SOME PROPERTIES AND GENERATING FUNCTION OF ORDERED PARTITIONS*

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1. Introduction and summary

'Ordered partitions' distinguish between the various ways of writing the same partition. They manifest themselves when one is interested in the number of ways a product of n distinct primes can be factored [8] or the number of ways of distributing n distinct objects into n boxes with empty boxes permitted [5] or the number of permutations of n elements with ordered cycles [4], etc. In statistical theory, they have been found useful for multiplying polykays [2] through their representation in terms of partitions. For a polykay $\{\alpha\}$, if the corresponding symmetric mean is denoted by $\langle\alpha\rangle$, then [3]

$$\langle\alpha\rangle=\{\alpha\}+\sum\{\beta_\alpha\}$$

where the summation is over all distinct subpartitions β_α of α. Similar multiplication procedure may in fact be developed for any sample symmetric function admitting a representation in terms of partitions.

In this paper, we study some properties of ordered partitions, and propose a technique for generating their complete set for a given unordered partition. We also discuss the number of ordered partitions of a given weight and its generating function [5], [8] from a statistical viewpoint. Some statistical applications of the theory of ordered partitions are indicated.

2. Definitions

DEFINITION 1. An ordered partition of weight m is a list of m symbols $\alpha=\alpha_1\cdots\alpha_m$ such that either $i\,\alpha\,j$ or $i\not\alpha\,j\;\forall(i,j)$; $i,j=1,2,\cdots,m$, where $i\,\alpha\,j$ or $i\not\alpha\,j$ according as $\alpha_i=\alpha_j$ or $\alpha_i\neq\alpha_j$.

Example. $1122345, 1212345, 2132451,\cdots$ are various ordered partitions from the 5-part partition $2,2,1,1,1$ of weight 7.

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DEFINITION 2. Let $\alpha=\alpha_1\cdots\alpha_m$ and $\beta=\beta_1\cdots\beta_m$ be two ordered partitions of weight m. α is said to be an ordered subpartition of β ($\alpha\leqq\beta$) iff $i\,\alpha\, j\Rightarrow i\,\beta\, j\ \forall\,(i, j)$; $i, j=1, 2,\cdots, m$.

Example. 112324 is an ordered subpartition of 112123.

α and β are said to be equivalent if $i\,\alpha\, j\Rightarrow i\,\beta\, j$ and $i\,\beta\, j\Rightarrow i\,\alpha\, j\ \forall\,(i, j)$; $i, j=1, 2,\cdots, m$.

DEFINITION 3. A lattice is a system $(S, \leqq, \wedge, \vee)$ where S is a set of elements, $\leqq$ is a binary relation on S satisfying the reflexive, antisymmetric and transitive laws, and any two elements $\alpha, \beta\in S$ have a g.l.b. $\alpha\wedge\beta$ and a l.u.b. $\alpha\vee\beta$.

DEFINITION 4. A first element 0 of an ordered set $(S, \leqq)$ is an element which satisfies the relation $\alpha\geqq 0\ \forall\,\alpha\in S$.

DEFINITION 5. A last element 1 of an ordered set $(S, \leqq)$ is an element which satisfies the relation $\alpha\leqq 1\ \forall\,\alpha\in S$.

If first element and/or last element exist, they are unique.

Example. In the set S_m of ordered partitions of weight m, the one-part partition $11\cdots1$ and the m-part partition $12\cdots m$ are the last and the first element respectively.

DEFINITION 6. An element $\beta\in S_m$ is said to be a complement of $\alpha\in S_m$ if $\alpha\wedge\beta=0$ and $\alpha\vee\beta=1$.

DEFINITION 7. A lattice S_m with 0 and 1 is said to be complemented if for every $\alpha\in S_m$, there is $\alpha'\in S_m$ such that $\alpha\wedge\alpha'=0$ and $\alpha\vee\alpha'=1$. α' is not necessarily unique.

DEFINITION 8. A subset of a lattice S_m in which every pair of elements has a g.l.b. and a l.u.b. is called a sublattice.

DEFINITION 9. Let $\boldsymbol{\alpha}_i$ denote the column vector formed by all ordered partitions of weight m_i. Then the Kronecker product $\boldsymbol{\alpha}_1\otimes\boldsymbol{\alpha}_2\otimes\cdots\otimes\boldsymbol{\alpha}_n$ is the column vector whose components are the ordered partition $\underbrace{11\cdots1}_{m_1}\underbrace{22\cdots2}_{m_2}\cdots\underbrace{nn\cdots n}_{m_n}$ and all its ordered subpartitions.

Example.

$$\begin{bmatrix}11\\12\end{bmatrix}\otimes\begin{bmatrix}11\\12\end{bmatrix}=\begin{bmatrix}1122\\1123\\2311\\1234\end{bmatrix}.$$

Throughout this paper partition (subpartition) will stand for ordered partition (ordered subpartition) unless otherwise mentioned.

3. Set of ordered partitions as a lattice

Here we outline the construction of g.l.b. and l.u.b. under which the set S_m of partitions of weight m, with the subpartition partial ordering, forms a lattice.

Let $\alpha=\alpha_1\cdots\alpha_m$, $\beta=\beta_1\cdots\beta_m$ be two partitions of weight m.

Let $\delta=\alpha\wedge\beta$, $\delta=\delta_1\cdots\delta_m$ be defined as the set of ordered pairs (α_1, β_1) $(\alpha_2, \beta_2)\cdots(\alpha_m, \beta_m)$ such that $i\,\delta\, j \Rightarrow i\,\alpha\, j,\ i\,\beta\, j\ \forall (i, j)$; $i, j=1, 2, \cdots, m$.

Let $\mu=\alpha\vee\beta$, $\mu=\mu_1\cdots\mu_m$ be defined as follows:

Set $\mu_1=1$. If for any $i=1, 2,\cdots, m$, $\exists k$ such that $1\,\alpha\, k$ and $k\,\beta\, i$, set $\mu_i=1$ (if $1\,\alpha\, i$ then $k=i$). Let I_1 be the set of all integers i such that $\mu_i=1$. If for $i_1\in I_1$ and for some $j=1, 2,\cdots, m$, $\exists k$ such that $i_1\,\alpha\, k$ and $k\,\beta\, j$, set $\mu_j=1$. Continue until unable to find such k for some j. If $\mu_1=\cdots=\mu_m=1$, we are done and μ is a one-part partition. Otherwise, let i_2 be the first integer not in I_1, let $\mu_{i_2}=2$. Let I_2 be the set different from I_1 and constructed in the same manner as I_1. If I_1 and I_2 exhaust the whole set $1, 2,\cdots, m$ then we are done. Otherwise continue the process with $I_3,\cdots$. With μ constructed as above, $i\,\mu\, j\Rightarrow\exists k$ such that $i\,\alpha\, k$ and $k\,\beta\, j$.

It can easily be seen [2] that, under this definition of g.l.b. and l.u.b., the set S_m of partitions of weight m, with the subpartition partial ordering, forms a lattice.

4. Some theorems

THEOREM 1. *The set S_m of all partitions of weight m, with the subpartition partial ordering, forms a complemented lattice.*

PROOF. It is known [2] that S_m is a lattice. In order to show that S_m is a complemented lattice, for any $\alpha\in S_m$, $\alpha\neq 0, 1$, construct $\beta\in S_m$ such that $\alpha\wedge\beta=0$, $\alpha\vee\beta=1$, as follows. If $\alpha_{1_{i_1}}=\alpha_{1_{i_2}}=\alpha_{1_{i_3}}=\cdots=\alpha_{1_{i_r}}$, set $\beta_{1_{i_1}}=\beta_1$, $\beta_{1_{i_2}}=\beta_2,\cdots$ where $\beta_1\neq\beta_2\cdots$. Since $\alpha\neq 0, 1$, the whole set of α's will not be exhausted. From rest of the α's choose $\alpha_{2_{i_1}}=\alpha_{2_{i_2}}=\cdots=\alpha_{2_{i_s}}$ and set $\beta_{2_{i_1}}=\beta_1$, $\beta_{2_{i_2}}=\beta_2,\cdots$. Continue until the whole set is exhausted. Obviously β is a partition of weight m and hence $\beta\in S_m$.

Let $\delta=\alpha\wedge\beta=(\alpha_1, \beta_1)(\alpha_2, \beta_2)\cdots(\alpha_m, \beta_m)$ where $i\not\alpha j$ or $i\not\beta j\Rightarrow i\not\delta j$. From the construction of β it is clear that whenever $i\not\alpha j$, $i\,\beta\, j$ and vice versa, we have $i\not\delta j\ \forall (i, j)$; $i, j=1, 2,\cdots, m$. Thus δ is a m-part partition of weight m, i.e. $\delta=0$.

Let $\mu=\alpha\vee\beta$, where μ is constructed such that if for any $(i, j)\,\exists k$

such that $i\alpha k$ and $k\beta j$ then $i\mu j$. For each (i, j); $i, j=1, 2, \cdots, m$, we can always find such k.

$$\begin{aligned}&\alpha_{1_{i_1}}=\alpha_{1_{i_2}}=\cdots=\alpha_{1_{i_r}}\\&\alpha_{2_{i_1}}=\alpha_{2_{i_2}}=\cdots=\alpha_{2_{i_s}}\\&\cdots\cdots\cdots\cdots\cdots\cdots\end{aligned}$$

(If $i\alpha j$ then $k=j$. Otherwise take the row containing α_i and the column containing α_j and set k as the suffix of the position where these two intersect.) Thus $i\mu j\,\forall\,(i, j)$; $i, j=1, 2, \cdots, m$ and hence μ is a one-part partition, i.e. $\mu=1$. Hence S_m is a complemented lattice. (If α was $0, 1$, choose $\beta=1, 0$).

Example. Let $\alpha\in S_{11}$, $\alpha=12314512267$. Here $\alpha_1=\alpha_4=\alpha_7$, set $\beta_1=1$, $\beta_4=2$, $\beta_7=3$. Also $\alpha_2=\alpha_8=\alpha_9$, so $\beta_2=1$, $\beta_8=2$, $\beta_9=3$. Similarly $\beta_3=1$, $\beta_5=1$, $\beta_6=1$, $\beta_{10}=1$, $\beta_{11}=1$, so that $\beta=11121132311$.

$$\begin{aligned}\delta=\alpha\wedge\beta=&(1,1)(2,1)(3,1)(1,2)(4,1)(5,1)(1,3)(2,2)(2,3)\\&(6,1)(7,1)\,.\end{aligned}$$

Clearly $\delta_1\neq\delta_2\neq\cdots\neq\delta_{11}$, so $\delta=\alpha\wedge\beta=0$.

$$\begin{aligned}&\mu=\alpha\vee\beta.\quad \alpha_1=\alpha_4=\alpha_7\Rightarrow\mu_1=\mu_4=\mu_7=1\;;\qquad \alpha_1=\alpha_4,\ \beta_4=\beta_8\Rightarrow\mu_1=\mu_8\;;\\&\alpha_1=\alpha_7,\ \beta_7=\beta_9\Rightarrow\mu_1=\mu_9\;;\qquad \alpha_1=\alpha_1,\ \beta_1=\beta_2=\beta_3=\beta_5=\beta_6=\beta_{10}=\beta_{11}\Rightarrow\\&\mu_1=\mu_2=\mu_3=\mu_5=\mu_6=\mu_{10}=\mu_{11}=1\,.\end{aligned}$$

Thus $\mu_1=\mu_2=\cdots=\mu_{11}$, so $\mu=\alpha\vee\beta=1$.

THEOREM 2. *The set of subpartitions of any partition of weight m forms a sublattice.*

PROOF. Let $\alpha=\alpha_1\cdots\alpha_m$ be any partition of weight m. Let S'_m be the set of all subpartitions of α. Since any partition is its own subpartition, $\alpha\in S'_m$.

Let $\beta, \gamma\in S'_m$ and $\delta=\beta\wedge\gamma$, $\mu=\beta\vee\gamma$. Form δ and μ as in Section 3. Now $\delta\leqq\alpha$ iff $i\delta j\Rightarrow i\alpha j\,\forall\,(i, j)$; $i, j=1, 2, \cdots, m$. But $i\delta j\Rightarrow i\beta j$ and $i\gamma j\Rightarrow i\alpha j$ since $\beta\leqq\alpha$, $\gamma\leqq\alpha$. Hence $\delta\leqq\alpha$ i.e. $\delta\in S'_m$.

Again $\mu\leqq\alpha$ iff $i\mu j\Rightarrow i\alpha j$. But $i\mu j\Rightarrow\exists\,k$ such that $i\beta k$ and $k\gamma j$ $\Rightarrow i\alpha k$ and $k\alpha j$ since $\beta\leqq\alpha$, $\gamma\leqq\alpha$, hence $i\alpha j$. Thus $\mu\leqq\alpha$ i.e. $\mu\in S'_m$. Hence S'_m is a sublattice.

It is interesting to note that the first element of the sublattice is the same as that of the lattice itself, whereas the last element is the partition whose subpartitions are being considered.

COROLLARY. *The components of the Kronecker product of $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n$ form a sublattice of the lattice of partitions of weight m_1+m_2+*

$\cdots + m_n$.

PROOF. Obvious on using Definition 9.

THEOREM 3. *If* $\alpha = \underbrace{111\cdots1}_{m_1}\,\underbrace{222\cdots2}_{m_2}\cdots\underbrace{nn\cdots n}_{m_n}$ *is a partition of weight* $m_1+m_2+\cdots+m_n$, *there are* $r_1\cdots r_n$ *subpartitions of* α, *where* r_i *is the number of partitions of weight* m_i.

PROOF. In α, if we replace $11\cdots1$ by any other partition of weight m_1, we get a subpartition of α, thus yielding r_1 subpartitions of α. In any of these, replace $22\cdots2$ by any other partition of weight m_2, thus obtaining $r_1\cdot r_2$ subpartitions of α. Continuing this process, we obtain $r_1\cdot r_2\cdot r_3\cdots r_n$ subpartitions of α.

COROLLARY. *A lattice of subpartitions of* $r\alpha$ *is isomorphic to the lattice of subpartitions of* α, *where* r *is an* r*-part partition of weight* r.

5. A scheme to generate ordered partitions

It seems desirable to develop a scheme to generate the partitions systematically, so that none are missed, since for higher weights the number of partitions increases fast. For weights 5, 6, 7, 8, there are 52, 203, 877, 4140 partitions respectively.

We illustrate the technique by generating all partitions originating from the unordered partition 332. The first partition, writing the symbols in ascending order, is 11122233. We write it as $1112\bar{2}\bar{\bar{2}}3\bar{3}$ to effect distinction between identical symbols, where necessary. The technique is to take one symbol at a time, and move it left successively till it reaches i) extreme left if the symbol has not occurred before, ii) identical symbol if it occurred earlier. In the example, no 1 is to be moved, 2 moves upto the extreme left, $\bar{2}$ upto 2, $\bar{3}$ upto 3, etc. All $8!/3!3!2!2!=280$ partitions, in the order of their generation, are listed in the Appendix. Details follow.

Start with 1's, no permutation is needed. Shift 2 one place left successively, till it reaches extreme left. We get the first row below. Now shift $\bar{2}$ one place left, till it gets next to 2, unless it is already so. Continue till $2\bar{2}$ reach the extreme left, obtaining

$$\begin{array}{llll}
1112\bar{2}\vdots\bar{\bar{2}}3\bar{3} & 112\vdots1\bar{2}\vdots\bar{\bar{2}}3\bar{3} & 12\vdots11\bar{2}\vdots\bar{\bar{2}}3\bar{3} & 2\vdots111\bar{2}\vdots\bar{\bar{2}}3\bar{3} \\
 & 112\vdots\bar{2}1\vdots\bar{\bar{2}}3\bar{3} & 12\vdots1\bar{2}1\vdots\bar{\bar{2}}3\bar{3} & 2\vdots11\bar{2}1\vdots\bar{\bar{2}}3\bar{3} \\
 & & 12\vdots\bar{2}11\vdots\bar{\bar{2}}3\bar{3} & 2\vdots1\bar{2}11\vdots\bar{\bar{2}}3\bar{3} \\
 & & & 2\vdots\bar{2}111\vdots\bar{\bar{2}}3\bar{3}
\end{array}.$$

These $4+3+2+1=10$ partitions appear in the first row of the Appendix.

Normally we would shift $\bar{\bar{2}}$ one place left till it gets next to $\bar{2}$, but when we have runs of the same length, e.g. 111, 222, we do not do so. Shifting $\bar{\bar{2}}$ here would result in repeated partitions, e.g., $211\bar{2}\bar{\bar{2}}13\bar{3}$ is the same partition as $12\bar{2}11\bar{\bar{2}}3\bar{3}$.

Shifting 3 one place left successively, till it reaches the extreme left, in all partitions in the first row of the Appendix, we get the next 6 rows. Thus we obtain 70 partitions.

Next shift $\bar{3}$ one place left in the above partitions till it gets next to 3, unless already so, and continue till $3\bar{3}$ reach extreme left. We obtain successively the next 6, 5, 4, 3, 2, 1 rows in the Appendix, accounting for 280 partitions in all.

We might note that there would still be 280 partitions if we were looking at the unordered partition 333. Here the partitions would be all those in the Appendix, with a 3 appended at the end of each. The $\bar{\bar{3}}$ would not be moved since this is a case of runs of same length. In general, partitions from $\underbrace{rr\cdots r}_{n}$ may be obtained by attaching an n at the end of all partitions from $\underbrace{rr\cdots r}_{n-1}\,r-1$, there being $(nr)!/(r!)^n n! = (nr-1)!/(r!)^{n-1}(r-1)!(n-1)!$ of them.

Dwyer and Schaeffer of the University of Michigan, using a similar scheme, have written a computer program for generating partitions, and have used it to generate all partitions through weight 8.

6. Generating function for ordered partitions

Let $P(x)=e^x \sum_{m=0}^{\infty} (N_m x^m/m!)$, where N_m denotes the number of partitions of weight m, with $N_0=1$. Then, we observe upon differentiation,

$$P^{(m)}(0)=\sum_{r=0}^{m}\binom{m}{r}N_r=N_{m+1}\ .$$

Thus, $P(x)$ is the generating function of the number of partitions. If $N^r N^s\cdots$ signifies $N_{r+s+\cdots}$, one could write

$$P(x)=e^x e^{Nx}=e^{(N+1)x}\ ,$$

so that

$$N_{m+1}=P^{(m)}(0)=(N+1)^m\ .$$

THEOREM 4.

$$N_m e=\sum_{r=1}^{\infty}\frac{r^m}{r!}=\sum_{r=0}^{\infty}\frac{r^m}{r!}\ .$$

PROOF. The result holds for $m=0,1$. Let us assume that the result is true for all $m\leqq n$, then we show it is true for $n+1$. Thus

$$\begin{aligned}\sum_{r=1}^{\infty}\frac{r^{n+1}}{r!}&=\sum_{r=0}^{\infty}\frac{(r+1)^{n+1}}{(r+1)!}\\&=\sum_{r=0}^{\infty}\frac{(r+1)^{n}}{r!}\\&=\sum_{r=0}^{\infty}\frac{1}{r!}\sum_{s=0}^{n}\binom{n}{s}r^{s}\\&=\sum_{s=0}^{n}\binom{n}{s}\sum_{r=0}^{\infty}\frac{r^{s}}{r!}\\&=\sum_{s=0}^{n}\binom{n}{s}eN_{s}\\&=e\sum_{s=0}^{n}\binom{n}{s}N_{s}\\&=eN_{n+1}\,.\end{aligned}$$

COROLLARY 1. *N_m is the mth ordinary moment of $P(1)$, the Poisson distribution with parameter 1, as $N_m=e^{-1}\sum_{r=0}^{\infty}(r^m/r!)$ according to the theorem.*

COROLLARY. 2. *The generating function $P(x)$ is $e^{(e^x-1)}$, the moment generating function of the $P(1)$ distribution.*

COROLLARY 3.

$$N_{n+1}=\frac{1}{e}\left(1^n+\frac{2^n}{1!}+\frac{3^n}{2!}+\cdots\right)\qquad \textit{(Dobinsky's formula) [5].}$$

PROOF.

$$\begin{aligned}N_{n+1}&=(n+1)\text{th moment of } P(1)\\&=\sum_{r=0}^{\infty}e^{-1}\frac{r^{n+1}}{r!}=\frac{1}{e}\sum_{r=1}^{\infty}\frac{r^{n}}{(r-1)!}\,.\end{aligned}$$

Remark 1. By considering $N_{m+1}=\sum_{r=0}^{m}\binom{m}{r}N_r$, it can be shown that if

$$A=\begin{pmatrix}1 & -1 & 0 & 0 & \cdots & 0\\ 1 & \binom{1}{1} & -1 & 0 & \cdots & 0\\ 1 & \binom{2}{1} & \binom{2}{2} & -1 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ 1 & \binom{m}{1} & \binom{m}{2} & \binom{m}{3} & \cdots & \binom{m}{m}\end{pmatrix},$$

then, $N_{m+1}=|A|$ and $N_r=|B|$, where B is the submatrix of A obtained by retaining the first r rows and columns.
For example,

$$N_4=\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & \binom{1}{1} & -1 & 0 \\ 1 & \binom{2}{1} & \binom{2}{2} & -1 \\ 1 & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \end{vmatrix}=15 .$$

Remark 2. Stirling numbers $s(n, k)$ of the second kind for a given weight are generated by, [4],

$$e^{\lambda(e^x-1)}=\sum_{n=0}^{\infty}\frac{x^n}{n!}\left(\sum_{k=0}^{n}s(n, k)\lambda^k\right) .$$

In our context, $s(n, k)$ denotes the number of ordered partitions obtained from the k-part partition of weight n, i.e.,

$$s(n, k)=\sum_{\Sigma \Pi_i=k}(p_1^{\Pi_1}\cdots p_r^{\Pi_r}) ,$$

where

$$\sum p_i\Pi_i=n \qquad \text{and} \qquad (p_1^{\Pi_1}\cdots p_r^{\Pi_r})=\frac{n!}{(p_1!)^{\Pi_1}\cdots(p_r!)^{\Pi_r}\Pi_1!\cdots\Pi_r!} .$$

This provides a method for computing Stirling numbers. Moreover, by considering the moment generating function of $P(\lambda)$,

$$\mu_n'=\sum_{k=0}^{n}s(n, k)\lambda^k$$

and for $P(1)$,

$$\mu_n'=\sum_{k=0}^{n}s(n, k)=N_n .$$

7. Applications

Using the theory of this paper, we can define generalized h-statistics [6], which estimate products of central moments unbiasedly just as polykays estimate products of cumulants.

If ( ) denotes a generalized h-statistic, then for a partition α of weight m, the mth degree symmetric mean

$$\langle\alpha\rangle=\sum_{\beta_\alpha\leq\alpha}(\beta_\alpha)$$

with β_α having at most as many parts of length >1 as those of α. Thus, for example, we have

$$\begin{aligned}\langle 1111r\rangle &= (1111r)+(1112r)+(1121r)+(1211r)+(2111r)\\ &\quad+(1123r)+(1213r)+(1231r)+(2113r)\\ &\quad+(2131r)+(2311r)+(1234r)\,.\end{aligned}$$

Now if $\langle\alpha\rangle$ and (α) denote the vectors of symmetric means and all h-statistics of any weight m, then

$$\langle\alpha\rangle = \Delta(\alpha)$$

where Δ is a non-singular upper triangular matrix with $\Delta_{ij}=1$ if $\alpha^j \leqq \alpha^i$ and α^j has at most as many parts of length >1 as those of α^i, and $\Delta_{ij}=0$ otherwise.

As for polykays [2], we have developed a method, using ordered partitions and its properties, to express powers and products of generalized h-statistics as linear combinations of such statistics. Also we have extended Carney's method [2] for obtaining double products of polykays to multiple products [7].

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Appendix

Ordered Partitions Originating from 332

11122233	11212233	12112233	21112233	11221233	12121233	21121233	12211233	21211233	22111233
11122323	11212323	12112323	21112323	11221323	12121323	21121323	12211323	21211323	22111323
11123223	11213223	12113223	21113223	11223123	12123123	21123123	12213123	21213123	22113123
11132223	11231223	12131223	21131223	11232123	12132123	21132123	12231123	21231123	22131123
11312223	11321223	12311223	21311223	11322123	12312123	21312123	12321123	21321123	22311123
13112223	13121223	13211223	23111223	13122123	13212123	23112123	13221123	23121123	23211123
31112223	31121223	31211223	32111223	31122123	31212123	32112123	31221123	32121123	32211123
11122332	11212332	12112332	21112332	11221332	12121332	21121332	12211332	21211332	22111332
11123232	11213232	12113232	21113232	11223132	12123132	21123132	12213132	21213132	22113132
11132232	11231232	12131232	21131232	11232132	12132132	21132132	12231132	21231132	22131132
11312232	11321232	12311232	21311232	11322132	12312132	21312132	12321132	21321132	22311132
13112232	13121232	13211232	23111232	13122132	13212132	23112132	13221132	23121132	23211132
31112232	31121232	31211232	32111232	31122132	31212132	32112132	31221132	32121132	32211132
11123322	11213322	12113322	21113322	11223312	12123312	21123312	12213312	21213312	22113312
11132322	11231322	12131322	21131322	11232312	12132312	21132312	12231312	21231312	22131312
11312322	11321322	12311322	21311322	11322312	12312312	21312312	12321312	21321312	22311312
13112322	13121322	13211322	23111322	13122312	13212312	23112312	13221312	23121312	23211312
31112322	31121322	31211322	32111322	31122312	31212312	32112312	31221312	32121312	32211312
11133222	11233122	12133122	21133122	11233212	12133212	21133212	12233112	21233112	22133112
11313222	11323122	12313122	21313122	11323212	12313212	21313212	12323112	21323112	22313112
13113222	13123122	13213122	23113122	13123212	13213212	23113212	13223112	23123112	23213112
31113222	31123122	31213122	32113122	31123212	31213212	32113212	31223112	32123112	32213112
11331222	11332122	12331122	21331122	11332212	12331212	21331212	12332112	21332112	22331112
13131222	13132122	13231122	23131122	13132212	13231212	23131212	13232112	23132112	23231112
31131222	31132122	31231122	32131122	31132212	31231212	32131212	31232112	32132112	32231112
13311222	13312122	13321122	23311122	13312212	13321212	23311212	13322112	23312112	23321112
31311222	31312122	31321122	32311122	31312212	31321212	32311212	31322112	32312112	32321112
33111222	33112122	33121122	33211122	33112212	33121212	33211212	33122112	33212112	33221112