A χ^2 GOODNESS-OF-FIT TEST FOR MARKOV RENEWAL PROCESSES

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A Markov Renewal Process (M.R.P.) is a process similar to a Markov chain, except that the time required to move from one state to another is not fixed, but is a random variable whose distribution may depend on the two states between which the transition is made. For an M.R.P. of m ($<\infty$) states we derive a goodness-of-fit test for a hypothetical matrix of transition probabilities. This test is similar to the test Bartlett has derived for Markov chains. We calculate the first two moments of the test statistic and modify it to fit the moments of a standard χ^2. Finally, we illustrate the above procedure numerically for a particular case of a two-state M.R.P.

1. Introduction

A Markov Renewal Process (M.R.P.) with m ($<\infty$) states is one which records at each time t, the number of times a system visits each of the m states in the time interval $(0, t)$. The system moves from state to state according to a Markov chain with transition probability matrix $P_0=[p_{ij}]$, but, unlike a Markov chain, the time between transitions is not constant. Rather, the holding time in each state is a random variable whose distribution function (d.f.) may depend on the two states between which the move is made. The observations of an M.R.P. are also different from those of a Markov chain in that they consist of both the observed states of the system $J_0, J_1, J_2, \cdots$ and the successive holding times X_0 $(=0)$, $X_1, X_2, \cdots$ in each state. The M.R.P. makes a transition from J_{k-1} to J_k after remaining in J_{k-1} for time X_k. Also, one should note that an M.R.P. with $m=1$ is an ordinary Renewal process.

In this paper we derive a large sample, goodness-of-fit test for a

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hypothetical matrix P_0 of transition probabilities for an M.R.P. An M.R.P. has been used by Pyke [17] for some problems in counter theory. Çinlar [4] has discussed the use of M.R.P.'s in queueing theory. Another very interesting application of M.R.P.'s is one given by Perrin and Sheps [14], in which they have considered human reproduction a stochastic process and have analyzed the process as an M.R.P. Finally, M.R.P.'s have been used by many other authors as models for problems in inventory, reliability, maintenance, and so on. The test derived in this paper is expected to be useful in such problems.

Two papers by Pyke ([15] and [16]) have been the catalyst for many papers on M.R.P.'s which have appeared since then. However, most of the work on M.R.P.'s up to this time has dealt with only the underlying distribution theory and limit theorems. With the exception of one paper by Moore and Pyke [12] on estimating transition distributions, very little has been done in the area of statistical inference for M.R.P.'s. This is contrasted by the fact that many papers have been written on statistical analysis of Markov chains—Bartlett [1], Bhat [2], Billingsley [3], Good [6], Patankar [13], and Whittle [18], to name a few. It seems appropriate, then, to consider a statistical inference problem for M.R.P.'s, namely, a goodness of fit test for a hypothetical model.

2. Notation and previous results

Throughout the paper we use the notation of Pyke ([15] and [16]) as far as possible, and we assume the underlying Markov chain to be irreducible, aperiodic, and recurrent. Thus, we have

$$\begin{aligned} \mathrm{P}\{J_n=j,\ X_n\leqq x\,|\,J_0, J_1,\cdots, J_{n-1}=i\} \\ =\mathrm{P}\{J_n=j,\ X_n\leqq x\,|\,J_{n-1}=i\} \\ =p_{ij}F_{ij}(x)\ , \end{aligned} \tag{2.1}$$

where the p_{ij} are transition probabilities and $F_{ij}(x)$ are holding-time distributions, and we set $Q_{ij}(x)=p_{ij}F_{ij}(x)$, the transition d.f. of the M.R.P. The Laplace-Stieltjes Transform (L.-S.T.) of any mass function will be denoted by the corresponding small letter; for example,

$$q_{ij}(s)=\int_0^\infty e^{-sx}dQ_{ij}(x)\ , \qquad s>0\ . \tag{2.2}$$

We let $N_j(t)$ represent the number of visits to state j ($j=1,\cdots,m$) in the interval $(0,t)$, and the vector

$$N(t)=[N_1(t),\cdots,N_m(t)]'\ .$$

Now, Z_t will be the state of the system at time t, and the set of initial probabilities will be given by $\mathrm{P}[Z_0=i]=a_i$, $(i=1,\cdots,m)$. Also,

$$M_{ij}(t)=\mathrm{E}[N_j(t)\mid Z_0=i] , \tag{2.3}$$

$$R_{ij}(t)=\mathrm{E}[N_j(t)\{N_j(t)-1\}\mid Z_0=i] , \tag{2.4}$$

and

$$C^i_{jk}(t)=\mathrm{E}[N_j(t)N_k(t)\mid Z_0=i] , \qquad j\neq k , \tag{2.5}$$

with corresponding L.-S.T.'s $m_{ij}(s)$, $r_{ij}(s)$, and $c^i_{jk}(s)$. Furthermore,

$$G_{ij}(t)=\mathrm{P}\{N_j(t)>0\mid Z_0=i\} , \tag{2.6}$$

with L.-S.T. $g_{ij}(s)$. $G_{ij}(t)$ is called the first passage time of state j, and $G_{jj}(t)$, the recurrence time.

We shall denote by P_0, $F(x)$, $Q(x)$, $G(t)$, $M(t)$, $R(t)$, $f(s)$, $q(s)$, $g(s)$, $m(s)$, and $r(s)$ matrices whose (i, j)th elements are respectively p_{ij}, $F_{ij}(x)$, $Q_{ij}(x)$, $G_{ij}(t)$, $M_{ij}(t)$, $R_{ij}(t)$, $f_{ij}(s)$, $q_{ij}(s)$, $g_{ij}(s)$, $m_{ij}(s)$, and $r_{ij}(s)$. For $m\times m$ matrices $A=[a_{ij}]$, ${}_dA$ will mean the diagonal matrix

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & & \vdots \\ & & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix} ,$$

I will be the identity matrix, 0 the null matrix, and $\boldsymbol{e}$ a column vector of ones. Also, E will denote $\boldsymbol{e}\boldsymbol{e}'$, and δ_{ij} will be the Kronecker delta.

Kshirsagar and Gupta [10] have found the L.-S.T. of the probability generating function (p.g.f.) of $N(t)$ as follows:

$$\boldsymbol{\Psi}=[I-q(s)\xi]^{-1}[I-q(s)]\boldsymbol{e} , \tag{2.7}$$

where $\xi=\mathrm{diag}(\xi_1,\cdots,\xi_m)$, and $|\xi_i|<1$ for $i=1,\cdots,m$. By differentiation of (2.7), they have found the L.-S.T.'s of the first two moments of $N_j(t)$ as individual elements of

$$m(s)=q(s)[I-q(s)]^{-1}=[I-q(s)]^{-1}-I , \tag{2.8}$$

$$r(s)=2m(s)\,{}_dm(s) , \tag{2.9}$$

and the cross-product moments as

$$c^i_{jk}(s)=m_{jk}(s)m_{ij}(s)+m_{kj}(s)m_{ik}(s) . \tag{2.10}$$

The quantities (2.9) and (2.10) will be used later to find the variances and covariances of the $N_j(t)$. Pyke's ([16]) generating function expression for the first passage time distribution is

$$g(s)=q(s)[I-q(s)]^{-1}\{_d[(I-q(s))^{-1}]\}^{-1}, \tag{2.11}$$

from which one may evaluate the mean recurrence times, say b_{ii} ($i=1,\cdots,m$), by taking limits of appropriate quantities as s tends to zero.

A generalization of an M.R.P. that we wish to consider is an Equilibrium Markov Renewal Process (E.M.R.P.). When $m=1$, this process reduces to an Equilibrium Renewal Process as discussed by Cox [5]. This process has the characteristic that instead of observing it from its start, we begin observing only after a sufficiently long time has elapsed; then the distribution of the first state and holding time is different from all the subsequent ones. Pyke [16] has proved that the initial probabilities for this process are

$$\tilde{a}_i=\frac{\eta_i}{b_{ii}}, \qquad i=1,\cdots,m, \tag{2.12}$$

where η_i is the mean of the d.f. $\sum_{j=1}^{m} Q_{ij}(x)$. Pyke has also proved the transition d.f. for the first transition to be

$$\tilde{Q}_{ij}(t)=\frac{p_{ij}}{\eta_i}\int_0^t [1-F_{ij}(y)]dy \qquad (i, j=1,\cdots,m). \tag{2.13}$$

After the first transition is observed, the transition d.f. becomes $Q_{ij}(x)$, as before.

For the E.M.R.P. the L.-S.T.'s of the moments of $N(t)$ are given by Kshirsagar and Gupta [10] as:

$$\tilde{m}(s)=\tilde{q}(s)[I-q(s)]^{-1}, \tag{2.14}$$

$$\tilde{r}(s)=2\tilde{m}(s)_d m(s), \tag{2.15}$$

and

$$\tilde{c}^i_{jk}(s)=\tilde{m}_{ij}(s)m_{jk}(s)+\tilde{m}_{ik}(s)m_{kj}(s). \tag{2.16}$$

Using (2.12), (2.13), and (2.14) they have shown that the unconditional expectation of $N(t)$ is given by

$$\mathrm{E}[N_1(t),\cdots,N_m(t)]=t\left[\frac{1}{b_{11}},\cdots,\frac{1}{b_{mm}}\right], \tag{2.17}$$

a result analogous to the corresponding renewal process result discussed by Cox [5]. Later in the paper we shall use (2.15) and (2.16) to obtain the unconditional variances and covariances of $N(t)$.

Many of the results discussed so far have been available only in terms of L.-S.T.'s of the quantities involved. Unfortunately, except for the very simplest of cases, inversion of these L.-S.T.'s is not possible. However, asymptotic expressions may be obtained for large t by expanding the L.-S.T.'s in powers of s and using Tauberian arguments, as outlined by Cox [5] in the case of ordinary renewal processes. The quantity $[I-q(s)]^{-1}$ appears often, especially in the moments of $N(t)$, so its expansion will be very important in what follows.

Kshirsagar and Gupta [9], Hunter [7], and Keilson [8], have independently obtained expansions for $[I-q(s)]^{-1}$, all by different methods, but a new result of Kshirsagar and Gupta [11] is more desirable than the first three because it is given in terms of the basic quantities $q(s)$, and because it is easier to calculate. Let $\boldsymbol{U}'=[U_1,\cdots,U_m]$ be the vector of stationary state probabilities for the Markov chain; that is,

$$\boldsymbol{U}'P_0=\boldsymbol{U}' .$$

Then, let $L=\boldsymbol{e}\boldsymbol{U}'$, $Z=(I-P_0+L)$, and $k_r=\boldsymbol{U}'P_r\boldsymbol{e}$, where $P_r=\int_0^\infty x^r dQ(x)$, the matrix of rth moments of the transition d.f. $Q(x)$. Then, Kshirsagar and Gupta [11] have shown that

$$[I-q(s)]^{-1}=\frac{1}{s}A_{-1}+A_0+sA_1+o(s) , \tag{2.18}$$

where

$$A_{-1}=\frac{1}{k_1}L , \tag{2.19}$$

$$A_0=\left(I-\frac{1}{k_1}LP_1\right)Z\left(I-\frac{1}{k_1}P_1L\right) , \tag{2.20}$$

and

$$\begin{aligned} A_1=&\left\{-ZP_1+\frac{1}{k_1}LP_1ZP_1+\frac{1}{2k_1}LP_2\right\}A_0 \\ &+\frac{1}{2k_1}\left\{ZP_2-\frac{1}{k_1}LP_1ZP_2-\frac{1}{3k_1}LP_3\right\}L . \end{aligned} \tag{2.21}$$

For convenience we denote individual elements of A_r as $a_{ij}^{(r)}$.

Now, using the expansion and (2.8), (2.9), and (2.10) we may find the means of the $N_j(t)$'s as

$$M(t)=t\cdot A_{-1}+(A_0-I)+o(1) . \tag{2.22}$$

If we let $V(t)=[V_{ij}(t)]$, where $V_{ij}(t)=\mathrm{Var}\,\{N_j(t)\,|\,Z_0=i\}$, then

$$\begin{aligned} V(t)&=R(t)+M(t)-[M(t)]^2 \\ &=t[2A_{-1_d}A_0-A_{-1}]+[A_{0_d}A_0-A_0\Box A_0+{}_dA_0-A_0+A_{-1_d}A_1 \\ &\quad +A_{1_d}A_{-1}]+o(1) , \end{aligned} \tag{2.23}$$

where $C\Box D=[c_{ij}d_{ij}]$, a matrix of products of individual elements. Also, if we let $\Gamma^i_{jk}(t)=\mathrm{Cov}\,[N_j(t),\, N_k(t)\,|\,Z_0=i]$, then

$$\begin{aligned}(2.24)\quad \Gamma^i_{jk}(t)&=C^i_{jk}(t)-M_{ij}(t)M_{ik}(t)\\ &=t\frac{1}{k_1}[U_j a^{(0)}_{jk}+U_k a^{(0)}_{kj}]+\Big[\frac{1}{k_1}\{U_j(a^{(1)}_{jk}+a^{(1)}_{ik})+U_k(a^{(1)}_{ij}+a^{(1)}_{kj})\}\\ &\quad +a^{(0)}_{jk}(a^{(0)}_{ij}+a^{(0)}_{ik}-\delta_{ij}-\delta_{ik})-(a^{(0)}_{ij}-\delta_{ij})(a^{(0)}_{ik}-\delta_{ik})\Big]+o(1)\ .\end{aligned}$$

Now, for the E.M.R.P. we may find the unconditional variances and covariances of $N_j(t)$ by applying the expansion to (2.15) and (2.16), inverting and multiplying by the vector of initial probabilities (2.12). After some simplification, we obtain

$$(2.25)\qquad V_j(t)=\mathrm{Var}\,[N_j(t)]=t\Big[\frac{1}{k_1}U_j(2a^{(0)}_{jj}-1)\Big]+\frac{2}{k_1}U_j a^{(1)}_{jj}+o(1)\ ,$$

and

$$\begin{aligned}(2.26)\quad \Gamma_{jk}(t)&=\mathrm{Cov}\,[N_j(t),\, N_k(t)]\\ &=t\frac{1}{k_1}[U_j(a^{(0)}_{jk}-\delta_{jk})+U_k(a^{(0)}_{kj}-\delta_{kj})]+\Big[\frac{1}{k_1}(U_j a^{(1)}_{jk}+U_k a^{(1)}_{kj})\Big]+o(1)\ .\end{aligned}$$

We are now prepared to apply these results to our inference problem.

3. The χ^2 goodness of fit test

Bartlett [1] has considered a χ^2 test for the goodness of fit of a hypothetical matrix of transition probabilities in the case of a Markov chain. The procedure consists of observing a Markov chain with m states and computing the classical χ^2 test statistic, namely $\sum_{j=1}^{m}(n_j-m_j)^2/m_j$, where n_j denotes the number of visits to the jth state of the chain, and m_j denotes the expected number of visits to the jth state, assuming the hypothetical matrix of transition probabilities to be the true one. The test is carried out using the standard χ^2 distribution with $m-1$ degrees of freedom. The validity of the approach lies in the asymptotic normality of the n_j, which Bartlett proved in the same paper.

Patankar [13] has modified Bartlett's procedure by calculating the expectation and variance of the χ^2, say A and $2B$, respectively, taking $A\chi^2/B$ to have an approximate χ^2 with A^2/B degrees of freedom. This is a better approximation in that the first two moments of the modified statistic agree exactly with those of a χ^2.

For the M.R.P. we use the $N_j(t)$'s as the observations and note that Moore and Pyke [12] have shown that as t becomes large the $N_j(t)$ are asymptotically normally distributed, with means $M_{ij}(t)$, variances

$V_{ij}(t)$, and covariances $\Gamma^i_{jk}(t)$. Thus, for an M.R.P. the goodness-of-fit statistic takes the form

$$\chi_0^2=\sum_{j=1}^{m}\frac{[N_j(t)-M_{ij}(t)]^2}{M_{ij}(t)} . \tag{3.1}$$

There are several differences between this statistic and Bartlett's statistic for Markov chains. First, one observes a fixed (total) number of transitions in a Markov chain, whereas in an M.R.P. we have observed the process for a fixed but large length of time, so that the number of transitions is random. In a Markov chain a transition will always occur after every unit length of time, but this is not necessarily so in an M.R.P. Since, in the Markov chain, $\sum_{j=1}^{m} n_j=n$, the fixed total number of transitions, there is a linear constraint on the variables, so the χ^2 has $m-1$ degrees of freedom. No such restriction exists for the M.R.P. Finally, in the M.R.P. the expectations $M_{ij}(t)$ are available only in terms of the expansions rather than as functions of the $N_j(t)$ and p_{ij}, and the matrix of known d.f.'s is involved.

Rewriting the statistic as a quadratic form, we have

$$\chi_0^2=\boldsymbol{Y}'Q\boldsymbol{Y} , \tag{3.2}$$

where $\boldsymbol{Y}'=[N_1(t)-M_{i1}(t),\cdots, N_m(t)-M_{im}(t)]$, and

$$Q=\operatorname{diag}\left[\frac{1}{M_{i1}(t)},\cdots,\frac{1}{M_{im}(t)}\right] .$$

Then, it is well-known that

$$\mathrm{E}\,(\chi_0^2)=\operatorname{tr}\,(VQ) , \tag{3.3}$$

and

$$\operatorname{Var}\,(\chi_0^2)=2\operatorname{tr}\,\{(VQ)^2\} , \tag{3.4}$$

where

$$V=\begin{bmatrix} V_{i1}(t) & \Gamma^i_{12}(t) & \cdots & \Gamma^i_{1m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^i_{m1}(t) & \Gamma^i_{m2}(t) & \cdots & V_{im}(t) \end{bmatrix}$$

is the matrix of variances and covariances of the Y's given the initial state i. Then, multiplying the matrices and taking the trace yields

$$\mathrm{E}\,(\chi_0^2)=\sum_{j=1}^{m}\frac{V_{ij}(t)}{M_{ij}(t)} , \tag{3.5}$$

and

$$\text{Var}\,(\chi_0^2)=2\sum_{j,k=1}^{m}(1-\delta_{jk})\frac{[\Gamma_{jk}^{i}(t)]^2}{M_{ij}(t)M_{ik}(t)}+2\sum_{j=1}^{m}\left[\frac{V_{ij}(t)}{M_{ij}(t)}\right]^2, \tag{3.6}$$

for a fixed initial state i. Substituting the expansions (2.22), (2.23) and (2.24) into (3.5) and (3.6) and simplifying, we have

$$A=\text{E}\,(\chi_0^2)=\sum_{j=1}^{m}(2a_{jj}^{(0)}-1)+o(1)\ , \tag{3.7}$$

and

$$\begin{aligned}2B=\text{Var}\,(\chi_0^2)=2\sum_{j,k=1}^{m}(1-\delta_{jk})&\left\{\frac{U_j(a_{jk}^{(0)})^2}{U_k}+2a_{jk}^{(0)}a_{kj}^{(0)}+\frac{U_k(a_{kj}^{(0)})^2}{U_j}\right\}\\&+2\sum_{j=1}^{m}\{4(a_{jj}^{(0)})^2-4a_{jj}^{(0)}+1\}+o(1)\ .\end{aligned} \tag{3.8}$$

Then, taking $\chi'^2=A\chi_0^2/B$ to be approximate χ^2 with A^2/B degrees of freedom, the moments of χ'^2 exactly fit those of the data if the hypothetical probabilities p_{ij} are the true ones.

For the E.M.R.P. we calculate χ_E^2, say, the same way as before. The moments are found, however, using (2.17), (2.25), and (2.26) in (3.5) and (3.6). After some simplification, we have

$$A_E=\text{E}\,(\chi_E^2)=\sum_{j=1}^{m}(2a_{jj}^{(0)}-1)+o(1)\ , \tag{3.9}$$

and

$$\begin{aligned}2B_E=\text{Var}\,(\chi_E^2)=2\sum_{j,k=1}^{m}(1-\delta_{jk})&\left\{\frac{U_j}{U_k}(a_{jk}^{(0)}-\delta_{jk})^2+2(a_{jk}^{(0)}-\delta_{jk})(a_{kj}^{(0)}-\delta_{kj})\right.\\&\left.+\frac{U_k}{U_j}(a_{kj}^{(0)}-\delta_{kj})^2\right\}+2\sum_{j=1}^{m}\{4(a_{jj}^{(0)})^2-4a_{jj}^{(0)}+1\}+o(1)\ .\end{aligned} \tag{3.10}$$

First note that (3.9) is identical to (3.7). Then, careful evaluation of the Kronecker deltas in (3.10) reveals that it (3.10) is identical to (3.8). This satisfies the intuition that, if t is large, the process reaches equilibrium in the interval $(0, t)$, whether we begin observing from the start or not. Having so noted and satisfied our intuition, we then modify χ_E^2 as before by taking $\chi''^2=A_E\chi_E^2/B_E$ to obtain an approximate χ^2 with A_E^2/B_E degrees of freedom to carry out the test.

To consider the question of how large t should be for application of the asymptotic formulas, we return for a moment to ordinary one-state renewal processes. Although no mathematically rigorous proof has been given, Cox [5] argues roughly that a minimum requirement is that

$$t>\frac{\mu^3}{\sigma^2}\ ,$$

where μ and σ^2 are the mean and variance, respectively, of the life distribution. To apply this result to M.R.P.'s, we note first that if we consider only transitions into state j, then we have an ordinary renewal process with life distribution $G_{jj}(t)$, the recurrence-time distribution. The above minimum requirement now becomes

$$t > \max_j \left\{ \frac{b_{jj}^3}{v_{jj}^2} \right\} , \qquad j=1,\cdots,m ,$$

where b_{jj} and v_{jj}^2 are respectively the mean and variance of the recurrence-time distribution.

4. The χ^2 statistic for a two-state M.R.P.

Direct evaluation of $[I-q(s)]^{-1}-I$ and inversion gives

(4.1)

$$M(t)=\begin{bmatrix} b\alpha t+(c_{22}\alpha-\alpha^2\beta b-1)+o(1) & (1-a)\alpha t-\alpha c_{21}-\alpha^2\beta(1-a)+o(1) \\ b\alpha t-(c_{12}\alpha+\alpha^2\beta b)+o(1) & (1-a)\alpha t+c_{11}\alpha-(1-a)\alpha^2\beta-1+o(1) \end{bmatrix}$$

where $p_{11}=1-p_{12}=a$; $p_{21}=1-p_{22}=b$; $[c_{ij}]=[p_{ij}\mu_{ij}]=P_1$;

(4.2) $[d_{ij}]=[p_{ij}(\mu_{ij}^2+\sigma_{ij}^2)]=P_2$; μ_{ij} and σ_{ij}^2 are respectively means and variances of $F_{ij}(x)$; $1/\alpha=(1-a)(c_{21}+c_{22})+b(c_{11}+c_{12})$; $\beta=\det(P_1)-1/2\{(1-a)(d_{21}+d_{22})+b(d_{11}+d_{12})\}$.

Again, direct evaluation using (2.9), (2.10), (2.23) and (2.24) yields expansions for the variances and covariances for $m=2$ as follows:

(4.3)

$$\begin{aligned} V_{11}(t)&=t[2\alpha b(c_{22}\alpha-\alpha^2\beta b-1)+\alpha b]+o(t) , \\ V_{12}(t)&=t(1-a)\alpha[2\alpha(c_{11}-c_{12})-2\alpha^2\beta(1-a)+2\alpha c_{21}-1]+o(t) , \\ V_{21}(t)&=t(\alpha b)[2\alpha(c_{22}-c_{21})-2\alpha^2\beta b+2\alpha c_{12}-1]+o(t) , \\ V_{22}(t)&=t(1-a)\alpha[2\{\alpha c_{11}-\alpha^2\beta(1-a)-1\}+1]+o(t) , \\ \Gamma_{12}^1(t)&=\Gamma_{21}^1(t)=t[\alpha(1-a)\{-\alpha c_{12}-\alpha^2\beta b\}-\alpha b\{\alpha(2c_{12}-c_{21}) \\ &\qquad -\alpha^2\beta(1-a)\}]+o(t) , \end{aligned}$$

and

$$\begin{aligned} \Gamma_{12}^2(t)=\Gamma_{21}^2(t)=t[\alpha b\{-\alpha c_{12}-\alpha^2\beta(1-a)-1\}-\alpha(1-a) \\ \cdot \{\alpha c_{12}+\alpha^2\beta b\}]+o(t) . \end{aligned}$$

We may now calculate our test statistic as

(4.4)
$$\chi_0^2=\frac{[N_1(t)-M_{i1}(t)]^2}{M_{i1}(t)}+\frac{[N_2(t)-M_{i2}(t)]^2}{M_{i2}(t)} ,$$

with

$$\text{(4.5)} \qquad \mathrm{E}\,(\chi_0^2)=\frac{V_{i1}(t)}{M_{i1}(t)}+\frac{V_{i2}(t)}{M_{i2}(t)}$$

and

$$\text{(4.6)} \qquad \mathrm{Var}\,(\chi_0^2)=2\left\{\left[\frac{V_{i1}(t)}{M_{i1}(t)}\right]^2+\left[\frac{V_{i2}(t)}{M_{i2}(t)}\right]^2+2\frac{[\Gamma_{12}^i(t)]^2}{M_{i1}(t)M_{i2}(t)}\right\} ,$$

for a given initial state i. Substituting (4.1) and (4.3) into (4.5) and (4.6) and simplifying yields

$$\text{(4.7)} \qquad \mathrm{E}\,(\chi_0^2)=2\alpha(c_{11}-c_{12}+c_{21}+c_{22})-2\alpha^2\beta[(1-a)+b]-2+o(1) ,$$

and

$$\text{(4.8)} \qquad \mathrm{Var}\,(\chi_0^2)=[2c_{22}\alpha-2\alpha^2\beta b-1]^2+[2c_{11}\alpha-2\alpha^2\beta(1-a)-1]^2 \\ +2\left[\frac{1-a}{b}\{-\alpha c_{12}-\alpha^2\beta b\}^2+2\{\alpha c_{12}+\alpha^2\beta b\}\{\alpha(2c_{12}-c_{21}) \right. \\ \left. +\alpha^2\beta(1-a)\}+\frac{b}{1-a}\{\alpha(2c_{12}-c_{21})+\alpha^2\beta(1-a)\}^2\right]+o(1) ,$$

when i, the initial state, is equal to one. Now, for $i=2$, we have

$$\text{(4.9)} \qquad \mathrm{E}\,(\chi_0^2)=2\alpha(c_{11}+c_{12}-c_{21}+c_{22})-2\alpha^2\beta[(1-a)+b]-2+o(1) ,$$

and

$$\text{(4.10)} \qquad \mathrm{Var}\,(\chi_0^2)=[2\alpha(c_{22}-c_{21}+c_{12})-2\alpha^2\beta b-1]^2+[2\alpha c_{11}-2\alpha^2\beta(1-a)-1]^2 \\ +2\left[\frac{b}{1-a}\{-\alpha c_{12}-\alpha^2\beta(1-a)-1\}^2+2\{\alpha c_{12}+\alpha^2\beta(1-a)+1\} \right. \\ \left. \cdot\{\alpha c_{12}+\alpha^2\beta b\}+\frac{1-a}{b}\{\alpha c_{12}+\alpha^2\beta b\}^2\right]+o(1) .$$

5. Numerical examples

To illustrate our method we present here some tests carried out numerically on samples generated from a particular M.R.P. We wish to test the matrix

$$P_0=\begin{bmatrix} .3 & .7 \\ .6 & .4 \end{bmatrix} ,$$

using a matrix of distribution functions of the form

$$F(x)=\begin{bmatrix} 1-e^{-x/2} & 1-e^{-(x/2)^2} \\ 1-e^{-x/2} & 1-e^{-(x/2)^2} \end{bmatrix} .$$

For this case we have

$$P_1=\begin{bmatrix} 0.600 & 1.239 \\ 1.200 & 0.708 \end{bmatrix} \quad \text{and} \quad P_2=\begin{bmatrix} 2.400 & 2.800 \\ 4.800 & 1.600 \end{bmatrix}.$$

From (4.2) we have $1/\alpha=2.439$ and $\beta=-4.862$. In each case we observe the process for $t=80$ units of time. We list the initial state and the observed number of visits to each state of the system, but not the observed sequence of states or sojourn times, since they are not needed in the actual computation. We use (4.1), (4.4), (4.7), (4.8), (4.9), and (4.10) and tabulate the results below.

Modified χ^2 test of goodness of fit for a two-state M.R.P.

Initial state	$N_1(t)$	$N_2(t)$	Observed χ_0^2	$\mathrm{E}(\chi_0^2)=A$	$\mathrm{Var}(\chi_0^2)=2B$	Modified χ'^2	Degrees of freedom
2	12	18	3.990	1.230	1.644	5.970	1.84
2	14	20	1.667	1.230	1.644	2.499	1.84
1	31	12	12.130	1.165	1.368	20.667	1.98
2	17	18	1.360	1.230	1.644	2.040	1.84
1	18	20	0.528	1.165	1.368	0.901	1.98

With the exception of the third sample, the values of the modified statistic are insignificant when compared to the 95% point of the standard χ^2 with two degrees of freedom, and there is no reason to doubt the fit.

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