OPTIMAL BALANCED 2^7 FRACTIONAL FACTORIAL DESIGNS OF RESOLUTION V, WITH $N \leqq 42$

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Summary

In this paper, we present a class of fractional factorial designs of the 2^7 series, which are of resolution V. Such designs allow the estimation of the general mean, the main effects and the two factors interactions (29 parameters in all for the 2^7 factorial) assuming that the higher order effects are negligible. For every value of N (the number of runs) such that $29 \leqq N \leqq 42$, we give a resolution V design that is optimal (with respect to the trace criterion) within the subclass of balanced designs. Also, for convenience of analysis, we present for each design, the covariance matrix of the estimates of the various parameters. As a by product, we establish many interesting combinatorial theorems concerning balanced arrays of strength four (which are generalizations of orthogonal arrays of strength four, and also of balanced incomplete block designs with block sizes not necessarily equal).

1. Introduction and preliminaries

Since we shall be concerned with 2^7 fractional factorial designs of resolution V, we shall have 29 ($=\nu$, say) parameters in all. These include the general mean μ, the main effect A_i ($i=1,\cdots,7$), and the 2-factor interactions A_{ij} ($i<j$; $i, j=1,\cdots,7$); the higher order effects are assumed negligible. A 2^7 design T is of resolution V, if and only if the above 29 parameters are estimable. The (29×29) covariance matrix of the estimates of the parameters obtained by using T, may be denoted by V_T. If the corresponding "information matrix" is M_T, then $V_T=(M_T)^{-1}$. (Recall that M_T occurs in the normal equations: $M_T\ \boldsymbol{p}=\boldsymbol{z}$, where $\boldsymbol{p}$ (29×1) is the vector of parameters, and the elements of $\boldsymbol{z}$ (29×1) are linear functions of the observations.) A design T is 'balanced' if V_T is invariant with respect to the permutation of factors, i.e. the quantities $\mathrm{Var}(\hat{A}_i)$, $\mathrm{Var}(\hat{A}_{ij})$, $\mathrm{Cov}(\hat{\mu}, \hat{A}_i)$, $\mathrm{Cov}(\hat{\mu}, \hat{A}_{ij})$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_j)$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_{ij})$, $\mathrm{Cov}(\hat{A}_i, \hat{A}_{jk})$, $\mathrm{Cov}(\hat{A}_{ij}, \hat{A}_{ik})$ and $\mathrm{Cov}(\hat{A}_{ij}, \hat{A}_{kl})$ are independent

of i, j, k, l (assumed to be all distinct). To give an equivalent condition for balance, we first observe that a 2^m design T with N runs can be represented by a $(m\times N)$ matrix T with elements 0 and 1, where the rows correspond to factors and columns correspond to runs or assemblies. Then it is known (e.g. Srivastava [6]) that a necessary and sufficient condition for T to be balanced is the following. For every $(4\times N)$ submatrix T_0 of T, and for every (4×1) vector $\boldsymbol{u}$ in which 1 occurs i times $(i=0, 1, \cdots, 4)$ and 0 occurs $(4-i)$ times, the number of times $\boldsymbol{u}$ occurs as a column of T_0 must be μ_i (a nonnegative integer). The vector $\boldsymbol{\mu}'=(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$ is called the index set of the balanced design T, which is also called a 'balanced array of strength 4'. In case the μ_i are all equal to (say) μ, T becomes an orthogonal array of strength 4, index μ, 7 'constraints', and N 'assemblies'.

Srivastava and Chopra [7] studied the characteristic polynomial of M_T for the general 2^m factorial, when T is balanced. In particular, they obtained $\operatorname{tr} V_T$ as a function of $\boldsymbol{\mu}'$, which is given below for $m=7$ for later use.

THEOREM 1.1. *Let there be a balanced 2^7 design T with index set $\boldsymbol{\mu}'=(\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$. Then, we have*

$$\text{(1.1)}\quad \text{(a)}\qquad \operatorname{tr} V_T=\frac{c_2}{c_3}+\frac{6c_4}{c_5}+\frac{7}{8\mu_2}\,,$$

where

$$\text{(1.2)}\qquad \begin{aligned} c_2&=3\gamma_1^2+32\gamma_1\gamma_3+20\gamma_1\gamma_5+39\gamma_3^2+60\gamma_3\gamma_5-7\gamma_2^2-3(2\gamma_2+5\gamma_4)^2\\ c_3&=\gamma_1^3-126\gamma_3^3+39\gamma_1\gamma_3^2+16\gamma_1^2\gamma_3+10\gamma_1^2\gamma_5+60\gamma_1\gamma_3\gamma_5-7\gamma_1\gamma_2^2-70\gamma_2^2\gamma_5\\ &\quad+14\gamma_2^2\gamma_3+210\gamma_2\gamma_3\gamma_4-3\gamma_1(2\gamma_2+5\gamma_4)^2\\ c_4&=2\gamma_1+2\gamma_3-4\gamma_5\\ c_5&=(\gamma_1-\gamma_3)(\gamma_1+3\gamma_3-4\gamma_5)-5(\gamma_2-\gamma_4)^2\,,\end{aligned}$$

where

$$\text{(1.3)}\qquad \begin{aligned} \gamma_1&=N=\mu_0+4\mu_1+6\mu_2+4\mu_3+\mu_4=\mu''+4\mu'+6\mu_2\\ \gamma_2&=(\mu_4-\mu_0)+2(\mu_3-\mu_1)=-\mu_0''-2\mu_0'\\ \gamma_3&=\mu_4-2\mu_2+\mu_0=\mu''-2\mu_2\\ \gamma_4&=(\mu_4-\mu_0)-2(\mu_3-\mu_1)=-\mu_0''+2\mu_0'\\ \gamma_5&=\mu_0-4\mu_1+6\mu_2-4\mu_3+\mu_4=\mu''-4\mu'+6\mu_2\,,\end{aligned}$$

where

$$\mu''=\mu_0+\mu_4\,,\quad \mu'=\mu_1+\mu_3\,,\quad \mu_0''=\mu_0-\mu_4\,,\quad \mu_0'=\mu_1-\mu_3\,;$$

and

$$c_2 \geqq 0\,,\quad c_3 \geqq 0\,,\quad c_4 \geqq 0 \quad \textit{and} \quad c_5 \geqq 0\,. \tag{1.4}$$

The above study helps in the analytical (and to some extent, the combinatorial) aspects of the problem of finding optimal balanced fractional factorial designs. However, the combinatorial aspect of this problem, being more difficult, needed further investigation. To help in this, some studies were made in Srivastava [5]; we quote some of these below for later use. To avoid repetition, we shall assume, for Theorems 1.2–1.6, that we are considering a balanced array $T\,(m \times N)$ of strength 4 and index set $\boldsymbol{\mu}' = (\mu_0, \mu_1, \mu_2, \mu_3, \mu_4)$.

THEOREM 1.2 (a) *Let $m=5$. A necessary and sufficient condition for the existence of T is that there exists an integer d such that*

$$\begin{aligned} d &\geqq \phi_{11} \equiv \phi_{11}(\boldsymbol{\mu}) \equiv \max\,(0,\, \mu_4-\mu_3,\, \mu_4-\mu_3+\mu_2-\mu_1) \\ d &\leqq \phi_{12} \equiv \phi_{12}(\boldsymbol{\mu}) \equiv \min\,(\mu_4,\, \mu_2-\mu_3+\mu_4,\, \mu_0-\mu_1+\mu_2-\mu_3+\mu_4)\,. \end{aligned} \tag{1.5}$$

(b) *Let $m=6$. A necessary and sufficient condition for the existence of T is that there exists an integer d_0 such that*

$$\begin{aligned} d_0 &\geqq \phi_{21} \equiv \phi_{21}(\boldsymbol{\mu}) \equiv \max\,(0,\, d_{1\cdot 2}+\theta_1,\, d_{1\cdot 4}+\theta_2,\, d_{1\cdot 6}+\theta_3)\,, \\ d_0 &\leqq \phi_{22} \equiv \phi_{22}(\boldsymbol{\mu}) \equiv \min\,(d_6,\, d_{4\cdot 6}-\theta_4,\, d_{2\cdot 6}-\theta_5)\,, \end{aligned} \tag{1.6}$$

where $d_{i\cdot j} = d_i + d_{i+1} + \cdots + d_j$, $j \geqq i$; for each i, $d = d_i$ satisfies (1.5); *$d_1 \geqq d_2 \geqq \cdots \geqq d_6$; and the θ's are given by*

$$\begin{aligned} &\theta_1 = -\mu_4\,,\quad \theta_2 = -\mu_2+2\mu_3-3\mu_4\,,\quad \theta_3 = -\mu_0+2\mu_1-3\mu_2+4\mu_3-5\mu_4\,, \\ &\theta_4 = -\mu_3+2\mu_4\,,\quad \theta_5 = -\mu_1+2\mu_2-3\mu_3+4\mu_4\,. \end{aligned} \tag{1.7}$$

(c) *The conditions $\phi_{21} \leqq \phi_{22}$ are equivalent to the following* 12 *inequalities:*

$$\begin{array}{c} (\mathrm{a}) \\ (\mathrm{b}) \\ (\mathrm{c}) \\ (\mathrm{d}) \\ (\mathrm{e}) \\ (\mathrm{f}) \\ (\mathrm{g}) \\ (\mathrm{h}) \\ (\mathrm{i}) \\ (\mathrm{j}) \\ (\mathrm{k}) \\ (\mathrm{l}) \end{array} \begin{bmatrix} d_6 \\ -d_1-d_2+d_6 \\ -d_1-d_2-d_3-d_4+d_6 \\ -d_1-d_2-d_3-d_4-d_5 \\ d_4+d_5+d_6 \\ -d_1-d_2+d_4+d_5+d_6 \\ -d_1-d_2-d_3+d_5+d_6 \\ -d_1-d_2-d_3 \\ d_2+d_3+d_4+d_5+d_6 \\ -d_1+d_3+d_4+d_5+d_6 \\ -d_1+d_5+d_6 \\ -d_1 \end{bmatrix} \geqq \begin{bmatrix} 0 \\ -\mu_4 \\ -\mu_2+2\mu_3-3\mu_4 \\ -\mu_0+2\mu_1-3\mu_2+4\mu_3-5\mu_4 \\ -\mu_3+2\mu_4 \\ \mu_4-\mu_3 \\ -\mu_2+\mu_3-\mu_4 \\ -\mu_0+2\mu_1-3\mu_2+3\mu_3-3\mu_4 \\ -\mu_1+2\mu_2-3\mu_3+4\mu_4 \\ -\mu_1+2\mu_2-3\mu_3+3\mu_4 \\ -\mu_1+\mu_2-\mu_3+\mu_4 \\ -\mu_0+\mu_1-\mu_2+\mu_3-\mu_4 \end{bmatrix}. \tag{1.8}$$

Now, for $m=6$, and for a fixed (given) $\boldsymbol{\mu}'$, let there be g distinct values of the vector $(d_0, \boldsymbol{d}')=(d_0, d_1, \cdots, d_6)$ which satisfy (1.6). Let these be denoted by $(d_{0r}, d_{1r}, \cdots, d_{6r})$, $r=1, \cdots, g$. (Note that, in particular, $g=1$, if $\phi_{11}=\phi_{12}$ and $\phi_{21}=\phi_{22}$.) Let $\bar{d}_r=(1/6)(d_{1r}+d_{2r}+\cdots+d_{6r})$. Define

$$
\begin{aligned}
(1.9)\qquad \delta_{r0}&=(\mu_0-2\mu_1+3\mu_2-4\mu_3+5\mu_4)-6\bar{d}_r+d_{0r}=-\theta_3-6\bar{d}_r+d_{0r}\ ,\\
\delta_{r1}&=(\mu_1-2\mu_2+3\mu_3-4\mu_4)+5\bar{d}_r-d_{0r}=-\theta_5+5\bar{d}_r-d_{0r}\ ,\\
\delta_{r2}&=(\mu_2-2\mu_3+3\mu_4)-4\bar{d}_r+d_{0r}=-\theta_2-4\bar{d}_r+d_{0r}\ ,\\
\delta_{r3}&=(\mu_3-2\mu_4)+3\bar{d}_r-d_{0r}=-\theta_4+3\bar{d}_r-d_{0r}\ ,\\
\delta_{r4}&=\mu_4-2\bar{d}_r+d_{0r}=-\theta_1-2\bar{d}_r+d_{0r}\ ,\\
\delta_{r5}&=\bar{d}_r-d_{0r}\ ,\\
\delta_{r6}&=d_{0r}\ .
\end{aligned}
$$

THEOREM 1.3. *Suppose T exists with $m=7$. Let x_i $(i=0, 1, \cdots, 7)$ denote the number of columns of T each of which is of weight i. (By 'weight i', we mean that the column has i ones and $(7-i)$ zeros in it.) Then the x_i must satisfy the following "Single Diophantine Equations" (SDE).*

$$
\begin{aligned}
(1.10)\qquad &(\mathrm{a})\quad 35x_0+15x_1+5x_2+x_3=35\mu_0\\
&(\mathrm{b})\quad 5x_1+5x_2+3x_3+x_4=35\mu_1\\
&(\mathrm{c})\quad 5x_2+9x_3+9x_4+5x_5=105\mu_2\\
&(\mathrm{d})\quad x_3+3x_4+5x_5+5x_6=35\mu_3\\
&(\mathrm{e})\quad x_4+5x_5+15x_6+35x_7=35\mu_4\ .
\end{aligned}
$$

(In the sequel, a design T with $x_0=x_7=0$ is called a 'trim' design.)

THEOREM 1.4. *Suppose T exists with $m=7$. Also suppose $\boldsymbol{\mu}'$ is such that (for 6-rowed arrays) g distinct values of $(d_0, \boldsymbol{d}')$ correspond to it. (Clearly $g>0$, since T exists and T is 7-rowed.) Then there exist nonnegative integers $y_1, \cdots, y_g$, with $\sum_1^g y_r=7$, such that the following "Triple Diophantine Equations" (TDE) are satisfied. Below, $\pi_j=\sum_{r=1}^g \delta_{rj}y_r$, $(j=0, 1, \cdots, 6)$.*

$$
\begin{aligned}
(1.11)\qquad &(\mathrm{a})\quad 7x_0+x_1=\pi_0\ , &&(\mathrm{b})\quad 6x_1+2x_2=6\pi_1\ ,\\
&(\mathrm{c})\quad 5x_2+3x_3=15\pi_2\ , &&(\mathrm{d})\quad 4x_3+4x_4=20\pi_3\ ,\\
&(\mathrm{e})\quad 3x_4+5x_5=15\pi_4\ , &&(\mathrm{f})\quad 2x_5+6x_6=6\pi_5\ ,\\
&(\mathrm{g})\quad x_6+7x_7=\pi_6\ .
\end{aligned}
$$

THEOREM 1.5. *Under the conditions of Theorem* 1.4, *suppose that* $g=1$. *Then the TDE are reduced to what we call "Double Diophantine Equations" (DDE). These correspond to equations* (1.11) *with* $\pi_j=7\delta_j$ $(j=0, 1, \cdots, 6)$. *(Here the suffix* 1 *in* δ_{1j} $(j=0, \cdots, 6)$ *has been dropped for convenience.)*

Using some of the above results, optimal balanced 2^m designs were obtained for $m=4, 5, 6$, and for each value of N in a practical range, in Srivastava and Chopra [8]. By 'optimal' we mean that in the class of all balanced designs, T is chosen such that $\mathrm{tr}\, V_T$ is a minimum. In this paper, we shall obtain similar designs for the case $m=7$.

For brevity, we omit detailed introduction to the theory of optimal balanced designs. The reader interested in previous work should look into the bibliography at the end, and the further references therein.

2. Optimal balanced designs

In Table I, we give the optimal balanced 2^7 designs of resolution V, for every N, with $29 \leqq N \leqq 42$. In general, for a given N, one could have more than one distinct (non-isomorphic) balanced designs. However, in our case, for every N, there is only one optimal design, the value of μ' corresponding to which is indicated in the 2nd column of Table I.

Fortunately, the structure of the optimal designs obtained here is quite simple. Each design is expressible simply by a set of 8 nonnegative integers $(\lambda_0, \lambda_1, \cdots, \lambda_7)$, such that in a design with these parameters, every column vector with i zeros $(i=0, 1, \cdots, 7)$ and $(7-i)$ ones

Table I Optimal balanced designs of the 2^7 series

N	μ'	λ_7	λ_6	λ_5	λ_4	λ_3	λ_2	λ_1	λ_0
29	43113	1	0	1	0	0	0	1	0
30	53113	2	0	1	0	0	0	1	0
31	63113	3	0	1	0	0	0	1	0
32	73113	4	0	1	0	0	0	1	0
33	73114	4	0	1	0	0	0	1	1
34	83114	5	0	1	0	0	0	1	1
35	93114	6	0	1	0	0	0	1	1
36	43126	1	0	1	0	0	0	2	0
37	53126	2	0	1	0	0	0	2	0
38	63126	3	0	1	0	0	0	2	0
39	73126	4	0	1	0	0	0	2	0
40	83126	5	0	1	0	0	0	2	0
41	83127	5	0	1	0	0	0	2	1
42	83128	5	0	1	0	0	0	2	2

Table II Covariance matrices of optimal balanced 2^7 factorial designs

μ'	N	tr V	V $(\hat{\mu})$	Cov $(\hat{\mu}, \hat{A}_i)$	Cov $(\hat{\mu}, \hat{A}_{ij})$	V $(\hat{A}_i)$	Cov $(\hat{A}_i, \hat{A}_j)$	Cov $(\hat{A}_i, \hat{A}_{ij})$	Cov $(\hat{A}_i, \hat{A}_{jk})$	V $(\hat{A}_{ij})$	Cov $(\hat{A}_{ij}, \hat{A}_{ik})$	Cov $(\hat{A}_{ij}, \hat{A}_{kl})$
43113	29	1.4861	.0763	.0086	−.0086	.0503	−.0043	.0043	−.0034	.0503	−.0043	.0078
53113	30	1.4479	.0624	.0052	−.0052	.0494	−.0052	.0052	−.0026	.0494	−.0052	.0039
63113	31	1.4351	.0578	.0040	−.0040	.0491	−.0054	.0054	−.0023	.0491	−.0054	.0026
73113	32	1.4288	.0555	.0034	−.0034	.0490	−.0056	.0056	−.0021	.0490	−.0056	.0019
73114	33	1.4248	.0555	.0033	−.0035	.0487	−.0059	.0054	−.0023	.0489	−.0057	.0016
83114	34	1.4210	.0540	.0029	−.0032	.0486	−.0060	.0055	−.0022	.0488	−.0058	.0013
93114	35	1.4185	.0531	.0026	−.0029	.0485	−.0061	.0056	−.0021	.0488	−.0058	.0010
43126	36	1.3098	.0703	.0065	−.0091	.0345	−.0026	.0000	−.0019	.0475	−.0052	.0109
53126	37	1.2717	.0564	.0030	−.0056	.0336	−.0034	.0008	−.0010	.0466	−.0060	.0059
63126	38	1.2589	.0517	.0018	−.0044	.0333	−.0037	.0011	−.0007	.0463	−.0063	.0042
73126	39	1.2526	.0494	.0013	−.0039	.0332	−.0039	.0013	−.0006	.0462	−.0065	.0034
83126	40	1.2487	.0480	.0009	−.0035	.0331	−.0039	.0013	−.0005	.0461	−.0065	.0029
83127	41	1.2467	.0477	.0011	−.0034	.0330	−.0040	.0013	−.0006	.0460	−.0066	.0026
83128	42	1.2452	.0475	.0012	−.0033	.0329	−.0041	.0012	−.0006	.0460	−.0066	.0024

occurs exactly λ_i times. Thus, in Table I, we simply present the λ's for each design.

Also, by definition, the covariance matrix V_T of a balanced design T has at most 10 distinct elements. For each design T in Table I, the distinct elements of V_T are given in Table II.

It would be useful to mention here briefly the nature of our process of obtaining optimal designs. First of all, for each N, we shall obtain all the values of $\boldsymbol{\mu}'$ for which trim designs can possibly exist i.e. all $\boldsymbol{\mu}'$ which satisfy the conditions (1.4), (1.5), (1.8), (1.11) with $x_0=x_7=0$. Next, suppose that for some value of N, say $N=N_0$, we have a value of $\boldsymbol{\mu}'$, say $\boldsymbol{\mu}'=\boldsymbol{\mu}^{*\prime}=(\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$, which corresponds to a possibly existent trim design. Then, for any $N_1>N_0$, there corresponds to this trim design a not-trim design with $\boldsymbol{\mu}'=(\mu_0^*+\alpha, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*+\beta)$, where $\alpha\geqq 0$, $\beta\geqq 0$, and $\alpha+\beta=N_1-N_0$. The reason is that given the trim design, the said not-trim design could be obtained by adjoining α columns of weight 0 each, and β columns of weight 7 each. Thus, for each N, we obtain the set of all values of $\boldsymbol{\mu}'$ which correspond to possibly existent designs (both trim and otherwise). For any given N, we then calculate $\mathrm{tr}\, V$ corresponding to every such value of $\boldsymbol{\mu}'$, and find the value of $\boldsymbol{\mu}'$ which minimizes $\mathrm{tr}\, V$. Finally, we construct designs which correspond to such optimal values of $\boldsymbol{\mu}'$.

We now begin the combinatorial investigations.

3. Combinatorial analysis of trim designs leading to optimal balanced designs

In this section, we make certain investigations on trim designs, which in turn is helpful in obtaining optimal designs.

Recall the x_i $(i=0, 1, \cdots, 7)$ of Theorem 1.3. Since we shall consider trim designs alone, we have $x_0=x_7=0$. Now, define

$$\begin{aligned} (3.1)\qquad & u=x_1+x_6\,, \quad v=x_2+x_5\,, \quad w=x_3+x_4\,, \\ & u'=x_1-x_6\,, \quad v'=x_2-x_5\,, \quad w'=x_3-x_4\,, \\ & \mu'=\mu_1+\mu_3\,, \quad \mu_0'=\mu_1-\mu_3\,, \quad \mu''=\mu_0+\mu_4\,, \quad \mu_0''=\mu_0-\mu_4\,. \end{aligned}$$

Equations (1.10 a, e), (1.10 b, d), (1.10 c), (1.10 a)-(1.10 c) and (1.10 b)-(1.10 d) give

$$\begin{aligned} (3.2)\qquad & \text{(a)}\ \ 15u+5v+w=35\mu''\,, \qquad \text{(b)}\ \ 15u'+5v'+w'=35\mu_0''\,, \\ & \text{(c)}\ \ 5u+5v+4w=35\mu'\,, \qquad \text{(d)}\ \ 5u'+5v'+2w'=35\mu_0'\,, \\ & \text{(e)}\ \ 5v+9w=105\mu_2\,. \end{aligned}$$

Equations (3.2 a, c, e), when solved for u, v, w, give

(3.3) $u=9\mu_2+5\mu''-8\mu'\geqq 0$ i.e. $\mu''\geqq\frac{8}{5}\mu'-\frac{9}{5}\mu_2$.

(3.4) $v=3[9\mu'-11\mu_2-3\mu'']\geqq 0$ i.e. $\mu''\leqq 3\mu'-\frac{11}{3}\mu_2$.

(3.5) $w=5[6\mu_2-3\mu'+\mu'']\geqq 0$ i.e. $\mu''\geqq 3\mu'-6\mu_2$.

Similarly (3.2 b, d) give

(3.6a) $10u'-w'=35(\mu_0''-\mu_0')$ i.e. $35\,|\,(10u'-w')$

(3.6b) $2v'+w'=7(3\mu_0'-\mu_0'')$ i.e. $7\,|\,(2v'+w')$.

Finally, we observe that

(3.7) (i) $(\mu'\pm\mu_0')$, $(\mu''\pm\mu_0'')$, $(u\pm u')$, $(v\pm v')$, $(w\pm w')$ are even,

(ii) $|u'|\leqq u$, $|v'|\leqq v$, $|w'|\leqq w$, $|\mu_0'|\leqq\mu'$, $|\mu_0''|\leqq\mu''$.

Also, from (3.3)-(3.5) we have

(3.8) $$\mu'\geqq\frac{4}{3}\mu_2\,,\qquad \mu''\geqq\frac{1}{3}\mu_2\,.$$

THEOREM 3.1. *The number of assemblies N in a balanced trim design with $m\geqq 7$ and index set $\boldsymbol{\mu}'$ must satisfy*

(3.9) $$\max\left(\frac{21}{5}\mu_2+\frac{28}{5}\mu',\,7\mu'\right)\leqq N\leqq\min\left(14\mu_2+\frac{7}{3}\mu'',\,\frac{7}{3}\mu_2+7\mu'\right)\,.$$

PROOF. This follows immediately from (3.3)-(3.5) and (1.3).

THEOREM 3.2. *If T is a trim design with $m\geqq 7$, then $\mu_2=3$ and* 4 *respectively implies $N\geqq 35$ and* 51. *Also $\mu_2\geqq 5$ implies $N\geqq 60$.*

PROOF. The results for $\mu_2=3$, 4, 5 follow from (3.3), (3.8), and (3.9).

Since we will be interested in constructing optimal designs with $29\leqq N\leqq 42$ we shall not, in view of Theorem 3.2, consider designs with $\mu_2\geqq 4$.

THEOREM 3.3. (a) *There does not exist any trim design with $m=7$, $\mu_2=1$ and the following set of values of N:* (i) $24\leqq N\leqq 27$, (ii) $31\leqq N\leqq 34$, (iii) $38\leqq N\leqq 41$ *and* (iv) $45\leqq N\leqq 48$. (b) *For the remaining values of N in the range $27\leqq N\leqq 50$ (i.e. when $N=28, 29, 30$; $35, 36, 37$; $42, 43, 44$; $49, 50$), the values of (μ',μ'') corresponding to which designs might possibly exist are* (4,6), (4,7), (4,8), (5,9), (5,10), (5,11), (6,12), (6,13), (6,14), (7,15) *and* (7,16).

PROOF. These results follow from Theorem 3.1.

THEOREM 3.4. *If a trim design T with $m=7$ and $\mu_2=1$ exists, then $\mu''\geqq\mu'$.*

PROOF. Let $x=\mu_0-\mu_1$, $y=\mu_4-\mu_3$. From (1.5), we have $\phi_{12}=\min(\mu_4, 1+y, 1+y+x)\geqq d\geqq\max(0, +y, 1+y-\mu_1)=\phi_{11}$. This implies $1+y+x\geqq0$. If possible, let $\mu_1+\mu_3>\mu_0+\mu_4$, i.e. $x+y<0$. Thus we can have either (a) $x=0$, $y=-1$, or (b) $x=-1$, $y=0$. We shall present the proof only for case (a). The proof for (b) follows from that of (a) simply by interchanging 0 and 1 in T. Now, for $(x, y)=(0, -1)$, we get $\phi_{11}=\phi_{12}=0$. Hence $d_i=0$ for all i, and $\boldsymbol{d}'$ is the zero vector. Next using (1.8 c, d, e), we find that the only permissible value of $\boldsymbol{\mu}'$ is (0, 0, 1, 2, 1). However, because of Theorem 3.1, this value also does not correspond to any existent trim design.

THEOREM 3.5. *Let there exist a trim design T with $m\geqq7$, and $\mu_2=1$. Then $\mu_0\geqq\mu_1$ and $\mu_4\geqq\mu_3$.*

PROOF. We just showed that, under the stated conditions, we must have $\mu_0+\mu_4\geqq\mu_1+\mu_3$. Now, let $\mu_0<\mu_1$. Then, by the last theorem, $\mu_4>\mu_3$, and $x<0$, $y>0$. From (3.10), we then have $\phi_{11}=y\leqq1+y+x$, implying $x+1\geqq0$. Thus $x=-1$, and $\phi_{11}=\phi_{12}=y=d_1=d_6$. Then (1.8 h, j) shows that $\boldsymbol{\mu}'$ is of the form $(1, 2, 1, \mu_4-y, \mu_4)$, and hence $\phi_{21}=\phi_{22}=+2y-\mu_4=d_0$, $\bar{d}=+y$. Thus $\delta_0=\delta_1=\delta_3=0$, $\delta_2=1$, which contradicts the DDE (Theorem 1.5).

THEOREM 3.6. *For a trim design T with $m=7$ and $\mu_2=1$, one of the following sets of conditions must be satisfied:*

(i) $\{\mu_0=\mu_1=0, \mu_3\geqq3, 3\mu_3=\mu_4+6\}$,

(ii) $\{\mu_0=3\mu_1, \mu_3\geqq3$ *and* $3\mu_3=\mu_4+6\}$, *or* $\{3\mu_1=\mu_0+6, \mu_1\geqq3, \mu_4=3\mu_3\}$,

(iii) $\{\mu_1\geqq3, \mu_3=\mu_4=0$ *and* $3\mu_1=\mu_0+6\}$.

PROOF. From (1.5) and Theorem 3.5, we have $\phi_{11}=1+y$, or y, according as $\mu_1=0$, or $\geqq1$; $\phi_{12}=1+y$, or μ_4, according as $\mu_3\geqq1$, or $\mu_3=0$. The pair $(\phi_{11}, \phi_{12})=(1+y, \mu_4)$ with $\mu_1=\mu_3=0$ is not possible because $\phi_{11}\leqq\phi_{12}$. We discuss the remaining three cases one by one. (Case I) $(\phi_{11}, \phi_{12})=(1+y, 1+y)$ with $\mu_1=0$, $\mu_3\geqq1$. Then $d_1=d_6=1+y$; and (1.8 c, i) gives $\mu_3\geqq2$, and $\mu_4\geqq2\mu_3-3$. Hence $(d_0; \bar{d})=(\mu_4-2\mu_3+3; 1+\mu_4-\mu_3)$, so that $\boldsymbol{\delta}'\equiv(\mu_0, 0, 0, 0, 1, \mu_3-2, \mu_4-2\mu_3+3)$. Using this in the DDE gives $\mu_0=0$, and $x_5=21$, $x_6=7(\mu_3-3)=7\delta_6$, which leads to (i). (Case II) $(\phi_{11}, \phi_{12})=(y, 1+y)$ with $\mu_1\geqq1$, $\mu_3\geqq1$, we find that (1.8 f, g) imply $d_1=d_6=1+y$ or $d_1=d_6=y$. Take the first case. In order that (1.8) be satisfied, we must have $\mu_3\geqq2$, $\mu_0\geqq2\mu_1$ and $\mu_4+3\geqq2\mu_3$. Then $\phi_{21}=\phi_{22}=\mu_4-2\mu_3+3$ and $\boldsymbol{\delta}'=(\mu_4-2\mu_3+3, \mu_3-2, 1, 0, 0, \mu_1, \mu_0-2\mu_1)$. Substituting these in the

DDE, we get $\mu_0=3\mu_1$, $\mu_3\geqq 3$ and $3\mu_3=\mu_4+6$. The other result of (ii), is similarly obtained starting with $d_1=d_6=y$. (Case III). Now, let $(\phi_{11}, \phi_{12})=(\mu_4, \mu_4)$, where $\mu_1\geqq 1$ and $\mu_3=0$. Here $d_0=\mu_4$, $\mu_0\geqq 2\mu_1-3$, $\mu_1\geqq 2$ and $\boldsymbol{\delta}'=(\mu_4, 0, 0, 0, 1, \mu_1-2, \mu_0-2\mu_1+3)$. This implies $x_2=21$, $x_3=x_4=x_5=x_6=0$, $x_1=7\delta_0=7(\mu_1-3)$, leading to (iii).

COROLLARY 3.1. *For a trim design T with $m=7$ and $\mu_2=1$, μ'' is divisible by* 3.

This corollary in conjunction with Theorem 3.3 (b) and a re-application of Theorem 3.6 gives

COROLLARY 3.2. *If a trim design with $m=7$ and $\mu_2=1$, exists, then (μ', μ'') must take one of the values* (4,6), (5,9), (6,12) *and* (7,15), *with $N=28$, 35, 42, and 49, respectively. This shows that N must be a multiple of* 7, *and that corresponding to a given such N, the index set $\boldsymbol{\mu}'$ must be of the form $(3\rho, \rho, 1, N/7-\rho, 3N/7-3\rho-6)$, where the integer ρ satisfies $N/7-3\geqq\rho\geqq 0$. (Note that if an array with index set $(\mu_0^*, \mu_1^*, \mu_2^*, \mu_3^*, \mu_4^*)$ exists, then one with the 'reverse' index set $(\mu_4^*, \mu_3^*, \mu_2^*, \mu_1^*, \mu_0^*)$ can be obtained from it by interchanging* 0 *and* 1. *In the above, we have therefore presented one index set in each case and ignored the reverse one.)*

THEOREM 3.7. *Let T be a trim design with $m=7$, $\mu_2=2$. Then* (a) $\mu'\geqq 3$, (b) *when $\mu'=4$, the possible values of N are* 31, 32, *and for $\mu'=5$, N must equal* 37, 38 *or* 39, (c) *$\mu'\geqq 6$ implies $N\geqq 42$.*

PROOF. Part (a) follows from (3.8). Also, when $\mu'=3$, (3.9) gives $(126/5)\leqq N\leqq(77/3)$, which is impossible since N is an integer. When $\mu'=4$, (3.9) gives $N=31$ or 32. For $\mu'=5$, we get $37\leqq N\leqq 39$, and for $\mu'\geqq 6$ we get $N\geqq 42$.

THEOREM 3.8. *Let T be a trim design with $m=7$ and $\mu_2=2$. Then $\mu''\geqq\mu'$.*

PROOF. Let $x=\mu_1-\mu_0$, $y=\mu_3-\mu_4$, so that $x+y=\mu'-\mu''$. Conditions (1.5) then reduce to

$$(3.10)\quad \phi_{12}=\min(\mu_4, 2-y, 2-y-x)\geqq d\geqq\max(0, -y, 2-y-\mu_1)=\phi_{11}\,.$$

If possible let $\mu'>\mu''$, i.e. $x+y\geqq 1$. This can hold under any one of the following conditions: (a) $x>0$, $y>0$, (b) $x=0$, $y>0$, (c) $x>0$, $y<0$, (d) $x>0$, $y=0$, (e) $x<0$, $y>0$. We shall present the proof only for the cases (a), (b) and (c). Cases (d) and (e) follow from (b) and (c) respectively by symmetry, i.e. by interchanging 0 and 1 in T.

Consider (a). From (3.10), we have $\mu''-\mu'\geqq d-2\geqq -2$. Thus $x+$

$y \leqq 2$, which for case (a) implies $x=y=1$, and $d=0$. Hence we get $\phi_{12}=\min(\mu_4, 1, 0) \geqq 0 \geqq \max(0, -1, 1-\mu_1)=\phi_{11}$. Now $\mu_1 > \mu_0 \geqq 0$. Hence $\phi_{11}=0$. Thus $\phi_{12}=0$, and $d_1=d_6=0$. Using this in (1.8 d), we get $\mu_4=0$ or 1. Again, since $x=y=1$, (1.8 d) gives $\mu_0-\mu_4 \leqq 0$. Hence $(\mu_0, \mu_4)=(0, 0)$, (0, 1) or (1, 1), which implies $\mu'' \leqq 2$. Hence, because of Theorem 3.7 (a), no array exists under case (a).

Case (b). As in case (a), using (3.9), we get $x=0$ and $y=2$ or 1. For $y=2$, we find $\phi_{11}=\phi_{12}=0$, so that $d_1=0$. Then (1.8 h, c, e) implies $\mu_1=0$, $\mu_4=2$, and hence $\mu''=2$, which is impossible by Theorem 3.7 (a).

For $x=0$, $y=1$, (3.9) reduces to $\phi_{12}=\min(\mu_4, 1) \geqq d \geqq \max(0, 1-\mu_1)=\phi_{11}$. Hence $(\phi_{11}, \phi_{12})=(1, 1)$, (0, 0) or (0, 1). Consider the value (1,1). Then $d_1=d_6=1$, $\mu_1=0$. Hence $\mu_0=0$. Using (1.10) with $x_0=x_7=0$, and $\mu_2=2$, $\mu_0=\mu_1=0$, we find $x_5=42$, and $\mu_3 \geqq 6$, $\mu_4 \geqq 6$. But (1.8 e) gives $\mu_4 \leqq 4$, a contradiction.

Consider next $(\phi_{11}, \phi_{12})=(0, 0)$, so that $\mu_4=0$, and hence $\mu_3=1$. Equations (1.10 e, d, c) then give $x_2=-21$, a contradiction. Next, take the value (0, 1). Then, $\mu_4 \geqq 1$, $\mu_1 \geqq 1$, and $1 \geqq d_1 \geqq \cdots \geqq d_6 \geqq 0$. From (1.8 h), we have $d_1+d_2+d_3 \leqq 3-\mu_1 \leqq 2$. This and (1.8 f), implies that $\boldsymbol{d}'=(0, 0, 0, 0, 0, 0)$ or $(1, 0, 0, 0, 0, 0)$. These values of $\boldsymbol{d}'$; using (1.8 e), imply $\mu_4=1$, and hence $\mu_3=2$.

Next, (1.8 i) gives $\mu_1 \geqq 2$. Finally, using (1.8 d), we find that when $d_1=0$, $\boldsymbol{\mu}'=(2, 2, 2, 2, 1)$ or $(3, 3, 2, 2, 1)$, and when $d_1=1$, $d_2=0$, we have $\boldsymbol{\mu}'=(2, 2, 2, 2, 1)$. Now $\boldsymbol{\mu}'=(3, 3, 2, 2, 1)$ implies $N=36$ which is not possible by Theorem 3.7 (c). Also, the value (2, 2, 2, 2, 1) is not possible, since it would imply the existence of the (orthogonal) array (2, 2, 2, 2, 2), which is known not to exist.

Case (c). Here $x>0$, $y<0$, and, from (3.10), $2 \geqq x+y \geqq 1$, and $x \leqq 2$. Hence $x=2$, $y=-1$. Hence $\mu_1 \geqq x \geqq 2$. Also $\mu_4 \geqq -y=1$. Hence $\phi_{11}=\phi_{12}=1=d_1=d_6$. Then (1.8 e, d) give $\mu_4 \leqq 2$, and (1.8 d, i) imply $2 \leqq \mu_1-\mu_4 \leqq 3$. Hence the possible values of $\boldsymbol{\mu}'$ are (2, 4, 2, 1, 2), (3, 5, 2, 1, 2), (1, 3, 2, 0, 1) and (2, 4, 2, 0, 1). Of these, the first three imply respectively $N=36$, 41 and 26, and are rejected by Theorem 3.7 (b, c). The last one is rejected by (1.10 e, d), since $x_7=0$, and the other x's are non-negative. This completes the proof of the theorem.

THEOREM 3.9. *Let T be an array having $\mu_2=2$, $m \geqq 7$, $\mu_0+\mu_4=\mu_1+\mu_3$. Then $\mu_0=\mu_1$, and $\mu_3=\mu_4$.*

PROOF. Define x and y as in the last theorem, so that (3.10) still holds. Suppose the lemma does not hold, and suppose $x<0$. (The proof for the case when $x>0$ follows by symmetry.) Then $y=-x>0$. From (3.10), $y \leqq 2$. Hence $y=1$ or 2.

Case I: $y=2$. Then $\phi_{12}=0=\phi_{11}$. Hence $d_1=0$, and (1.8 c, e, h) imply $\mu_4=2$, and $\mu_1\leqq 2$. Hence $\boldsymbol{\mu}'=(\mu_1+2, \mu_1, 2, 4, 2)$ with $\mu_1\leqq 2$. For each such $\boldsymbol{\mu}'$, (1.7) and (1.6) give $\phi_{21}=\phi_{22}=0=d_0$; and hence $g=1$. Then (1.9) gives $\delta_3=\delta_5=0$, $\delta_4=2$, contradicting the DDE.

Case II: $y=1$. Here again, we can have $(\phi_{11}, \phi_{12})=(0, 0)$, $(1, 1)$ and $(0, 1)$. For the value $(0, 0)$, (3.10) implies $\mu_4=0$. Hence $\boldsymbol{\mu}'$ is of the form $(\mu_0, \mu_0-1, 2, 1, 0)$ giving $N=12+5\mu_0$, and $\mu'=\mu_0-1$. Using (3.9), we must then have $7(\mu_0-1)\leqq 12+5\mu_0\leqq 7(\mu_0-1)+(14/3)$, which implies $\mu_0 \leqq(19/2)$, and $\mu_0\geqq(43/6)$. Hence $\mu_0=8$ or 9, contradicting (1.10).

When $(\phi_{11}, \phi_{12})=(1, 1)$, we have $\mu_0=1$, and $\boldsymbol{\mu}'=(1, 0, 2, 1+\mu_4, \mu_4)$. Also, (1.8 c, e) give $3\leqq\mu_4\leqq 4$; and hence $N\leqq 37$. But, since $\mu_1=0$, (1.10 a, b, e) give $x_5=42>37$, a contradiction.

Finally, take $(\phi_{11}, \phi_{12})=(0, 1)$. Then $\mu_0\geqq 2$, $\mu_4\geqq 1$, and (3.9) gives $4 \leqq\mu'=\mu''\leqq 6$. Also $\boldsymbol{\mu}'=(\mu_0, \mu_0-1, 2, \mu_4+1, \mu_4)$. Using this in (1.10), we get $10x_1-2x_3-x_4=35$, $5(x_2+x_5)+9(x_3+x_4)=210$, and $x_3+2x_4-10x_6=35$. This gives $(x_1+x_6)\cdot(10/3)=(210/9)-(5/9)(x_2+x_5)=(x_3+x_4)$, and hence $N =42-(1/2)(x_3+x_4)$. When $\mu_4=6$, we get $N=42$, and hence $x_3+x_4=x_1+x_6=0$. Hence $35=10x_1-2x_3-x_4=-x_4$, a contradiction. Similarly, when $\mu'=5$, we have $N=37$, and hence $x_3+x_4=10$ and $x_1+x_6=3$. Then $35= 10x_1-x_3-10$, or $x_1\geqq 4.5$, which contradicts $x_1+x_6=3$. Finally, when $\mu' =4$, the same steps lead to $x_3+x_4=20$, $x_1+x_2=6$, and hence $10x_1-x_3= 55$. This implies $x_1=6$, $x_2=0$, $x_3=5$, $x_4=15$, and (1.10a) becomes $90+ 5x_2+5=35\mu_0$. Hence $\mu_0\neq 2$. Thus $\mu_0=3$, $\mu_4=1$, $\boldsymbol{\mu}'=(3, 2, 2, 2, 1)$, $x_2=2$, $x_5=4$.

When $\boldsymbol{\mu}'=(3, 2, 2, 2, 1)$, (1.8 b, c) give $\boldsymbol{d}'=(1, 0, 0, 0, 0, 0)$ or $(0, 0, 0, 0, 0, 0)$. In each of these two cases, (1.7), (1.6) give $\phi_{21}=\phi_{22}=0$, and hence $d_0=2$. Thus to the value $(3, 2, 2, 2, 1)$ of $\boldsymbol{\mu}'$, there correspond 2 possible values of $(d_0, \boldsymbol{d}')$. Hence $g=2$. From (1.9) $\delta_{15}=1/6$, and $\delta_{25}=0$, so that (1.11 f) gives $6\pi_5=6\cdot((1/6)y_1+0\cdot y_2)=2x_5+6x_6=2(4)+6(0)=8$, or $y_1 =8$. But, $y_1+y_2=7$, and y's are nonnegative integers. Thus we have a contradiction, and the proof of the theorem is completed.

THEOREM 3.10. *If T is a trim design with $m=7$ and $\mu_2=2$, then $\mu'\neq 4$.*

PROOF. Suppose $\mu'=4$. Then by Theorem 3.7 (b), $N=31$ or 32, i.e. $\mu''=3$ or 4. By Theorem 3.8, the case $\mu''=3$ is rejected. Also when $\mu''=4$, Theorem 3.9 gives $\boldsymbol{\mu}'=(\mu_1, \mu_1, 2, \mu_3, \mu_3)$. The possible values of (μ_1, μ_3) are $(0, 4)$, $(4, 0)$, $(1, 3)$, $(3, 1)$ and $(2, 2)$. Of these, the first two values are impossible because of (1.4) since $c_5\geqq 0$. Also $(\mu_1, \mu_3)=(2, 2)$ is impossible, since the orthogonal array $(2, 2, 2, 2, 2)$ does not exist when $m=7$. We are therefore left with the value $(1, 3)$, the case $(3, 1)$ being dealt with by symmetry. When $\mu_1=1$, (3.10) gives $(\phi_{11}, \phi_{12})=(1,$

2). Inequalities (1.8 c, f, g, j) imply $d_1=d_6=1$. But then (1.8 i) is contradicted. This completes the proof.

THEOREM 3.11. *If T is a trim design with $m=7$ and $\mu_2=2$, then $N\neq 37$.*

PROOF. Suppose $N=37$. Then $\mu'=5$. As in the last theorem, because of (1.4), and also in view of Theorem 3.9, the only values of $\boldsymbol{\mu}'$ which need to be considered (aside from symmetry considerations) are (1, 1, 2, 4, 4) and (1, 2, 2, 3, 3). Since in both cases, $\mu''=\mu'$, we use (1.10) as in the proof of Theorem 3.9 to obtain $x_3+x_4=10$, $x_1+x_6=3$, and $10x_1-x_3=45$, which leads to the contradiction $x_1\geqq 4.5$, and $x_1\leqq 3$.

THEOREM 3.12. *If T is a trim design with $m=7$, and $\mu_2=2$, then $N\neq 38$.*

PROOF. Suppose $N=38$. Theorem 3.7 gives $\mu'=5$, and hence $\mu''=6$. Using this in (3.2 a, c, e), we get $u=x_1+x_6=8$, $v=x_2+x_5=15$, $w=x_3+x_4=15$. Now, let the symbol $a|b$ mean "a divides b" where a and b are integers. Then from (1.10 a, b), we get $5|x_3$, $5|x_4$, and $7|(x_1-2x_2+x_3)$. Similarly, from (1.10 c, d) and (3.6 a), we have $7|(x_4-2x_5+x_6)$, and $7|\{2(x_2-x_5)+(x_3-x_4)\}$. Using these facts, it can be easily checked that the only possible solutions $\boldsymbol{x}'=(x_1, x_2, x_3, x_4, x_5, x_6)$ of (1.10) are, apart from (0, 1) symmetry, given by the following four vectors=(6, 8, 10, 5, 7, 2), (5, 13, 0, 15, 2, 3), (2, 14, 5, 10, 1, 6), (3, 9, 15, 0, 6, 5). Of these, the first two give $\boldsymbol{\mu}'=(4, 3, 2, 2, 2)$, and the last two correspond to $\boldsymbol{\mu}'=(3, 3, 2, 2, 3)$.

Consider first $\boldsymbol{\mu}'=(4, 3, 2, 2, 2)$. Then $(\phi_{11}, \phi_{12})=(0, 2)$, $\boldsymbol{\theta}'\equiv(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5)=(-2, -4, -6, 2, 3)$, and (1.6) becomes $\phi_{22}=\min(d_6, d_{4\cdot 6}-2, d_{2\cdot 6}-3)\geqq\phi_{21}=\max(0, d_{1\cdot 2}-2, d_{1\cdot 4}-4, d_{1\cdot 6}-6)$. Hence $d_{1\cdot 5}\leqq 6$, so that $d_4<2$. Also $d_{4\cdot 6}\geqq 2$, hence $d_4=1$, $d_5=1$, $d_6=0$ or 1; $d_{1\cdot 3}=d_{1\cdot 6}-d_{4\cdot 6}\leqq 4$, hence $d_2=1$. Also, $d_{1\cdot 2}\leqq 2+d_6$, so that $d_6=0$ implies $d_1=1$. Hence $\boldsymbol{d}'$ can have three values, namely (1, 1, 1, 1, 1, 0), (2, 1, 1, 1, 1, 1) and (1, 1, 1, 1, 1, 1) with (ϕ_{21}, ϕ_{22}) respectively equal to (0, 0), (1, 1) and (0, 1), and $d_0=0$, 1, and (0 or 1). Thus $g=4$. For the four possible values of $(d_0, \boldsymbol{d}')$, we then have $(\delta_0, \delta_3, \delta_6)=(1, 3/6, 0)$, (0, 3/6, 1), (0, 1, 0) and (1, 0, 1). Hence, when $\boldsymbol{x}'=(5, 13, 0, 15, 2, 3)$, we get from (1.11) that $y_1+y_4=5$, $y_1+y_2+y_3=6$, $y_2+y_4=3$, with $y_1+y_2+y_3+y_4=7$. These give $y_1=4$, $y_2=2$, $y_3=0$, $y_4=1$. However, the 6-rowed array corresponding to $\boldsymbol{d}'=(2, 1, 1, 1, 1, 1)$ has only 2 columns of weight 5 or more, whereas $x_6=3$. This implies $y_2=0$, a contradiction. Similar is the case for $\boldsymbol{x}'=(6, 8, 10, 5, 7, 2)$. Hence there exists no array with $\boldsymbol{\mu}'=(4, 3, 2, 2, 2)$.

Now, let $\boldsymbol{\mu}'=(3, 3, 2, 2, 3)$, so that $(\phi_{11}, \phi_{13})=(1, 3)$, $\boldsymbol{\theta}'=(-3, -7, -10, 4, 7)$. As before, it is easily seen that in this case, $(d_0; \boldsymbol{d}')$ may have three values (1; 2, 2, 2, 2, 2, 1), (2; 2, 2, 2, 2, 2, 2) and (1; 2, 2, 2, 2, 1, 1), so that $g=3$. Also, we have respectively, $\delta_6=1, 2, 1$.

Now (1.11 g) gives $3=y_1+2y_2+y_3\geqq 7$, a contradiction. This completes the proof.

THEOREM 3.13. *If T is a trim design with $m=7$, and $\mu_2=2$, then $N\neq 39$.*

PROOF. We proceed as in the last theorem. Let $N=39$. Then $\mu'=5$, $\mu''=7$, and $u=13$, $v=6$, $w=20$. As before, utilizing the fact that certain linear functions of the x's are divisible by 7 or 5, it is easily checked that $\boldsymbol{x}'$ can have four possible values: (9, 1, 0, 20, 5, 4), (6, 2, 5, 15, 4, 7), (13, 2, 5, 15, 4, 0), (10, 3, 10, 10, 3, 3). These correspond respectively to $\boldsymbol{\mu}'=(4, 2, 2, 3, 3)$, $(3, 2, 2, 3, 4)$, $(6, 3, 2, 2, 1)$ and $(5, 3, 2, 2, 2)$.

Consider $\boldsymbol{\mu}'=(4, 2, 2, 3, 3)$, so that $(\phi_{11}, \phi_{12})=(0, 2)$, and $\phi_{22}=\min(d_6, d_{4\cdot 6}-3, d_{2\cdot 6}-5)\geqq\phi_{11}=\max(0, d_{1\cdot 2}-3, d_{1\cdot 4}-5, d_{1\cdot 6}-9)$. Hence $d_{1\cdot 5}\leqq 9$, so that $d_6<2$. When $d_6=0$, the inequality $(d_{4\cdot 6}\geqq 3)$ gives $d_4=2$, which implies $d_{1\cdot 2}=4>3+d_6$, a contradiction. Hence $d_6=1$. If $d_4=2$, then $d_{1\cdot 4}=8>5+d_6$; hence $d_4=1$. Now $d_{1\cdot 2}\leqq d_{4\cdot 6}$; hence $d_2=1$. Hence $(d_0; \boldsymbol{d}')$ has 2 values $(0; 2, 1, 1, 1, 1, 1)$ and $(0; 1, 1, 1, 1, 1, 1)$, and $\delta_6=0$ for both cases. Hence, (1.11 g) gives $x_6=0$, a contradiction.

For $\boldsymbol{\mu}'=(3, 2, 2, 3, 4)$, we have $(\phi_{11}, \phi_{12})=(1, 3)$, and $\phi_{21}=\max(0, d_{1\cdot 2}-4, d_{1\cdot 4}-8, d_{1\cdot 6}-13)\leqq\phi_{22}=\min(d_6, d_{4\cdot 6}-5, d_{2\cdot 6}-9)$. Thus $d_{1\cdot 5}\leqq 13$, and hence $d_5\leqq 2$. Now $d_3=3$ implies $d_{1\cdot 4}>8+d_6$; hence $d_3\leqq 2$. Also $d_{4\cdot 6}\geqq 5$. Hence $d_4=2=d_5$, $d_6=0$ or 1. Now $d_{1\cdot 2}\leqq d_{4\cdot 6}-1$. Thus $\boldsymbol{d}'$ has three possible values (2, 2, 2, 2, 2, 1), (3, 2, 2, 2, 2, 2) and (2, 2, 2, 2, 2, 2), with $(\phi_{21}, \phi_{22})=(0, 0)$, $(1, 1)$ and $(0, 1)$ respectively. Thus $g=4$, the value of δ_0 for the four cases being respectively 2, 1, 1 and 2. Hence (1.11 a) gives $x_1=2y_1+y_2+y_3+2y_4\geqq\sum_{i=1}^{4} y_i=7>6$, a contradiction.

When $\boldsymbol{\mu}'=(6, 3, 2, 2, 1)$, we have $(\phi_{11}, \phi_{12})=(0, 1)$, and from (1.6) it is easily seen that $\boldsymbol{d}'$ can have two possible values (1, 0, 0, 0, 0, 0) and (0, 0, 0, 0, 0, 0) with $\phi_{21}=\phi_{22}=0=d_0$ in each case, and $g=2$. Hence δ_5 equals 1/6 and 0 respectively. Hence from (1.11 f), $y_1=2x_5+6x_6=8$ (using the value of $\boldsymbol{x}'$ corresponding to the present value of $\boldsymbol{\mu}'$). Hence $y_1>7$, a contradiction.

Finally, let $\boldsymbol{\mu}'=(5, 3, 2, 2, 2)$. Then $(\phi_{11}, \phi_{12})=(0, 2)$, and $\phi_{21}=\max(0, d_{1\cdot 2}-2, d_{1\cdot 4}-4, d_{1\cdot 6}-7)\leqq\phi_{22}=\min(d_6, d_{4\cdot 6}-2, d_{2\cdot 6}-3)$. Hence $d_{1\cdot 3}\leqq 5$, so that $d_3\leqq 1$. Thus $d_{1\cdot 2}\leqq d_{4\cdot 6}\leqq 3$, so that $d_2\leqq 1$. But $d_{4\cdot 6}\geqq 2$. Hence $d_2=d_5=1$. Since $d_{1\cdot 2}\leqq 2+d_6$, we can not have $(d_1, d_6)=(2, 0)$. Thus the possible values of $\boldsymbol{d}'$ are (1, 1, 1, 1, 1, 1), (1, 1, 1, 1, 1, 0) and (2, 1, 1, 1, 1, 1) with $(\phi_{21}, \phi_{22})=(0, 1)$, $(0, 0)$ and $(1, 1)$. Hence $g=4$, and the value of (δ_5, δ_6) in the four cases is $(1, 0)$, $(0, 1)$, $(5/6, 0)$, and $(1/6, 1)$. Using (1.11 g), we get $y_2+y_4=3$. But $y_2=y_4=0$, since $x_6=3$, and the number of columns of weight 5 or 6 in the corresponding arrays is respectively

1 and 2. This completes the proof.

THEOREM 3.14. *If T is a trim design with $m=7$, and $\mu_2=2$, then $N\geqq 42$.*

PROOF. This follows from Theorems 3.7–3.13.

We next discuss designs with $\mu_2=3$. From (3.8), we have $\mu'\geqq 4$. When $\mu'=4$, (3.4), (3.8) imply $\mu''=1$. Next, using (3.2)–(3.8), we have $u=v=u'=v'=0$, $w=35$, $\mu_0''=\pm 1$, $w'=\pm 35$, $\mu_0'=\pm 2$. Therefore the only possible value of $\boldsymbol{\mu}'$, apart from an interchange of 0 and 1, is (1, 3, 3, 1, 0). This gives

THEOREM 3.15. *For $\mu_2=3$ and $\mu'=4$ the only possible value of $\boldsymbol{\mu}'$ for which trim designs may possibly exist is* (1, 3, 3, 1, 0), *apart from* (0, 1) *symmetry.*

THEOREM 3.16. *If a design T with $m=7$, $\mu_2=3$, and $\mu'=5$ exists, then $N=42$. Also, the corresponding values of $\boldsymbol{\mu}'$, apart from an interchange of* 0 *and* 1, *are* (1, 3, 3, 2, 3) *and* (4, 4, 3, 1, 0).

PROOF. From Theorem 3.1, $41\leqq N\leqq 42$. Now $N=41$ implies $\mu''=3$; this with (3.3)–(3.5) gives $u=2$, $v=9$ and $w=30$. Hence $u'=0$, -2, or 2. When $u'=0$, (3.6), (3.7) give $w'=0$, $\mu_0'=\mu_0''$ and $|7(3\mu_0'-\mu_0'')|\leqq 18$, which in turn implies $\mu_0'=\mu_0''=\pm 1$. Thus $\boldsymbol{\mu}'=(2, 3, 3, 2, 1)$ or (1, 2, 3, 3, 2), the second value being 'reverse' to the first. A similar argument for $u'=\pm 2$, leads to the same set of values for $\boldsymbol{\mu}'$. Now, for $\boldsymbol{\mu}'=(2, 3, 3, 2, 1)$, we have $(\phi_{11}, \phi_{12})=(0, 1)$, and by (1.8 b, c, f, g, i) all values of $\boldsymbol{d}'$ are rejected. Hence no trim design with $N=41$ assemblies exists. Next, we take $N=42$. Then $\mu''=4$. As above, we find $u=7$, $v=0$, $w=35$, and $(\mu_0', \mu_0'')=(\pm 3, \pm 4)$, $(\pm 1, \mp 2)$. This completes the proof.

THEOREM 3.17. *If T is a trim design with $m=7$, and $27\leqq N\leqq 41$, then the only values (apart from* (0, 1) *symmetry) of $\boldsymbol{\mu}'$ for which designs do exist are* (i) (1, 3, 3, 1, 0) *with $N=35$,* (ii) (0, 0, 1, 4, 6), (3, 1, 1, 3, 3), *with $N=28$, and* (iii) (0, 0, 1, 5, 9), (3, 1, 1, 4, 6), (6, 2, 1, 3, 3) *with $N=35$.*

PROOF. That the above are the only values of $\boldsymbol{\mu}'$ (with $29\leqq N<42$) for which designs may possibly exist is indicated by Theorems 3.14, 3.15, 3.16 and 3.6 (Corollary 3.2).

We now show that trim designs do exist for the above values of $\boldsymbol{\mu}'$. Let $m\geqq k$, and let $\Omega(m, k)$ denote the array with m rows and $\binom{m}{k}$ columns, such that the columns are distinct and are of 'weignt' k each (i.e. each column has k ones and $(m-k)$ zeros). Then, because of complete symmetry with respect to the rows, it is easily checked that $\Omega(m,$

k) forms a balanced array with $\boldsymbol{\mu}'=\left(\binom{m-4}{k}, \binom{m-4}{k-1}, \binom{m-4}{k-2}, \binom{m-4}{k-3}, \binom{m-4}{k-4}\right)$, where $\binom{a}{b}$ is defined to be zero if $a<b$ or $b<0$.

In particular, the array $\Omega(7, 3)$ has $\boldsymbol{\mu}'=(1, 3, 3, 1, 0)$. For $\boldsymbol{\mu}'=(0, 0, 1, 4, 6)$, we need $\{\Omega(7, 5)\oplus\Omega(7, 6)\}$, which means taking the two arrays $\Omega(7, 5)$ and $\Omega(7, 6)$ and simply adjoining them. Similarly, the remaining four values of $\boldsymbol{\mu}'$ given in the theorem correspond respectively to the arrays $\Omega(7, 1)\oplus\Omega(7, 5)$, $\Omega(7, 5)\oplus\Omega(7, 6)\oplus\Omega(7, 6)$, $\Omega(7, 1)\oplus\Omega(7, 5)\oplus\Omega(7, 6)$, $\Omega(7, 1)\oplus\Omega(7, 1)\oplus\Omega(7, 5)$. This completes the proof.

THEOREM 3.18. *Let T be a trim design with $m=7$, and $N=42$. Then,* (a) *apart from an interchange of* 0 *and* 1, *the possible values of $\boldsymbol{\mu}'$ are* (0, 0, 1, 6, 12), (3, 1, 1, 5, 9), (6, 2, 1, 4, 6), (9, 3, 1, 3, 3), (1, 3, 3, 2, 3), *and* (4, 4, 3, 1, 0); *and* (b) *the last two values of $\boldsymbol{\mu}'$ give rise to* $\operatorname{tr} V=\infty$.

PROOF. Part (b) follows from (1.1) and (1.2), since c_3 turns out to be zero for both of the above two cases with $\mu_2=3$. For part (a), we consider the three cases with $\mu_2=1$, 2, 3, separately.

For $\mu_2=1$, the four values of $\boldsymbol{\mu}'$ given above follow directly from Corollary 3.2. Also, when $\mu_2=3$, $N=42$ implies $\mu'\leqq 6$. For $\mu'=6$, we shall have $\mu''=0$, which is impossible in view of (1.10 a, e). Then (3.8), and Theorems 3.15 and 3.16 imply the two values of $\boldsymbol{\mu}'$ (for $\mu_2=3$) given above.

Finally, we show that the case $\mu_2=2$ is impossible. By Theorem 3.6, we have $\mu'=6$, and hence $\mu''=6$. Using (3.3)–(3.5), we get $u=w=0=w'$, $v=42$. Hence, (3.6 b) implies $7|2v'$, and the possible values of (x_2, x_5), and (x_5, x_2), are (21, 21), (28, 14), (35, 7), and (42, 0). It can be easily checked, using (1.10) and (1.1), that the values (21, 21) and (42, 0) give rise to singular designs (i.e. those for which $\operatorname{tr} V=\infty$). For the other cases, the possible values of $\boldsymbol{\mu}'$, apart from an interchange of 0 and 1, are (4, 4, 2, 2, 2) and (5, 5, 2, 1, 1). For these two cases, we have respectively, $(\psi_{11}, \psi_{12})=(0, 2)$ and (0, 1). It can be easily checked that each value of $\boldsymbol{d}'$ corresponding to any of these is rejected by using (1.8 b, d, e). This completes the proof.

THEOREM 3.19. *Let T be an optimal balanced design with N ($\geqq$29) runs and index set $\boldsymbol{\mu}'$. Then $\boldsymbol{\mu}'\neq(1+\alpha, 3, 3, 1, \beta)$, where α and β are nonnegative integers.*

PROOF. Let T_0 denote the design with $N=29$ in Table II, with $\boldsymbol{\mu}'=(4, 3, 1, 1, 3)$, and $\operatorname{tr} V_{T_0}=1.4861$. Using T_0, we can obtain a design T_0^* with N ($\geqq$29) runs by adjoining $(N-29)$ runs to T_0, each new run being simply a column of zeros. Hence $\operatorname{tr} V_{T_0^*}\leqq \operatorname{tr} V_{T_0}<\infty$. Now, in

(1.2), we have $c_5=(\gamma_1-\gamma_3)(\gamma_1+3\gamma_3-4\gamma_5)-5(\gamma_2-\gamma_4)^2=64\{5\mu_1\mu_3+\mu_2(\mu_1+\mu_3)-3\mu_2^2\}$. Hence when $\mu'=(1+\alpha, 3, 1, 1, \beta)$, we have $c_5=0$, and therefore from (1.1), $\text{tr}\, V_T=\infty>\text{tr}\, V_{T_0^*}$. Hence T could not be optimal. (Indeed, T is not even 'nonsingular', i.e. all the 29 parameters are not estimable using T). This completes the proof.

THEOREM 3.20. *If T is an optimal balanced design of resolution V with N ($\geqq 29$) runs, then the index set μ' of T is not of the form $(\alpha, 0, 1, 4, 6+\beta)$ or $(\alpha, 0, 1, 5, 9+\beta)$, where α and β are nonnegative integers.*

PROOF. From (1.1) and (1.2), we have $\text{tr}\, V_T \geqq (c_4/c_5)+(7/8\mu_2) \geqq (3\mu'-2\mu_2)/8\{5\mu_1\mu_3+\mu_2\mu'-3\mu_2^2\}$. The last expression equals 1.6875 and 1.5417 respectively for the two forms of μ' given in the theorem. Define T_0 and T_0^* as in the last theorem. Then $\text{tr}\, V_{T_0^*} \leqq \text{tr}\, V_{T_0}=1.4861<1.5417\leqq \text{tr}\, V_T$, so that T is not optimal. This completes the proof.

THEOREM 3.21. *Let T be an optimal balanced design of resolution V with N ($29\leqq N\leqq 42$) runs, and index set μ'. Then, apart from (0, 1) symmetry, μ' must be of one of the following three forms $(\alpha+3, 1, 1, 3, \beta+3)$, $(\alpha+3, 1, 1, 4, \beta+6)$, $(\alpha+6, 2, 1, 3, \beta+3)$, where α and β are nonnegative integers. (We may remark that the optimal values of μ' given in Tables* I *and* II *were obtained for any given N by a direct computation and comparison of the value of* tr V *(given by* (1.1)) *for the values of μ' of the above three forms, except for $N=42$, when the four values of μ' given in Theorem* 3.18 *were also considered.*

PROOF. This is a direct consequence of Theorems 3.17–3.19.

THEOREM 3.22 *The designs indicated in Table* I *for the various values of μ' given there, are unique.*

PROOF. As seen from Table I, the values of μ' there are of the form $(\alpha+3, 1, 1, 3, \beta+3)$ or $(\alpha+6, 2, 1, 3, \beta+3)$, where α and β are nonnegative integers. To prove the theorem, it is clearly sufficient to show that the trim designs corresponding to $\mu'=(3, 1, 1, 3, 3)$ and $(6, 2, 1, 3, 3)$ are unique. Now, when $\mu'=(3, 1, 1, 3, 3)$, equations (1.10), (3.3)–(3.5) give $x_2=21$, $x_6=7$, the other x's being zero. Hence, the design includes as a subset, 21 columns each of weight 2. Clearly, all of these 21 columns must be distinct, since if any column were repeated more than once, we would have $\mu_2>1$. Now, there are exactly 21 distinct columns possible, each of weight 2, namely the set $\Omega(7, 2)$. Thus $\Omega(7, 2)$ is a subset of the design. Similarly, all the 7 columns of weight 6 each must be distinct, so that $\Omega(7, 6)$ is also a subset of the design. Hence the design must be $\Omega(7, 2)\oplus\Omega(7, 6)$, which proves the uniqueness when $\mu'=(3, 1, 1, 3, 3)$. The uniqueness of the design $\Omega(7, 2)\oplus\Omega(7, 6)\oplus\Omega(7, 6)$ for the other values of μ' is similarly established.

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