ASYMPTOTIC FORMULAS FOR THE HYPERGEOMETRIC FUNCTION $\,_2F_1$
OF MATRIX ARGUMENT, USEFUL IN MULTIVARIATE ANALYSIS

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Summary

Let $S_i$ have the Wishart distribution $W_p(S_i, n_i)$ for $i=1, 2$. An asymptotic expansion of the distribution of $-2 \log \lambda = -2 \log \left[ \prod_{s=1}^{2} |S_s/n_s|^\nu/2 \cdot |(S_1 + S_2)/n|^{-\nu/2} \right]$ for large $n=n_1+n_2$ is derived, when $\Sigma_1 \Sigma^{-1}_2 = I + n^{-1/2} \theta$, based on an asymptotic solution of the system of partial differential equations for the hypergeometric function $\,_2F_1$, obtained recently by Muirhead [2]. Another asymptotic formula is also applied to the distributions of $-2 \log \lambda$ and $-\log |S_S(S_1+S_2)^{-1}|$ under fixed $\Sigma_1 \Sigma^{-1}_2$, which gives the earlier results by Nagao [4]. Some useful asymptotic formulas for $\,_2F_1$ were investigated by Sugiuira [7].

1. Preliminaries

Based on a random sample of size $n_i+1$ from $p$-variate normal population $N(\mu_i, \Sigma_i)$ for $i=1$ and 2, the modified likelihood ratio statistic for testing $\Sigma_1 = \Sigma_2$ against $\Sigma_1 \neq \Sigma_2$ can be expressed as

$$\lambda = |S_i/n_i|^{\nu/2} |S_j/n_j|^{\nu/2} |(S_1 + S_2)/n|^{-\nu/2},$$

where $S_i$ has the Wishart distribution $W_p(I, n_i)$ and $S_2$ has $W_p(I, n_2)$ independently with $n=n_1+n_2$. The diagonal matrix $\Gamma$ has the characteristic roots of $\Sigma_1 \Sigma^{-1}_2$ on the diagonal. The moments of $\lambda$ was given by Sugiuira [5] as

$$E[\lambda^k] = \left( \frac{n^n}{n_1^n n_2^n} \right)^{\nu/2} \frac{\Gamma_p(n/2)}{\Gamma_p(n_1/2) \Gamma_p(n_2/2)} \frac{\Gamma_p(n_1(1+h)/2) \Gamma_p(n_2(1+h)/2)}{\Gamma_p(n(1+h)/2)}$$

$$\cdot |\Gamma|^{\nu/2} \, _2F_1(n h/2, n_1(1+h)/2; n(1+h)/2; I-\Gamma),$$

which can be derived by starting the joint distribution of $(S_1, S_2)$ and using the formula
et\left[\frac{1}{2} (I-I^{-1})S_1\right]=\sum_{k=0}^\infty \Sigma_{\omega_i} C_i \left(\frac{1}{2} (I-I^{-1})S_1\right)/k!,

together with (12) and (22) in Constantine [1]. Kummer transformation formula
\( _2F_1(a_1, a_2; b; Z) = (I-Z)^{-a_1} _2F_1(b-a_1, a_2; b; -(I-Z)^{-1}) \) is also used in the final expression.

Put \( m_\alpha = \rho m_\alpha \) for \( \alpha = 1, 2 \) and the correction factor \( \rho = 1-(2p^3+3p-1)(n_1^{-1}+n_2^{-1}-n^{-1})/(6(p+1)) \) with \( m = m_1 + m_2 \). Asymptotic expansion of the distribution of \( -2\rho \log \lambda \) for large \( m \) and fixed \( \rho_\alpha = m_\alpha/m \> 0 \), when \( \Gamma = I + m^{-1}\theta \), was obtained by Sugiura [5] in terms of \( \chi^2 \)-distributions and when \( \Gamma \) is fixed, by Nagao [4] in terms of normal distribution function and its derivatives, in a more general case, namely for \( k \)-sample problem. When \( \Gamma = I + m^{-1/2}\theta \), however, both methods are not available and the present approach by system of partial differential equations for \( _2F_1 \) by Muirhead [3] is essentially useful.

Another asymptotic formula for \( _2F_1 \), based on the differential equations gives the asymptotic expansions of the distributions of \( -2\rho \log \lambda \) and \( -\log |S_\theta(S_1 + S_\theta)^{-1}| \) under fixed \( \Gamma \) obtained in Nagao [4] by different technique. The characteristic function of \( -\sqrt{n} \log |S_\theta(S_1 + S_\theta)^{-1}| \) can be written as

\[
(1.3) \quad \Gamma'\left(\frac{1}{2} n_2 - \sqrt{n} it\right)\Gamma'\left(\frac{1}{2} n_1\right)/\left[\Gamma'\left(\frac{1}{2} n_2\right)\Gamma'\left(\frac{1}{2} n - \sqrt{n} it\right)\right] \\
\cdot \quad _2F_1\left(-\sqrt{n} it, \frac{1}{2} n_1; \frac{1}{2} n - \sqrt{n} it; I - \Gamma^r\right),
\]

by the similar argument as for (1.2), which was remarked by Sugiura [6]. This statistic can be used to test \( \Sigma_1 = \Sigma_2 \) against \( \Sigma_1 \Sigma_1^{-1} \geq I \) for two normal populations.

2. General asymptotic formulas for \( _2F_1 \)

Muirhead [2] has proved that the hypergeometric function \( _2F_1(a, b; c; R) \) for \( R = \text{diag} (r_1, \ldots, r_p) \) can be characterized as the unique solution of the partial differential equation

\[
(2.1) \quad r_1 (1 - r_1) \frac{\partial F}{\partial r_1} + \left[ c - \frac{1}{2} (p-1) - \left( a + b + 1 - \frac{1}{2} (p-1) \right) r_1 \right] \\
\cdot \quad + \frac{1}{2} \sum_{i=2}^p r_i (1 - r_i) \frac{\partial F}{\partial r_i} - \frac{1}{2} \sum_{i=1}^p \frac{r_i (1 - r_i)}{r_i - r_j} \frac{\partial F}{\partial r_j} = \alpha F,
\]

subject to the condition that \( F \) is symmetric with respect to \( r_i \) and \( F \) is analytic at \( R = 0 \) with \( F(0) = 1 \).

Generalizing the characteristic function of \( -2\rho \log \lambda \) obtained by putting \( h = -2\rho it \) in (1.2) under \( \Gamma = I + m^{-1/2}\theta \), we shall consider \( G(R) = \)
log$_n^2 F_1(\alpha n; \beta, n + \beta_0; r, n + \gamma_0; n^{-1/2} R)$ for $\gamma_1 \neq 0$, which is the unique solution for

\[(2.2) \quad r_1(1 - r_1/\sqrt{n})\left\{\frac{\partial G}{\partial r_1} + \left(\frac{\partial G}{\partial r_1}\right)^2\right\} + \left[\gamma_1 n - (\alpha + \beta_1)\sqrt{n} r_1 - \frac{1}{2} (p - 1) + \gamma_0 \right.
\]
\[-\left(\beta_0 + 1 - \frac{1}{2} (p - 1)\right)r_1/\sqrt{n} + \frac{1}{2}\sum_{j=2}^p \frac{r_j}{r_1 - r_j}\]
\[-\frac{1}{2}\sum_{j=2}^p \frac{r_j^2}{r_1 - r_j} \sqrt{n} \frac{\partial G}{\partial r_1} - \frac{1}{2}\sum_{j=2}^p \frac{r_j}{r_1 - r_j} \frac{\partial G}{\partial r_j}\]
\[+ \frac{1}{2}\sum_{j=2}^p \frac{r_j^3}{r_1 - r_j} \frac{\partial G}{\partial r_j} \frac{1}{\sqrt{n}} = a\beta_1 n \sqrt{n} + a\beta_0 \sqrt{n},\]

subject to $G(0) = 0$ and $G$ is symmetric for $R$. Putting $G(R) = \sum_{k=1}^\infty Q_k(R)n^{-k}$ with $Q_k(0) = 0$ as in Sugiura [7], we can successively determine $Q_k(R)$ from (2.2). Equating the term of order $n\sqrt{n}$ in (2.2) yields $\gamma_1 \partial Q_{-1}/\partial r_1 = a\beta_1$, which implies $Q_{-1} = (a\beta_1/\gamma_1) \sum_{j=1}^p r_j$. We shall write

$s_j = \text{tr } R^j$ and put $\xi_1 = a + \beta_1 - a\beta_1/\gamma_1$, $\xi_2 = a + \beta_1 - 2a\beta_1/\gamma_1$, for abbreviation. Then equating the term of order $n, \sqrt{n}$ and 1 in (2.2) yields, after some computation:

$Q_0 = \frac{1}{2} (a\beta_1/\gamma_1^2)\xi_1 \sigma_1,$

\[(2.3) \quad Q_1 = \frac{1}{3} \sigma_1 (a\beta_1/\gamma_1^2)(a\beta_1 + \xi_1 \xi_2) + (a/\gamma_1)(\beta_0 - \beta_0 \sqrt{n}/\gamma_1)\sigma_1,
\]

$Q_2 = \frac{1}{4} (a\beta_1/\gamma_1)\sigma_1 \{a\beta_1 \xi_1 (2 - \xi_1/\gamma_1) + \xi_2 (a\beta_1 + \xi_1 \xi_2)\}$

$+ \frac{1}{2} (a/\gamma_1)\sigma_1 \left[ - (\beta_0/\gamma_1) \left(\gamma_0 + \frac{1}{2}\right) \xi_1 + (\beta_0 - \gamma_0 \beta_1/\gamma_1) \xi_2 + \beta_1 (\gamma_0 + \frac{1}{2}) \right]$

$+ \frac{1}{4} (a\beta_1/\gamma_1)\sigma_1 (1 - \xi_1/\gamma_1).$

Hence we have:

**Theorem 2.1.** An asymptotic formula of $_2 F_1(\alpha n, \beta, n + \beta_0; r, n + \gamma_0; n^{-1/2} R)$ for large value of $n$, when $\gamma_1 \neq 0$ is given by

\[(2.4) \quad \exp \left[ \sqrt{n} (a\beta_1/\gamma_1)\sigma_1 + \frac{1}{2} (a\beta_1/\gamma_1)\xi_1 \sigma_1 + n^{-1/2} Q_1 + n^{-1} Q_2 + O(n^{-2}) \right],
\]

where $s_j = \text{tr } R^j$, $\xi_1 = a + \beta_1 - a\beta_1/\gamma_1$ and $Q_1, Q_2$ are given by (2.3) with $\xi_2 = \xi_1 - a\beta_1/\gamma_1$.

Generalizing the characteristic function (1.3), an asymptotic formula
for \( \text{}_pF_1(\alpha\sqrt{\gamma}, \beta, \gamma + \sqrt{n} + \gamma_1\sqrt{n} + \gamma_2; \gamma_3\sqrt{n} + \gamma_1\sqrt{n} + \gamma_2; R) \), when \( \gamma_2 \neq 0 \) should be investigated. As in Muirhead [3], transforming the variables \( R = I - (I - (\beta_2/\gamma_1)R)^{-1} \) and putting \( F = [I - W]^{-\alpha/2}G \), we can rewrite the partial differential equation (2.1) for the function \( G(W) \). Further the transformation \( H = \log G \) is useful to simplify the right-hand side of the equation. Denoting \( a = \alpha\sqrt{\gamma} \), \( b = \beta_2\sqrt{\gamma} + \beta_1\sqrt{n} + \beta_0 \) and \( c = \gamma_1\sqrt{n} + \gamma_1\sqrt{\gamma} + \gamma_2 \), for abbreviation, the differential equation for \( H(W) \) is expressed by

\[
(2.5) \quad (-\beta_2/\gamma_1)[1 - (1 - \gamma_2/\beta_2)w_1]\left[a(a + 1) - 2(a + 1)(1 - w_1) \frac{\partial H}{\partial w_1} \right. \\
+ (1 - w_1) \left( \frac{\partial H}{\partial w_1} + \left( \frac{\partial H}{\partial w_1} \right)^2 \right) \left] - (\beta_2/\gamma_1) \left[ c - (p - 1)/2 \right] (1 - w_1) \right. \\
- \left. \left[ a + b + 1 - (p - 1)/2 \right] (\gamma_2/\beta_2)w_1 + \frac{1}{2} \sum_{j=1}^{\gamma_2} \frac{w_j(1 - w_j)}{w_1 - w_j} \right. \\
\cdot \left[ 1 - (1 - \gamma_2/\beta_2)w_1 \right] \left[ -a + (1 - w_1) \frac{\partial H}{\partial w_1} + \frac{\beta_2}{2 \gamma_2} \right. \\
\left. \cdot \sum_{j=1}^{\gamma_2} \frac{(1 - w_j)(1 - w_j)w_j}{w_1 - w_j} \right] \left( 1 - \frac{\gamma_2}{\beta_2} \right) w_j \left( -a + (1 - w_j) \frac{\partial H}{\partial w_j} \right) \\
= ab.

Putting \( H(W) = \sum_{k=0}^{\infty} Q_k(w)n^{-k/2} \) with \( Q_k(0) = 0 \) for \( k = 0, 1, \ldots \), and equating the term of order \( n^{-1/2} \) with \( \sqrt{n} \), and 1 in (2.5), we can get:

**Theorem 2.2.** An asymptotic formula for \( \text{}_pF_1(\alpha\sqrt{\gamma}; \beta_2\sqrt{\gamma} + \beta_1\sqrt{n} + \beta_2; \gamma_2\sqrt{n} + \gamma_1\sqrt{n} + \gamma_2; R) \) for large value of \( n \), when \( \gamma_2 \neq 0 \) is given by

\[
(2.6) \quad |I - W|^{-\alpha/2} \exp \left[ -\alpha \delta_1 \sigma_1 + \frac{1}{2} \alpha^2 \delta_2 \sigma_1 + n^{-1/2} Q_1 + n^{-1} Q_2 + O(n^{-3/2}) \right],
\]

where \( \sigma_j = \text{tr } W^j \), \( W = I - (I - (\beta_2/\gamma_1)R)^{-1} \) and \( \delta_1 = \beta_1/\beta_2 - \gamma_1/\gamma_2 \), \( \delta_2 = \beta_2^{-1} - \gamma_2^{-1} \). The coefficients \( Q_1 \) and \( Q_2 \) are given by, using \( \delta_1 = \beta_1/\beta_2 - \gamma_1/\gamma_2 \),

\[
Q_1 = \alpha \sigma_1 (-\delta_2 + \gamma_1/\gamma_2) + \frac{1}{2} \alpha \sigma_2 \left[ \delta_1 \left( \frac{1}{2} - \alpha \gamma_1/\gamma_2 \right) + \delta_2 (\sigma_1 + \alpha \delta_2 - \sigma_1/\gamma_2) \right] \\
+ \frac{1}{3} \alpha^2 \sigma_3 \left[ -3 \delta_1 + \alpha (\gamma_2^{-1} - \delta_2) \right] + \frac{1}{2} \alpha \sigma_4 \delta_2^2 + \frac{1}{4} \alpha \delta_2 \sigma_1^2,
\]

\[
(2.7) \quad Q_2 = \left( \alpha/\gamma_2 \right) \sigma_1 [\gamma_2 \delta_1 + \gamma_2 (\delta_2 - \gamma_1 \delta_1/\gamma_2)] + \frac{1}{2} \alpha \sigma_2 \left[ -(\alpha/\gamma_2) [\delta_1^2 + (1 + \gamma_2) \delta_2] \right] \\
+ \frac{1}{2} \delta_1 (2 \delta_2 + \delta_2 - \gamma_2^{-1}) - \frac{1}{2} (\gamma_1/\gamma_2) \delta_2 (1 - \alpha \gamma_1/\gamma_2 - 2 \alpha \gamma_1 \gamma_2^{-1}) \\
+ (\alpha \delta_2 + \delta_1 - \alpha/\gamma_2) (\delta_2 - 2 \delta_1 \gamma_1/\gamma_2) + \frac{1}{3} \sigma_1 \left[ \alpha \delta_3 \delta_2 \left( -\frac{3}{2} + 6 \alpha \gamma_1 \gamma_2^{-1} \right) \right],
\]
\[ +2\alpha^2\gamma_{1}^{-1}(\gamma_{1}^{-1}-\delta_{1})\left(\sigma_{1}^{2}+\frac{1}{2}\sigma_{2}\right) \alpha^{2}-3\alpha^2\delta_{1}\delta_{2}-\alpha^2\delta_{1}(\delta_{2}-\gamma_{1}^{-1}) \]
\[ \cdot \left(1-2\alpha_{2}\gamma_{1}^{-1}\right)-\alpha\delta_{1}(\delta_{1}+\alpha\delta_{2}-\alpha/\gamma_{1})^{1} \right] \]
\[ +\frac{1}{4}\sigma_{1}\alpha^2\delta_{1}\left[\sigma_{1}^{2}+\frac{1}{2}\sigma_{1}\left(\frac{5}{2}-\alpha^2\gamma_{1}^{-1}-4\alpha_{2}\gamma_{1}^{-1}\right)\right]-2\alpha\delta_{1}(\gamma_{1}^{-1}-\delta_{1}) \]
\[ +\left(\Delta_{1}+\delta_{1}+\alpha\gamma_{1}^{-1}\right)(5\delta_{1}+\alpha\delta_{1}-\alpha\gamma_{1}^{-1}) \right] +\sigma_{1}\alpha^2\delta_{1}\left[-2\delta_{1}+\alpha\gamma_{1}^{-1}-\alpha\delta_{1} \right] \]
\[ +\frac{5}{6}\alpha^2\sigma_{1}\delta_{1}^{2}\sigma_{1}^{2}+\frac{1}{4}\alpha^2\delta_{1}^{2}\sigma_{1}^{2}\delta_{1}(\gamma_{1}^{-1}-\delta_{1}) \right] +\frac{1}{2}\alpha^2\delta_{1}^{2}\sigma_{1}\delta_{1}^{2} \]
\[ -\frac{1}{2}\alpha\delta_{1}\sigma_{1}\delta_{1}(\delta_{1}+\alpha\delta_{2}-\alpha\gamma_{1}^{-1}) +\frac{1}{2}\alpha^2\delta_{1}^{2}\sigma_{1}^{2}\delta_{1} +\frac{1}{8}\alpha^2\delta_{1}^{2}\sigma_{1}^{2}. \]

3. Asymptotic expansions

The characteristic function of the likelihood ratio statistic \(-2\rho \log \lambda\) under \(\Gamma=I+m^{-1/2}\theta\) is expressed by (1.2) as

\[
(3.1) \quad \left\{ \frac{m^n}{m_1^n}\right\}^{1-n}_{m_i^n/m_z^n} \Gamma_p(m_i/2+\Delta_i) \Gamma_p(m_1(1-2it)/2+\Delta_1) \frac{\Gamma_p(m_i(1-2it)/2+\Delta_i)}{\Gamma_p(m_i/2+\Delta_i) \Gamma_p(m_1/2+\Delta_1) \Gamma_p(m_1(1-2it)/2+\Delta_1)} \cdot |I+m^{-1/2}\theta|^{-n^H} \cdot _2F_1(-mit, m_1(1-2it)/2+\Delta_1; m(1-2it)/2+\Delta_1; -m^{-1/2}\theta),
\]

where \(\Delta_i=\rho_i\Delta=(n_i-m_i)/2\) and \(\rho_i=n_i/n\). By Stirling’s formula, the first factor in (3.1), namely, the product of gamma functions can be evaluated as

\[
(3.2) \quad \text{The first factor} = (1-2it)^{-i/2} + O(m^{-2})
\]

for \(f=p+1/2\). The second factor can easily be written by

\[
(3.3) \quad \log |I+m^{-1/2}\theta|^{-n^H} = \rho_i it \left( -\sqrt{m} \sigma_1 + \frac{1}{2} \sigma_2 - \frac{1}{3} m^{-1/2} \sigma_3 + \frac{1}{4} m^{-1/2} \sigma_4 \right) + O(m^{-5/2}),
\]

where \(\sigma_j=\text{tr} \theta^j\). Finally putting \(n=m\), \(\alpha=-it\), \(\beta_1=\rho_1(1-2it)/2\), \(\beta_2=\Delta_1\), \(\gamma_1=(1-2it)/2\), \(\gamma_2=\Delta_1\) and \(R=\theta\) in Theorem 2.1, we can evaluate the third factor in (3.1) by \(T\),

\[
(3.4) \quad \log _2F_1 = \sqrt{m} \rho_i it \sigma_1 + \frac{1}{4} \rho_1 \sigma_1 (-2it - \rho_1 + \rho_2 T^{-1}) - \frac{1}{6} \rho_1 \sigma_1 m^{-1/2} \]
\[ \cdot \left[ -2it - \rho_1 - \rho_2 + 3\rho_2 T^{-1} + \rho_5(1-2\rho_5) T^{-3} \right] \]
\[ + m^{-i} \left[ \frac{1}{8} \rho_1 \sigma_1 (-2it - \rho_1 - \rho_2 + 3\rho_2 T^{-1} + 6\rho_2 T^{-1} + 2\rho_2(3-5\rho_2) T^{-1} \right)
\]
\[ + \rho_2(1 - 5\rho_2 + 5\rho_3)T^{-3} + \frac{1}{4}\rho_1\rho_2\sigma_i^3\{ -2A + (1 + 4A)T^{-1} \\
- (1 + 2A)T^{-1}\} + \frac{1}{2}\rho_1\rho_2\sigma_i^2(T^{-1} - T^{-1}) + O(m^{-3/2}) \].

Combined with (3.2), (3.3) and (3.4), we can get the asymptotic expansion of the distribution of \(-2\rho \log \lambda\) in terms of noncentral \(\chi^2\)-distributions with noncentrality \(\delta = \rho_1\rho_2\sigma_i/4\) as

\[(3.5) \quad P(-2\rho \log \lambda < x) = P_f + \frac{1}{6}\rho_1\rho_2\sigma_i^3m^{-1/2}\{(1 + \rho_2)P_f - 3\rho_2P_{f+1} \\\n- (1 - 2\rho_2)P_{f+1}\} + m^{-1}\sum_{k=0} h_k P_{f+2k} + O(m^{-3/2}) \, ,
\]

where \(P_f = P(\chi_f^2(\delta) < x)\) for the noncentral \(\chi^2\)-variate and

\[h_0 = \frac{1}{8}\rho_1\rho_2\{ -\sigma_i(1 + \rho_2 + \rho_3) - 4A\sigma_i + \frac{1}{9}\rho_1\rho_2\sigma_i^3(1 + \rho_2)^2 \} \, ,
\]

\[h_1 = \frac{1}{8}\rho_1\rho_2\{ 6\rho_2^2\sigma_i + 2(1 + 4A)\sigma_i + 2\rho_i^2 - \frac{2}{3}\rho_1\rho_2^2(1 + \rho_2)\sigma_i^2 \} \, ,
\]

\[(3.6) \quad h_2 = \frac{1}{8}\rho_1\rho_2\{ 2\rho_2(3 - 5\rho_2)\sigma_i - 2(2A + 1)\sigma_i - 2\sigma_i^2 + \frac{1}{9}\rho_1\rho_2\sigma_i^3(13\rho_2^2 + 2\rho_2 - 2) \} \, ,
\]

\[h_3 = \frac{1}{8}\rho_1\rho_2\{ (1 - 5\rho_2 + 5\rho_2^2)\sigma_i + \frac{2}{3}\rho_1\rho_2^2(1 - 2\rho_2)\sigma_i \} \, ,
\]

\[h_4 = \frac{1}{72}\rho_1^2\rho_2^2\sigma_i^2(1 - 2\rho_2)^2 \, .
\]

Next we shall consider the distribution of \(-2\rho m^{-1/2} \log \lambda\) under fixed \(\Gamma\), the characteristic function of which can be expressed from (1.2) as

\[(3.7) \quad \left( \frac{m^m}{m^{n\cdot m^{n^2}}} \right)^{-itp/\sqrt{m}} \frac{\Gamma_p(m/2 + \Delta)\Gamma_p(m/2 - \rho_1\rho_2it\sqrt{m} + \Delta)}{\Gamma_p(m/2 + \Delta)\Gamma_p(m/2 + \Delta)\Gamma_p(m/2 - \sqrt{m}\rho_1it + \Delta)} \ \cdot \ |\Gamma|^{-\rho_1\rho_2it\sqrt{m}}F(-\sqrt{m}it, m/2 - \rho_1\rho_2it\sqrt{m} + \Delta; \ m/2 - \sqrt{m}\rho_1it + \Delta; I - \Gamma) \, .
\]

By Stirling's formula, the first factor in (3.7) can be evaluated as

\[(3.8) \quad \exp\left[\frac{1}{2}p(p + 1)itm^{-1/2} + \frac{1}{2}p(p + 1)(it)^{m^{-1}} + O(m^{-3/2})\right] \, .
\]

Putting \(n = m\), \(\alpha = -it\), \(\beta_2 = \rho_1/2\), \(\beta_1 = -\rho_1it\), \(\beta_0 = \Delta\), \(\gamma_1 = 1/2\), \(\gamma_1 = -it\), \(\gamma_0 = \Delta\) and \(R = I - \Gamma\) in Theorem 2.2, we can get an asymptotic formula for
the second factor in (3.7) with respect to \( W = I - \bar{\Gamma}^{-1} \) for \( \bar{\Gamma} = \rho_1 \Gamma + \rho_2 I \), which gives the following asymptotic expansion under fixed \( \Gamma \) in terms of the standard normal distribution function \( \Phi(x) \) and the derivatives:

\[
(3.9) \quad P \left( -2 \rho m^{-1/2} \log \frac{\lambda + \sqrt{m}}{\sqrt{m}} \log \frac{|\Gamma^j|}{|\bar{\Gamma}|} < \tau_j x \right) \\
= \Phi(x) + m^{-1/2} \left[ g_1 \Phi^{(1)}(x)/\tau_1 + g_2 \Phi^{(2)}(x)/\tau_1^2 \right] + m^{-1} \sum_{k=1}^{3} h_{2k} \Phi^{(2k)}(x)/\tau_1^{2k} \\
+ O(m^{-3/2}) ,
\]

where \( \tau_j^2 = 2(\rho_j/\rho_1)\sigma_j \) for \( \sigma_j = \text{tr} \ W^j = \text{tr} (I - \bar{\Gamma}^{-1})^j \) and

\[
g_1 = \frac{1}{2} (\rho_2/\rho_1) (\sigma_2 + \sigma_1) - \frac{1}{2} p(p+1) , \\
g_2 = 2(\rho_2/\rho_1) \left\{ -\sigma_2 + \frac{2}{3} (1 - \rho_2/\rho_1)\sigma_3 + (\rho_2/\rho_1)\sigma_1 \right\} ,
\]

\[
(3.10) \quad h_2 = \frac{1}{2} p(p+1) - 2(\rho_2/\rho_1)(1+D)\sigma_1 + 2(\rho_2/\rho_1)(1 - \rho_2/\rho_1)\sigma_2 + \frac{5}{2} (\rho_2/\rho_1)^2 \sigma_1 \\
- 2(\rho_2/\rho_1)^3 \sigma_1^2 + 2(\rho_2/\rho_1)(1 - \rho_2/\rho_1)\sigma_1 \sigma_2 + 2(\rho_2/\rho_1)^2 \sigma_1 \sigma_2 \\
+ \frac{1}{2} (\rho_2/\rho_1)^3 \sigma_1^3 + \frac{1}{2} g_1 ,
\]

\[
h_4 = 4(\rho_2/\rho_1)\sigma_2 - \frac{16}{3} (\rho_2/\rho_1)(1 - \rho_2/\rho_1)\sigma_3 + 2(\rho_2/\rho_1)(1 - 7\rho_2/\rho_1 + (\rho_2/\rho_1)^3) \sigma_4 \\
+ 8(\rho_2/\rho_1)^3 (1 - \rho_2/\rho_1)\sigma_5 + \frac{20}{3} (\rho_2/\rho_1)^3 \sigma_6 + g_3 ,
\]

\[
h_6 = \frac{1}{2} g_3 .
\]

After some computation, we can verify that the above result agrees with the Theorem 8.1 in Nagao [4] for \( k=2 \).

Finally for the characteristic function of \( -\sqrt{n} \log |S_2(S_1+S_2)^{-1}| \) given in (1.3), the first factor can be reduced to

\[
(3.11) \quad -\sqrt{n} itp \log \rho_1 - pt^3 \rho_1/\rho_2 + 2ptitn^{-1/2} \\
\cdot \left\{ \frac{1}{3} \rho_1 [(1+\rho_1/\rho_2)(it)^3 + \frac{1}{4} (p+1)\rho_1/\rho_2] + p(it)^3 n^{-1} \right\} \\
\cdot \left\{ \frac{2}{3} (\rho_1/\rho_2)(it)^3 (1+\rho_1^{-1}+\rho_2^{-1}) + \frac{1}{2} (p+1)\rho_1(1+\rho_1/\rho_2)^3 \right\} \\
+ O(n^{-5/2}) .
\]

The second factor can be expanded by Theorem 2.2, which implies

\[
(3.12) \quad P \left( -\sqrt{n} \log |S_2(S_1+S_2)^{-1}| - \sqrt{n} \log |\rho_2 \bar{\Gamma}| < z \tau_2 \right) 
\]
\[= \Phi(x) + n^{-1/2} \{ g_1 \Phi^{(1)}(x)/\tau_2 + g_2 \Phi^{(3)}(x)/\tau_2^3 \} + n^{-1} \sum_{k=1}^{3} h_{2k} \Phi^{(2k)}(x)/\tau_2^{2k} + O(n^{-1/2}),\]

where \( \tau_2 = 2(2\sigma_1 + (\rho_1/\rho_2)\sigma_2 + p\rho_1/\rho_2) \) for \( \sigma_j = \text{tr} (I - \tilde{I}^{-1})' \) with \( \tilde{I} = \rho_1 I + \rho_2 I \) and
\[
g_1 = -\frac{1}{2} \left( \frac{\rho_1}{\rho_2} \right) p(p+1) + \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right) (\sigma_1^2 + \sigma_2),
\]
\[
g_2 = -\frac{2}{3} \left( \frac{\rho_1}{\rho_2} \right) p(2 + \rho_1/\rho_2) - 4\sigma_1 + 4(1 - \rho_1/\rho_2)\sigma_2
\]
\[+ \frac{4}{3} \left( \frac{\rho_2}{\rho_1} \right) (4 - \rho_2/\rho_1)\sigma_3 + 2(\rho_2/\rho_1)^2 \sigma_4,
\]
\[
h_2 = \frac{1}{2} \left( \frac{p(p+1)}{\rho_1/\rho_2} \right) (2 + \rho_1/\rho_2) + (1 - 3\rho_2/\rho_1)\sigma_2 + 2(\rho_2/\rho_1)(2 - \rho_2/\rho_1)\sigma_3
\]
\[+ \frac{5}{2} \left( \frac{\rho_2}{\rho_1} \right)^3 \sigma_4 + (1 - 3\rho_2/\rho_1)\sigma_1^2 + 2(\rho_2/\rho_1)(2 - \rho_2/\rho_1)\sigma_1\sigma_2 + 2(\rho_2/\rho_1)^2 \sigma_1 \sigma_3
\]
\[+ \frac{1}{2} \left( \frac{\rho_2}{\rho_1} \right)^2 \sigma_1^2 + \frac{1}{2} g_1^2,
\]
\[
h_4 = \frac{2}{3} \left( \frac{p}{\rho_1/\rho_2} \right) (1 + \rho_2/\rho_1) + 8\sigma_1 - 4(5 - 3\rho_2/\rho_1)\sigma_2
\]
\[+ \frac{8}{3} \{ 5 - 15\rho_2/\rho_1 + 3(\rho_2/\rho_1)^2 \} \sigma_3 + (\rho_2/\rho_1) \{ 30 - 30\rho_2/\rho_1 + 2(\rho_2/\rho_1)^2 \} \sigma_4
\]
\[+ 8(\rho_2/\rho_1)^2 (3 - \rho_2/\rho_1)\sigma_5 + \frac{20}{3} (\rho_2/\rho_1)^3 \sigma_6 + g_4 g_5,
\]
\[
h_4 = \frac{1}{2} g_2^2.
\]

Though this result is the same as the Theorem 11.1 in Nagao [4], our expression is somewhat simpler.

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[6] Sugiura, N. (1969). Asymptotic expansions of the distributions of $|S_1S_2^{-1}|$ and $|S_4(S_1+S_2)^{-1}|$ for testing the equality of two covariance matrices, *Univ. of North Carolina, Mimeo Ser.*, No. 653.