\alpha+\beta_1)\sqrt{n}\,r_1-\frac{1}{2}(p-1)+\gamma_0$$
$$-\left(\beta_0+1-\frac{1}{2}(p-1)\right)r_1/\sqrt{n}+\frac{1}{2}\sum_{j=2}^{p}\frac{r_1}{r_1-r_j}$$
$$-\frac{1}{2}\sum_{j=2}^{p}\frac{r_1^2}{r_1-r_j}\frac{1}{\sqrt{n}}\Big]\frac{\partial G}{\partial r_1}-\frac{1}{2}\sum_{j=2}^{p}\frac{r_j}{r_1-r_j}\frac{\partial G}{\partial r_j}$$
$$+\frac{1}{2}\sum_{j=2}^{p}\frac{r_j^2}{r_1-r_j}\frac{\partial G}{\partial r_j}\frac{1}{\sqrt{n}}=\alpha\beta_1 n\sqrt{n}+\alpha\beta_0\sqrt{n}\ ,$$

subject to $G(0)=0$ and G is symmetric for R. Putting $G(R)=\sum_{k=-1}^{\infty}Q_k(R)n^{-k/2}$ with $Q_k(0)=0$ as in Sugiura [7], we can successively determine $Q_k(R)$ from (2.2). Equating the term of order $n\sqrt{n}$ in (2.2) yields $\gamma_1\partial Q_{-1}/\partial r_1=\alpha\beta_1$, which implies $Q_{-1}=(\alpha\beta_1/\gamma_1)\sum_{j=1}^{p}r_j$. We shall write $\sigma_j=\operatorname{tr}R^j$ and put $\xi_1=\alpha+\beta_1-\alpha\beta_1/\gamma_1$, $\xi_2=\alpha+\beta_1-2\alpha\beta_1/\gamma_1$ for abbreviation. Then equating the term of order n, $\sqrt{n}$ and 1 in (2.2) yields, after some computation:

$$Q_0=\frac{1}{2}(\alpha\beta_1/\gamma_1^2)\xi_1\sigma_2\ ,$$

$$(2.3)\quad Q_1=\frac{1}{3}\sigma_3(\alpha\beta_1/\gamma_1^3)(\alpha\beta_1+\xi_1\xi_2)+(\alpha/\gamma_1)(\beta_0-\beta_1\gamma_0/\gamma_1)\sigma_1\ ,$$

$$Q_2=\frac{1}{4}(\alpha\beta_1/\gamma_1^4)\sigma_4\{\alpha\beta_1\xi_1(2-\xi_1/\gamma_1)+\xi_2(\alpha\beta_1+\xi_1\xi_2)\}$$
$$+\frac{1}{2}(\alpha/\gamma_1^2)\sigma_2\left\{-(\beta_1/\gamma_1)\left(\gamma_0+\frac{1}{2}\right)\xi_1+(\beta_0-\gamma_0\beta_1/\gamma_1)\xi_2+\beta_1\left(\beta_0+\frac{1}{2}\right)\right\}$$
$$+\frac{1}{4}(\alpha\beta_1/\gamma_1^2)\sigma_1^2(1-\xi_1/\gamma_1)\ .$$

Hence we have:

THEOREM 2.1. *An asymptotic formula of* ${}_2F_1(\alpha n,\beta_1 n+\beta_0;\gamma_1 n+\gamma_0;n^{-1/2}R)$ *for large value of* n, *when* $\gamma_1\neq 0$ *is given by*

$$(2.4)\quad \exp\left[\sqrt{n}(\alpha\beta_1/\gamma_1)\sigma_1+\frac{1}{2}(\alpha\beta_1/\gamma_1^2)\xi_1\sigma_2+n^{-1/2}Q_1+n^{-1}Q_2+O(n^{-3/2})\right]\ ,$$

where $\sigma_j=\operatorname{tr}R^j$, $\xi_1=\alpha+\beta_1-\alpha\beta_1/\gamma_1$ *and* Q_1, Q_2 *are given by* (2.3) *with* $\xi_2=\xi_1-\alpha\beta_1/\gamma_1$.

Generalizing the characteristic function (1.3), an asymptotic formula for ${}_2F_1(\alpha\sqrt{n}, \beta_2 n+\beta_1\sqrt{n}+\beta_0; \gamma_2 n+\gamma_1\sqrt{n}+\gamma_0; R)$, when $\gamma_2\neq 0$ should be investigated. As in Muirhead [3], transforming the variables R to $W=I-(I-(\beta_2/\gamma_2)R)^{-1}$ and putting $F=|I-W|^{\alpha\sqrt{n}}G$, we can rewrite the partial differential equation (2.1) for the function $G(W)$. Further the transformation $H=\log G$ is useful to simplify the right-hand side of the equation. Denoting $a=\alpha\sqrt{n}$, $b=\beta_2 n+\beta_1\sqrt{n}+\beta_0$ and $c=\gamma_2 n+\gamma_1\sqrt{n}+\gamma_0$ for abbreviation, the differential equation for $H(W)$ is expressed by

$$\begin{aligned}(2.5)\quad & (-\beta_2/\gamma_2)\{1-(1-\gamma_2/\beta_2)w_1\}\Big[a(a+1)-2(a+1)(1-w_1)\frac{\partial H}{\partial w_1} \\ & +(1-w_1)^2\Big\{\frac{\partial^2 H}{\partial w_1^2}+\Big(\frac{\partial H}{\partial w_1}\Big)^2\Big\}\Big]-(\beta_2/\gamma_2)\Big[\{c-(p-1)/2\}(1-w_1) \\ & -\{a+b+1-(p-1)/2\}(\gamma_2/\beta_2)w_1+\frac{1}{2}\sum_{j=2}^{p}\frac{w_1(1-w_j)}{w_1-w_j} \\ & \cdot\{1-(1-\gamma_2/\beta_2)w_1\}\Big]\Big[-a+(1-w_1)\frac{\partial H}{\partial w_1}\Big]+\frac{1}{2}\frac{\beta_2}{\gamma_2} \\ & \cdot\sum_{j=2}^{p}\frac{(1-w_1)(1-w_j)w_j}{w_1-w_j}\Big\{1-\Big(1-\frac{\gamma_2}{\beta_2}\Big)w_j\Big\}\Big\{-\frac{a}{1-w_j}+\frac{\partial H}{\partial w_j}\Big\} \\ & =ab\,.\end{aligned}$$

Putting $H(W)=\sum_{k=0}^{\infty}Q_k(w)n^{-k/2}$ with $Q_k(0)=0$ for $k=0,1,\cdots$, and equating the term of order n, $\sqrt{n}$, and 1 in (2.5), we can get:

THEOREM 2.2. *An asymptotic formula for* ${}_2F_1(\alpha\sqrt{n}; \beta_2 n+\beta_1\sqrt{n}+\beta_0; \gamma_2 n+\gamma_1\sqrt{n}+\gamma_0; R)$ *for large value of* n, *when* $\gamma_2\neq 0$ *is given by*

$$(2.6)\quad |I-W|^{\alpha\sqrt{n}}\exp\Big[-\alpha\delta_1\sigma_1+\frac{1}{2}\alpha^2\delta_2\sigma_2+n^{-1/2}Q_1+n^{-1}Q_2+O(n^{-3/2})\Big],$$

where $\sigma_j=\operatorname{tr}W^j$, $W=I-(I-(\beta_2/\gamma_2)R)^{-1}$ *and* $\delta_1=\beta_1/\beta_2-\gamma_1/\gamma_2$ $\delta_2=\beta_2^{-1}-\gamma_2^{-1}$. *The coefficients* Q_1 *and* Q_2 *are given by, using* $\delta_0=\beta_0/\beta_2-\gamma_0/\gamma_2$,

$$\begin{aligned} & Q_1=\alpha\sigma_1(-\delta_0+\gamma_1\delta_1/\gamma_2)+\frac{1}{2}\alpha\sigma_2\Big\{\delta_2\Big(\frac{1}{2}-\alpha\gamma_1/\gamma_2\Big)+\delta_1(\delta_1+\alpha\delta_2-\alpha/\gamma_2)\Big\} \\ & \quad+\frac{1}{3}\alpha^2\delta_2\sigma_3\{-3\delta_1+\alpha(\gamma_2^{-1}-\delta_2)\}+\frac{1}{2}\alpha^3\sigma_4\delta_2^2+\frac{1}{4}\alpha\delta_2\sigma_1^2\,, \\ (2.7) & \\ & Q_2=(\alpha/\gamma_2)\sigma_1[\gamma_0\delta_1+\gamma_1(\delta_0-\gamma_1\delta_1/\gamma_2)]+\frac{1}{2}\alpha\sigma_2\Big[-(\alpha/\gamma_2)\{\delta_1^2+(1+\gamma_0)\delta_2\} \\ & \quad+\frac{1}{2}\delta_1(2\delta_0+\delta_2-\gamma_2^{-1})-\frac{1}{2}(\gamma_1/\gamma_2)\delta_2(1-\alpha\gamma_1^{-1}-2\alpha\gamma_1\gamma_2^{-1}) \\ & \quad+(\alpha\delta_2+\delta_1-\alpha/\gamma_2)(\delta_0-2\delta_1\gamma_1/\gamma_2)\Big]+\frac{1}{3}\sigma_3\Big[\alpha\delta_1\delta_2\Big(-\frac{3}{2}+6\alpha\gamma_1\gamma_2^{-1}\end{aligned}$$

$$+2\alpha^2\gamma_2^{-1}\Big)+(\gamma_2^{-1}-\delta_2)\Big(\delta_1^2+\frac{1}{2}\delta_2\Big)\alpha^2-3\alpha^2\delta_0\delta_2-\alpha^2\delta_2(\delta_2-\gamma_2^{-1})$$
$$\cdot(1-2\alpha\gamma_1\gamma_2^{-1})-\alpha\delta_1(\delta_1+\alpha\delta_2-\alpha/\gamma_2)^2\Big]$$
$$+\frac{1}{4}\sigma_4\alpha^2\delta_2\Big[\delta_1^2+\delta_2\Big(\frac{5}{2}-\alpha^2\gamma_2^{-1}-4\alpha\gamma_1\gamma_2^{-1}\Big)-2\alpha\delta_1(\gamma_2^{-1}-\delta_2)$$
$$+(\alpha\delta_2+\delta_1-\alpha\gamma_2^{-1})(5\delta_1+\alpha\delta_2-\alpha\gamma_2^{-1})\Big]+\sigma_5\alpha^3\delta_2^2[-2\delta_1+\alpha\gamma_2^{-1}-\alpha\delta_2]$$
$$+\frac{5}{6}\alpha^4\delta_2^3\sigma_6+\frac{1}{4}\alpha\sigma_1^2[\delta_1(\delta_2-\gamma_2^{-1})-\gamma_2^{-1}\delta_2(\gamma_1+\alpha)]$$
$$-\frac{1}{2}\alpha\delta_2\sigma_1\sigma_2[\delta_1+\alpha\delta_2-\alpha\gamma_2^{-1}]+\frac{1}{2}\alpha^2\delta_2^2\sigma_1\sigma_3+\frac{1}{8}\alpha^2\delta_2^2\sigma_2^2\ .$$

3. Asymptotic expansions

The characteristic function of the likelihood ratio statistic $-2\rho\log\lambda$ under $\Gamma=I+m^{-1/2}\theta$ is expressed by (1.2) as

$$(3.1)\qquad \Big(\frac{m^m}{m_1^{m_1}m_2^{m_2}}\Big)^{-pit}\frac{\Gamma_p(m/2+\Delta)\Gamma_p(m_1(1-2it)/2+\Delta_1)\Gamma_p(m_2(1-2it)/2+\Delta_2)}{\Gamma_p(m_1/2+\Delta_1)\Gamma_p(m_2/2+\Delta_2)\Gamma_p(m(1-2it)/2+\Delta)}$$
$$\cdot\,|I+m^{-1/2}\theta|^{-m_1it}\cdot{}_2F_1(-mit,\ m_1(1-2it)/2+\Delta_1;$$
$$m(1-2it)/2+\Delta;\ -m^{-1/2}\theta)\ ,$$

where $\Delta_i=\rho_i\Delta=(n_i-m_i)/2$ and $\rho_i=n_i/n$. By Stirling's formula, the first factor in (3.1), namely, the product of gamma functions can be evaluated as

$$(3.2)\qquad \text{The first factor}=(1-2it)^{-f/2}+O(m^{-2})$$

for $f=p(p+1)/2$. The second factor can easily be written by

$$(3.3)\qquad \log|I+m^{-1/2}\theta|^{-m_1it}=\rho_1it\Big(-\sqrt{m}\,\sigma_1+\frac{1}{2}\sigma_2-\frac{1}{3}m^{-1/2}\sigma_3+\frac{1}{4}m^{-1}\sigma_4\Big)$$
$$+O(m^{-3/2})\ ,$$

where $\sigma_j=\operatorname{tr}\theta^j$. Finally putting $n=m$, $\alpha=-it$, $\beta_1=\rho_1(1-2it)/2$, $\beta_0=\Delta_1$, $\gamma_1=(1-2it)/2$, $\gamma_0=\Delta$ and $R=-\theta$ in Theorem 2.1, we can evaluate the third factor in (3.1) as, denoting $(1-2it)$ by T,

$$(3.4)\qquad \log{}_2F_1=\sqrt{m}\,\rho_1it\sigma_1+\frac{1}{4}\rho_1\sigma_2(-2it-\rho_2+\rho_2T^{-1})-\frac{1}{6}\rho_1\sigma_3m^{-1/2}$$
$$\cdot[-2it-\rho_2-\rho_2^2+3\rho_2^2T^{-1}+\rho_2(1-2\rho_2)T^{-2}]$$
$$+m^{-1}\Big[\frac{1}{8}\rho_1\sigma_4\{-2it-\rho_2-\rho_2^2-\rho_2^3+6\rho_2^3T^{-1}+2\rho_2^2(3-5\rho_2)T^{-2}$$

$$+\rho_2(1-5\rho_2+5\rho_2^2)T^{-3}\}+\frac{1}{4}\rho_1\rho_2\sigma_2\{-2\Delta+(1+4\Delta)T^{-1}$$
$$-(1+2\Delta)T^{-2}\}+\frac{1}{4}\rho_1\rho_2\sigma_1^2(T^{-1}-T^{-2})\Big]+O(m^{-3/2})\ .$$

Combined with (3.2), (3.3) and (3.4), we can get the asymptotic expansion of the distribution of $-2\rho\log\lambda$ in terms of noncentral χ^2-distributions with noncentrality $\delta=\rho_1\rho_2\sigma_2/4$ as

$$\text{(3.5)}\qquad \mathrm{P}(-2\rho\log\lambda<x)=P_f+\frac{1}{6}\rho_1\rho_2\sigma_3 m^{-1/2}\{(1+\rho_2)P_f-3\rho_2P_{f+2}$$
$$-(1-2\rho_2)P_{f+4}\}+m^{-1}\sum_{k=0}^{4}h_kP_{f+2k}+O(m^{-3/2})\ ,$$

where $P_f=\mathrm{P}(\chi_f^2(\delta)<x)$ for the noncentral χ^2-variate and

$$h_0=\frac{1}{8}\rho_1\rho_2\left\{-\sigma_4(1+\rho_2+\rho_2^2)-4\Delta\sigma_2+\frac{1}{9}\rho_1\rho_2\sigma_3^2(1+\rho_2)^2\right\}\ ,$$

$$h_1=\frac{1}{8}\rho_1\rho_2\left\{6\rho_2^2\sigma_4+2(1+4\Delta)\sigma_2+2\sigma_1^2-\frac{2}{3}\rho_1\rho_2^2(1+\rho_2)\sigma_3^2\right\}\ ,$$

$$\text{(3.6)}\qquad h_2=\frac{1}{8}\rho_1\rho_2\left\{2\rho_2(3-5\rho_2)\sigma_4-2(2\Delta+1)\sigma_2-2\sigma_1^2+\frac{1}{9}\rho_1\rho_2\sigma_3^2(13\rho_2^2+2\rho_2-2)\right\}\ ,$$

$$h_3=\frac{1}{8}\rho_1\rho_2\left\{(1-5\rho_2+5\rho_2^2)\sigma_4+\frac{2}{3}\rho_1\rho_2^2(1-2\rho_2)\sigma_3^2\right\}\ ,$$

$$h_4=\frac{1}{72}\rho_1^2\rho_2^2\sigma_3^2(1-2\rho_2)^2\ .$$

Next we shall consider the distribution of $-2\rho m^{-1/2}\log\lambda$ under fixed Γ, the characteristic function of which can be expressed from (1.2) as

$$\text{(3.7)}\qquad \left(\frac{m^m}{m_1^{m_1}m_2^{m_2}}\right)^{-itp/\sqrt{m}}$$
$$\cdot\frac{\Gamma_p(m/2+\Delta)\Gamma_p(m_1/2-\rho_1it\sqrt{m}+\Delta_1)\Gamma_p(m_2/2-\rho_2it\sqrt{m}+\Delta_2)}{\Gamma_p(m_1/2+\Delta_1)\Gamma_p(m_2/2+\Delta_2)\Gamma_p(m/2-\sqrt{m}\,it+\Delta)}$$
$$\cdot|\Gamma|^{-\rho_1it\sqrt{m}}\,{}_2F_1(-\sqrt{m}\,it,\ m_1/2-\rho_1it\sqrt{m}+\Delta_1;$$
$$m/2-it\sqrt{m}+\Delta;\ I-\Gamma)\ .$$

By Stirling's formula, the first factor in (3.7) can be evaluated as

$$\text{(3.8)}\qquad \exp\left[\frac{1}{2}p(p+1)itm^{-1/2}+\frac{1}{2}p(p+1)(it)^2m^{-1}+O(m^{-3/2})\right]\ .$$

Putting $n=m$, $\alpha=-it$, $\beta_2=\rho_1/2$, $\beta_1=-\rho_1it$, $\beta_0=\Delta_1$, $\gamma_2=1/2$, $\gamma_1=-it$, $\gamma_0=\Delta$ and $R=I-\Gamma$ in Theorem 2.2, we can get an asymptotic formula for the second factor in (3.7) with respect to $W=I-\tilde{\Gamma}^{-1}$ for $\tilde{\Gamma}=\rho_1\Gamma+\rho_2 I$, which gives the following asymptotic expansion under fixed Γ in terms of the standard normal distribution function $\Phi(x)$ and the derivatives:

$$
\begin{aligned}
(3.9)\quad & \mathrm{P}\,(-2\rho m^{-1/2}\log\lambda+\sqrt{m}\log(|\Gamma|^{\rho_1}/|\tilde{\Gamma}|)<\tau_1 x) \\
&=\Phi(x)+m^{-1/2}\{g_1\Phi^{(1)}(x)/\tau_1+g_3\Phi^{(3)}(x)/\tau_1^3\}+m^{-1}\sum_{k=1}^{3}h_{2k}\Phi^{(2k)}(x)/\tau_1^{2k} \\
&\quad +O(m^{-3/2}),
\end{aligned}
$$

where $\tau_1^2=2(\rho_2/\rho_1)\sigma_2$ for $\sigma_j=\mathrm{tr}\,W^j=\mathrm{tr}\,(I-\tilde{\Gamma}^{-1})^j$ and

$$g_1=\frac{1}{2}(\rho_2/\rho_1)(\sigma_2+\sigma_1^2)-\frac{1}{2}p(p+1),$$

$$g_3=2(\rho_2/\rho_1)\left\{-\sigma_2+\frac{2}{3}(1-\rho_2/\rho_1)\sigma_3+(\rho_2/\rho_1)\sigma_4\right\},$$

$$
\begin{aligned}
(3.10)\quad h_2=&\frac{1}{2}p(p+1)-2(\rho_2/\rho_1)(1+\Delta)\sigma_2+2(\rho_2/\rho_1)(1-\rho_2/\rho_1)\sigma_3+\frac{5}{2}(\rho_2/\rho_1)^2\sigma_4 \\
&-2(\rho_2/\rho_1)\sigma_1^2+2(\rho_2/\rho_1)(1-\rho_2/\rho_1)\sigma_1\sigma_2+2(\rho_2/\rho_1)^2\sigma_1\sigma_3 \\
&+\frac{1}{2}(\rho_2/\rho_1)^2\sigma_2^2+\frac{1}{2}g_1^2,
\end{aligned}
$$

$$
\begin{aligned}
h_4=&4(\rho_2/\rho_1)\sigma_2-\frac{16}{3}(\rho_2/\rho_1)(1-\rho_2/\rho_1)\sigma_3+2(\rho_2/\rho_1)\{1-7\rho_2/\rho_1+(\rho_2/\rho_1)^2\}\sigma_4 \\
&+8(\rho_2/\rho_1)^2(1-\rho_2/\rho_1)\sigma_5+\frac{20}{3}(\rho_2/\rho_1)^3\sigma_6+g_1g_3,
\end{aligned}
$$

$$h_6=\frac{1}{2}g_3^2.$$

After some computation, we can verify that the above result agrees with the Theorem 8.1 in Nagao [4] for $k=2$.

Finally for the characteristic function of $-\sqrt{n}\log|S_2(S_1+S_2)^{-1}|$ given in (1.3), the first factor can be reduced to

$$
\begin{aligned}
(3.11)\quad & -\sqrt{n}\,itp\log\rho_2-pt^2\rho_1/\rho_2+2pitn^{-1/2} \\
&\cdot\left\{\frac{1}{3}\rho_1[(1+\rho_2)/\rho_2^2](it)^2+\frac{1}{4}(p+1)\rho_1/\rho_2\right\}+p(it)^2n^{-1} \\
&\cdot\left\{\frac{2}{3}(\rho_1/\rho_2)(it)^2(1+\rho_2^{-1}+\rho_2^{-2})+\frac{1}{2}(p+1)\rho_1(1+\rho_2)/\rho_2^2\right\} \\
&+O(n^{-3/2}).
\end{aligned}
$$

The second factor can be expanded by Theorem 2.2, which implies

$$(3.12)\quad \mathrm{P}\,(-\sqrt{n}\log|S_2(S_1+S_2)^{-1}|-\sqrt{n}\log|\rho_2\tilde{\Gamma}|<x\tau_2)$$

$$=\Phi(x)+n^{-1/2}\{g_1\Phi^{(1)}(x)/\tau_2+g_3\Phi^{(3)}(x)/\tau_2^3\}+n^{-1}\sum_{k=1}^{3}h_{2k}\Phi^{(2k)}(x)/\tau_2^{2k}$$
$$+O(n^{-3/2})\,,$$

where $\tau_2^2=2(2\sigma_1+(\rho_2/\rho_1)\sigma_2+p\rho_1/\rho_2)$ for $\sigma_j=\mathrm{tr}\,(I-\tilde{\Gamma}^{-1})^j$ with $\tilde{\Gamma}=\rho_1\Gamma+\rho_2 I$ and

$$g_1=-\frac{1}{2}(\rho_1/\rho_2)p(p+1)+\frac{1}{2}(\rho_2/\rho_1)(\sigma_1^2+\sigma_2)\,,$$

$$g_3=-\frac{2}{3}(\rho_1/\rho_2)p(2+\rho_1/\rho_2)-4\sigma_1+4(1-\rho_2/\rho_1)\sigma_2$$
$$+\frac{4}{3}(\rho_2/\rho_1)(4-\rho_2/\rho_1)\sigma_3+2(\rho_2/\rho_1)^2\sigma_4\,,$$

$$h_2=\frac{1}{2}p(p+1)(\rho_1/\rho_2)(2+\rho_1/\rho_2)+(1-3\rho_2/\rho_1)\sigma_2+2(\rho_2/\rho_1)(2-\rho_2/\rho_1)\sigma_3$$
$$+\frac{5}{2}(\rho_2/\rho_1)^2\sigma_4+(1-3\rho_2/\rho_1)\sigma_1^2+2(\rho_2/\rho_1)(2-\rho_2/\rho_1)\sigma_1\sigma_2+2(\rho_2/\rho_1)^2\sigma_1\sigma_3$$
$$+\frac{1}{2}(\rho_2/\rho_1)^2\sigma_2^2+\frac{1}{2}g_1^2\,,$$

$$h_4=\frac{2}{3}p(\rho_1/\rho_2)(1+\rho_2^{-1}+\rho_2^{-2})+8\sigma_1-4(5-3\rho_2/\rho_1)\sigma_2$$
$$+\frac{8}{3}\{5-15\rho_2/\rho_1+3(\rho_2/\rho_1)^2\}\sigma_3+(\rho_2/\rho_1)\{30-30\rho_2/\rho_1+2(\rho_2/\rho_1)^2\}\sigma_4$$
$$+8(\rho_2/\rho_1)^2(3-\rho_2/\rho_1)\sigma_5+\frac{20}{3}(\rho_2/\rho_1)^3\sigma_6+g_1g_3\,,$$

$$h_6=\frac{1}{2}g_3^2\,.$$

Though this result is the same as the Theorem 11.1 in Nagao [4], our expression is somewhat simpler.

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REFERENCES

[1] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis, *Ann. Math. Statist.*, **34**, 1270-1285.

[2] Muirhead, R. J. (1970). Systems of partial differential equations for hypergeometric functions of matrix argument, *Ann. Math. Statist.*, **41**, 991-1001.

[3] Muirhead, R. J. (1970). Asymptotic distributions of some multivariate tests, *Ann. Math. Statist.*, **41**, 1002-1010.

[4] Nagao, H. (1970). Asymptotic expansions of some test criteria for homogeneity of variances and covariance matrices from normal populations, *J. Sci. Hiroshima Univ.*, Ser. A-I, **34**, 153-247.

[5] Sugiura, N. (1969). Asymptotic non-null distributions of the likelihood ratio criteria for covariance matrix under local alternatives, *Univ. of North Carolina, Mimeo Ser.*, No. 609.

[6] Sugiura, N. (1969). Asymptotic expansions of the distributions of $|S_1S_2^{-1}|$ and $|S_2(S_1+S_2)^{-1}|$ for testing the equality of two covariance matrices, *Univ. of North Carolina, Mimeo Ser.*, No. 653.

[7] Sugiura, N. (1972). Asymptotic solutions of the hypergeometric function $_1F_1$ of matrix argument, useful in multivariate analysis, *Ann. Inst. Statist. Math.*, **24**, 517-524.