ON CONSTRUCTION OF FRACTIONAL REPLICATES AND ON ALIASING SCHEMES

U. B. PAIK AND W. T. FEDERER

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1. Introduction

This paper represents a contribution in furthering the knowledge about the combinatorial structure of fractional replicates and about aliasing schemes for irregular fractional replicates. A genralized method of constructing irregular fractional replicates is presented, and aliasing schemes for some main effect plans are presented. Special reference is made to construction of saturated fractional replicates for a set of main effect parameters. A special ordering of treatment observations and of single-degree-of-freedom parameters is described; using this ordering irregular fractions with prescribed aliasing schemes result. An invariant property of the information matrices of main effect fractional replicates and a semi-invariant property of the aliasing matrix for the 2^n-factorial are discussed.

In the second section we give a Kronecker product representation for the design matrix of an s^n-factorial composed of linear contrasts; the statistical model is described together with estimates of parameter effects and their associated variances. In the third section of the paper a discussion is given of some previous results of the authors on an invariance property of the information matrix and on a semi-invariant property of the aliasing structure matrix for the 2^n-factorial. These results and others cited here are needed in the development of the remainder of the paper.

In the fourth section we show how to rearrange the treatment order and the corresponding design matrix to achieve certain aliasing structure properties. The results are presented in three theorems. In section five the method of construction is outlined and illustrated with two examples. The possible values of the determinants of the information matrices for saturated main effect plans from 2^4 and 3^3 factorials are presented at the end of this section. In the last section, aliasing structure schemes are exhibited and an aliasing structure property is defined and discussed.

2. Basic notations and statistical model

In an s^n-factorial system (s is a prime number), the space of treatment combinations, Z, is represented by the set $Z=\{(i_1, i_2, \cdots, i_n): i_h=0, 1, \cdots, s-1$ for all $h=1, 2, \cdots, n\}$ which contains s^n points, say $N=s^n$. A standard ordering of points in Z is given by the relationship between the coordinate of a point $z_v=(i_1, i_2, \cdots, i_n)$, $v=0, 1, \cdots, N-1$, and the order subscript

$$v=\sum_{h=1}^{n} i_h s^{n-h} . \tag{2.1}$$

The addition operator $+$ between any two treatment combinations z_v and $z_{v'}$ is defined as follows: if $z_v=(i_1, i_2, \cdots, i_n)$ and $z_{v'}=(i'_1, i'_2, \cdots, i'_n)$ then $z_{v''}=z_v+z_{v'}=(i''_1, i''_2, \cdots, i''_n)$, where $i''_h=i_h+i'_h$, mod s, for all $h=1, 2, \cdots, n$. It follows immediately that the set Z is a group with respect to operator $+$. We denote by αz_v, $\alpha=0, 1, \cdots, s-1$, the addition of z_v itself α-times, i.e., $\alpha z_v=(\alpha i_1, \alpha i_2, \cdots, \alpha i_n)=(i'_1, i'_2, \cdots, i'_n)$, mod s.

The expected value of the random vector $y(Z)$ associated with the space of treatment combinations Z is given by

$$\mathrm{E}\,[y(Z)]=X\boldsymbol{B} , \tag{2.2}$$

where X is an $N\times N$ orthogonal matrix in the sense that $X'X$ is a diagonal matrix, $\boldsymbol{B}$ is the $N\times 1$ column vector of single degree of freedom parameters, $\beta_0, \beta_1, \cdots, \beta_{N-1}$, and $y(Z)$ is the $N\times 1$ column vector with covariance matrix $\sigma^2 I$. The parameters β_u have the usual interpretation of main effects and interactions of n factors. We distinguish between linear effects, quadratic effects, and effects of higher order. (Note: Any orthogonal set of contrasts may be utilized but we have arbitrarily selected the polynomial set.) We also distinguish between linear by linear interactions, linear by quadratic interactions, etc. We further describe the structure of the $s^n=N$ parameters, β_u, $u=0, 1, \cdots, N-1$, by considering the space B of N points where $B=\{(\alpha_1, \alpha_2, \cdots, \alpha_n): \alpha_h=0, 1, \cdots, s-1$ for all $h=1, 2, \cdots, n\}$. The correspondence between the parameters β_u and the points of B is given by the order relation specified by $u=\sum_{h=1}^{n} \alpha_h s^{n-h}$. We also introduce the addition operator $+$ on the space B. The unit element of this group $\beta_0=(0, 0, \cdots, 0)$ is the mean response of all the treatment combinations. The parameters $(0, 0, \cdots, \alpha_k, 0, \cdots, 0)$, $k=1, 2, \cdots, n$, where $\alpha_k\geqq 1$ in the kth position correspond to the kth factor α_kth degree main effect. Interactions correspond to points where coordinates are zero or non-zero with at least two coordinate non-zeros. Later, we also use the following notations: M for β_0 and $A^{\alpha_1}B^{\alpha_2}\cdots K^{\alpha_n}$ for $(\alpha_1, \alpha_2, \cdots, \alpha_n)$.

Let $X^{(s)}$ be the matrix of coefficients of orthogonal polynomials of order s, where the elements of the first column are all 1 and the inner product of any two different column vectors of $X^{(s)}$ is zero. This matrix $X^{(s)}$ corresponds to a factor level vector $(0, 1, \cdots, s-1)'$. The matrix X can be defined as:

$$X = X^{(s^n)} = X^{(s)} \otimes \cdots \otimes X^{(s)} ,$$

where $\otimes$ denotes the Kronecker product, i.e., if

$$X^{(s)} = \begin{bmatrix} 1 & \xi_{01} & \cdots & \xi_{0,s-1} \\ 1 & \xi_{11} & \cdots & \xi_{1,s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \xi_{s-1,1} & \cdots & \xi_{s-1,s-1} \end{bmatrix}$$

then

$$X^{(s^n)} = \begin{bmatrix} X^{(s^{n-1})} & \xi_{01} X^{(s^{n-1})} & \cdots & \xi_{0,s-1} X^{(s^{n-1})} \\ X^{(s^{n-1})} & \xi_{11} X^{(s^{n-1})} & \cdots & \xi_{1,s-1} X^{(s^{n-1})} \\ \vdots & \vdots & \ddots & \vdots \\ X^{(s^{n-1})} & \xi_{s-1,1} X^{(s^{n-1})} & \cdots & \xi_{s-1,s-1} X^{(s^{n-1})} \end{bmatrix} . \tag{2.3}$$

Note. Let $\boldsymbol{x}(\alpha_1, \alpha_2, \cdots, \alpha_n)$ be the column vector in $X^{(s^n)}$ corresponding to the parameter point $(\alpha_1, \alpha_2, \cdots, \alpha_n)$ and let $\boldsymbol{x}(\alpha_h)$, $\boldsymbol{x}(\alpha_h, \alpha_k)$, etc. represent $(0, \cdots, \alpha_h, 0, \cdots, 0)$, $(0, \cdots, \alpha_h, 0, \cdots, \alpha_k, 0, \cdots, 0)$ etc., respectively. Define a specialized product of two matrices $A_{m\times n} = \|a_{ij}\|$ and $B_{m\times n} = \|b_{ij}\|$, $i=1, 2, \cdots, m$; $j=1, 2, \cdots, n$, such that

$$A : B = \|c_{ij}\| , \qquad \text{where } c_{ij} = a_{ij} b_{ij} ,\ i=1, 2, \cdots, m;\ j=1, 2, \cdots, n .$$

Then, it is easily verified that

$$\boldsymbol{x}(\alpha_1, \alpha_2, \cdots, \alpha_n) = \boldsymbol{x}(\alpha_1) : \boldsymbol{x}(\alpha_2) : \cdots : \boldsymbol{x}(\alpha_n) . \tag{2.4}$$

Suppose that the vector $y(Z)$ and $\boldsymbol{B}$ are rearranged and partitioned as follows: $y(Z^*)' = (y(Z_p)', y(Z_{N-p})')$, $\boldsymbol{B}^{*\prime} = (\boldsymbol{B}_p', \boldsymbol{B}_{N-p}')$, where $y(Z_p)$ and $\boldsymbol{B}_p$ are $p\times 1 = (n(s-1)+1)\times 1$ observations and main effect parameter vectors, respectively, with the mean parameter as the first element of $\boldsymbol{B}_p$ and $N = s^n$. Then, write $\boldsymbol{y}_p$ and $\boldsymbol{y}_{N-p}$ for $y(Z_p)$ and $y(Z_{N-p})$, respectively, and consider the following expression:

$$\mathrm{E}\begin{bmatrix} \boldsymbol{y}_p \\ \boldsymbol{y}_{N-p} \end{bmatrix} = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{B}_p \\ \boldsymbol{B}_{N-p} \end{bmatrix} \tag{2.5}$$

such that X_{11} is a non-singular $p\times p$ matrix. The existence of $\boldsymbol{y}_p$ is easy to verify. From (2.5) we obtain

$$\mathrm{E}\,[\boldsymbol{y}_p] = [X_{11}, X_{12}]\,[\boldsymbol{B}_p', \boldsymbol{B}_{N-p}']' , \tag{2.6}$$

and the observations in $\boldsymbol{y}_p$ yield a saturated fractional replicate for the parameter vector $\boldsymbol{B}_p$.

Using the least squares method (e.g., Banerjee and Federer [1], [2], Zacks [14]), we obtain the following solution:

$$\hat{\boldsymbol{B}}_p^* = \hat{\boldsymbol{B}}_p + X_{11}^{-1}X_{12}\hat{\boldsymbol{B}}_{N-p} = X_{11}^{-1}\boldsymbol{y}_p \,, \tag{2.7}$$

where $\boldsymbol{B}_p^* = \boldsymbol{B}_p + X_{11}^{-1}X_{12}\boldsymbol{B}_{N-p}$, and $\hat{\boldsymbol{B}}_p + X_{11}^{-1}X_{12}\hat{\boldsymbol{B}}_{N-p} = X_{11}^{-1}\boldsymbol{y}_p$ denotes the least squares estimator of $\boldsymbol{B}_p + X_{11}^{-1}X_{12}\boldsymbol{B}_{N-p}$. Alternatively,

$$\begin{aligned} \hat{\boldsymbol{B}}_p^* &= \hat{\boldsymbol{B}}_p + (X_1'X_1)^{-1}X_{11}'(I+\lambda\lambda')X_{12}\hat{\boldsymbol{B}}_{N-p} \\ &= (X_1'X_1)^{-1}X_{11}'(I+\lambda\lambda')\boldsymbol{y}_p \,, \end{aligned} \tag{2.8}$$

where $X_1 = [X_{11}'X_{21}']'$, and $\lambda = -X_{12}X_{22}^{-1}$. We note that

$$\operatorname{Var}(\hat{\boldsymbol{B}}_p^*) = (X_{11}'X_{11})^{-1}\sigma^2 \,.$$

3. An invariant property of $|X_{11}'X_{11}|$

In an s^n-factorial, denote the matrix of coefficients of orthogonal polynomials of order s corresponding to a factor level vector $(0, 1, \cdots, s-1)'$ by $X^{(s)}$ and the matrix corresponding to $(i, i+1, \cdots, i-1)' = (0, 1, \cdots, s-1)' + (i, i, \cdots, i)'$, (mod s), by $X_i^{(s)}$. The following lemma has been proved by Paik and Federer [7].

LEMMA 3.1. *Let* $G = (X^{(s)\prime}X^{(s)})^{-1}X^{(s)\prime}X_1^{(s)} = (X^{(s)})^{-1}X_1^{(s)}$, *then* $X_i^{(s)} = X^{(s)}G^i$ *for* $i = 0, 1, \cdots, s-1$, *and the matrix* G *has the form*

$$\operatorname{diag}(1, C) = \begin{bmatrix} 1 & 0 \cdots 0 \\ 0 & \cdots\cdots \\ \vdots & \vdots \quad C \\ 0 & \cdot \end{bmatrix}, \tag{3.1}$$

$C^s = I_{(s-1)\times(s-1)}$, *and* $|C^i| = \pm 1$ *for all integer values of* i.

Example. For $X^{(2)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $C = -1$, and for $X^{(3)} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix}$, $C = \frac{1}{2}\begin{pmatrix} -1 & 3 \\ -1 & -1 \end{pmatrix}$.

Let $Z_p(s^n)$ be a saturated main effect plan; write this as Z_p, represented by a submatrix of Z such as a $p\times n$ matrix in an s^n-factorial and X_{11} by a $p\times p$ coefficient matrix of the main effect parameters corresponding to the plan Z_p; let $J_p(i_1, i_2, \cdots, i_n)$ be a $p\times n$ matrix effect with the hth column having elements i_h for all $h = 1, 2, \cdots, n$ and $X_{11,v}$ be a $p\times p$ coefficient matrix of the main effect parameters corresponding to the plan $Z_{p,v}=Z_p+J_p(i_1, i_2, \cdots, i_n)$, (mod s), where the order subscript $v=\sum_{h=1}^{n} i_h s^{n-h}$. Then $X_{11,v}=X_{11}(\text{diag}(1, C^{i_1}, C^{i_2}, \cdots, C^{i_n}))$.

The following theorem also has been proved by Paik and Federer [7].

THEOREM 1. *If Z_p is a saturated main effect plan, then $Z_{p,v}$ also is a saturated main effect plan and* $|X_{11,v}'X_{11,v}|=|X_{11}'X_{11}|$.

The meaning of the theorem is that if Z_p is not a subgroup (in the algebraic sense) of Z in an s^n-factorial, $Z_p+J_p(i_1, i_2, \cdots, i_n)$, $i_h=0, 1, \cdots, s-1$ for all $h=1, 2, \cdots, n$, produces s^n different main effect plans, but determinants of the information matrices have the same value. (It appears that Webb [12] and Paik [5] were among the first to be aware of the fact that several plans gave the same value for the determinant and that any one could be as useful as any other.)

A main effect plan $Z_{p,v}$ of an s^n-factorial is said to be *independent* (nonisomorphic) of a main effect plan Z_p if $Z_{p,v}$ cannot be constructed by the procedure $Z_{p,v}=Z_p+J_p(i_1, i_2, \cdots, i_n)$, $i_h=0, 1, \cdots, s-1$ for all $h=1, 2, \cdots, n$. If $Z_{p,v}$ and Z_p are not independent then the plan $Z_{p,v}$ is an element of the set, $S(Z_p)=\{Z_p+J_p(i_1, i_2, \cdots, i_n): i_h=0, 1, \cdots, s-1$ for all $h=1, 2, \cdots, n\}$. The set $S(Z_p)$ is said to be the main effect plan set *generated* by Z_p. Using this criterion, we may list every independent main effect plan from an s^n-factorial. Paik and Federer [9] present a complete list of the generators for main effect plans for 2^2, 2^3, and 2^4-factorials. Since there are $n(s-1)$ main effect parameters in an s^n-factorial, the total number of main effect plans is $\binom{s^n}{n(s-1)+1}$ and the total number of generators of main effect plans is $\binom{s^n-1}{n(s-1)}\Big/(n(s-1)+1)$ for $n(s-1)+1$ not equal to a multiple of s. Thus for the 2^4-factorial there are $\binom{15}{4}\Big/5=273$ generators. Raktoe and Federer [11] have determined the total number of generators for main effect plans for all s^n.

Also, it should be noted that a semi-invariant property of the aliasing matrix for a 2^n-factorial has been proved by Paik and Federer [8]. The semi-invariant property of $X_{11}^{-1}X_{12}$ is defined such that the matrix $X_{11}^{-1}X_{12}$ remains unchanged, except for signs of some elements, under the procedure $Z_{p,v}=Z_p+J_p(i_1, i_2, \cdots, i_n)$. This means that the aliasing structure does not change under the procedure.

4. Rearranging the treatment order and the corresponding design matrix

From equation (2.7) or (2.8), we note that the inverse of X_{11}, or of X_{22}, is needed to obtain the solution. Also, we see later that if the size of the fraction is less than s^{n-1} in an s^n-factorial, then we may use the matrix $X^{(s^{n-1})}$ instead of the $N \times N$ matrix $X^{(s^n)}$ to obtain a solution such as (2.7) or (2.8). Also, we shall see in this case that the method of constructing a saturated fractional replicate resolves itself into the problem of selecting the smallest number of treatments from those corresponding to the orthogonal matrix $X^{(s^{n-k})}$ for some $k \geqq 1$. However, in this case, the mean effect will be confounded with the main effect A. This is the reason for rearranging the treatment order in Z with some higher order defining contrast before constructing a fractional replication, i.e., we shall require the mean effect to be unconfounded with the main effects.

Now consider rearranging the treatment order in vector Z with some defining contrast in an s^n-factorial. The s^n-1 degrees of freedom among the s^n treatment combinations may be partitioned into $(s^n-1)/(s-1)$ sets of $s-1$ degrees of freedom. Each set of $s-1$ degrees of freedom is given by the contrast among s sets of s^{n-1} treatment combinations specified by the following equations:

$$\alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n = j \ , \qquad j = 0, 1, \cdots, s-1, \ \bmod s \ , \tag{4.1}$$

where the right-hand sides of these equations are elements of the Galois Field $GF(s)$. The α_h's are positive integers between 0 and $s-1$, not all equal to zero, and all additions and multiplications are done within the Galois Field $GF(s)$, then the interaction $A^{\alpha_1}B^{\alpha_2}\cdots K^{\alpha_n}$ corresponds to the equation whose left-hand side is $\alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n$, i.e., the $(j+1)$th set of $s-1$ degrees of freedom given the defining contrast $M \doteq A^{\alpha_1}B^{\alpha_2}\cdots K^{\alpha_n}$, where M denotes the mean parameter and $\doteq$ means completely confounded with, may be expressed as: $M_j \doteq (A^{\alpha_1}B^{\alpha_2}\cdots K^{\alpha_n})_j$ which satisfies the following condition: $\alpha_1 i_1 + \alpha_2 i_2 + \cdots + \alpha_n i_n = j$, mod s, where $i_h = 0, 1, \cdots, s-1$ for all $h = 1, 2, \cdots, n$. For a defining contrast $M \doteq AB^{\alpha_2}\cdots K^{\alpha_n}$, the identity relationships are written as:

$$M_j \doteq (AB^{\alpha_2}\cdots K^{\alpha_n})_j \ , \qquad j = 0, 1, \cdots, s-1 \ . \tag{4.2}$$

Let the set of treatment combinations for fixed $i_1 = \gamma$, $\gamma = 0, 1, \cdots, s-1$, be $\{\gamma, i_2, \cdots, i_n\}$, then the $(k + \gamma s^{n-1})$th treatment corresponds to $M_{j+\gamma=t, \bmod s}$ in the set of $\{\gamma, i_2, \cdots, i_n\}$, for $0 \leqq k \leqq s^{n-1}-1$.

THEOREM 4.1. *In an s^n-factorial (s is a prime number), if the treatment order in Z is rearranged to correspond to the defining contrasts $M_j \doteq (AB^{\alpha_2}\cdots K^{\alpha_n})_j$, $j = 0, 1, \cdots, s-1$, then the following form of the corresponding linear orthogonal comparisons matrix X^* can be obtained by rearranging the row vectors in X, i.e.*

$$(4.3)\qquad X^*=\begin{bmatrix} X_{00}^* & X_{01}^* & \cdots & X_{0,s-1}^* \\ X_{00}^* & X_{11}^* & \cdots & X_{1,s-1}^* \\ \vdots & \vdots & \ddots & \vdots \\ X_{00}^* & X_{s-1,1}^* & \cdots & X_{s-1,s-1}^* \end{bmatrix}$$

where $X_{00}^*=X^{(s^{n-1})}$ *and* X_{ij}^*, $i, j=0, 1, \cdots, s-1$, *are all* $s^{n-1}\times s^{n-1}$ *matrices.*

PROOF. Let $X_{(\gamma)}^{(s^{n-1})}$ be an $s^{n-1}\times s^{n-1}$ matrix of the first column matrices in $X^{(s^n)}$ defined by (2.3) corresponding to a treatment combination set $\{(\gamma, i_2, \cdots, i_n)\colon i_h=0, 1, \cdots, s-1$ for all $h=2, 3, \cdots, n\}$. The first column matrices in (2.3) can be written as $(X_{(0)}^{(s^{n-1})\prime}, \cdots, X_{(s-1)}^{(s^{n-1})\prime})'$. Let $k_t^{(\gamma)}=\left\{t : t=\sum_{h=2}^{n} i_h s^{n-h}\right\}=\{0, 1, \cdots, s^{n-1}-1\}$ be the sequence of the row order numbers in $X_{(\gamma)}^{(s^{n-1})}$, and let $\{k_t^{(\gamma)}\}_j$ be the sub-sequence of the row order numbers in $X_{(\gamma)}^{(s^{n-1})}$, $\{k_t^{(\gamma)}\}=\{0, 1, \cdots, s^{n-1}-1\}$, corresponding to defining contrast $M_j\doteq(AB^{\alpha_2}\cdots K^{\alpha_n})_j$. Then, the sequence $\{k_t^{(\gamma)}\}$ can be partitioned as

$$\{k_t^{(\gamma)}\}=\{\{k_t^{(\gamma)}\}_0, \{k_t^{(\gamma)}\}_1, \cdots, \{k_t^{(\gamma)}\}_{s-1}\}\ .$$

Suppose $m\in\{k_t^{(\gamma)}\}_j$ and $m'\in\{k_t^{(\gamma')}\}_j$, then it may be easily verified that $m\neq m'$ if $\gamma\neq\gamma'$. This means that the set of sequences

$$(4.4)\qquad \{\{k_t^{(0)}\}_j, \{k_t^{(1)}\}_j, \cdots, \{k_t^{(s-1)}\}_j\}\qquad \text{given } j\ ,$$

consists of s^{n-1} non-negative integers less than or equal to $s^{n-1}-1$, and none of the integers is equal to another one. Hence,

$$(4.5)\qquad \{\{k_t^{(0)}\}_j, \{k_t^{(1)}\}_j, \cdots, \{k_t^{(s-1)}\}_j\}=\{\{k_t^{(0)}\}_0, \{k_t^{(0)}\}_1, \cdots, \{k_t^{(0)}\}_{s-1}\}\ .$$

Let $\{\boldsymbol{k}^{(\gamma)}\}_j$ be the set of row vectors corresponding to M_j in $X_{(\gamma)}^{(s^{n-1})}$, then

$$(4.6)\qquad \begin{bmatrix} \{\boldsymbol{k}^{(0)}\}_j \\ \{\boldsymbol{k}^{(1)}\}_j \\ \vdots \\ \{\boldsymbol{k}^{(s-1)}\}_j \end{bmatrix} \sim \begin{bmatrix} \{\boldsymbol{k}^{(0)}\}_0 \\ \{\boldsymbol{k}^{(0)}\}_1 \\ \vdots \\ \{\boldsymbol{k}^{(0)}\}_{s-1} \end{bmatrix} = X_{(0)}^{(s^{n-1})}\ ,$$

where the notation $\sim$ means that if we rearrange the row vector order properly in the left-hand side of the matrix of the $\sim$ notation, then this matrix will be the same as $X_{(0)}^{(s^{n-1})}$. This proves the theorem.

Remark. Let

$$(4.7)\qquad \boldsymbol{x}_{ij}(j, j_2, \cdots, j_n)$$

be the column vector in X_{ij}^* corresponding to the parameter $A^jB^{j_2}, \cdots, K^{j_n}$, where $j_h=0, 1, \cdots, s-1$ for $h=2, 3, \cdots, n$. We may obtain the following relations:

$$(4.8)\qquad \begin{aligned} \boldsymbol{x}_{ij}(j,\ j_2,\cdots,\ j_n) &= \boldsymbol{x}_{00}(0,\ j_2,\cdots,\ j_n) : \boldsymbol{x}_{ij}(j,0,0,\cdots,0) \\ &= \boldsymbol{x}_{00}(0,0,\cdots,0) : \boldsymbol{x}_{00}(0,j_2,0,\cdots,0) : \cdots \\ &\quad : \boldsymbol{x}_{00}(0,0,\cdots,j_n) : \boldsymbol{x}_{ij}(j,0,\cdots,0)\ . \end{aligned}$$

THEOREM 4.2. *In an s^n-factorial, let $X_{0\cdot}^* = [X_{00}^*, X_{01}^*, \cdots, X_{0,s-1}^*]$ be the $s^{n-1}\times s^n$ matrix corresponding to the defining contrast $M \doteq (AB^{\alpha_2}\cdots K^{\alpha_n})_0$, where at least two of $\alpha_2,\cdots,\alpha_n$ are not zero, then the mean and main effect columns in $X_{0\cdot}^*$ are orthogonal to each other.*

PROOF. Let U_{11} be a matrix which is constructed using the mean and main effect columns in $X_{0\cdot}^*$ and $\boldsymbol{u}_h(j)$ be the column vector corresponding to $(0,\cdots,j_h,\ 0,\cdots,0)$ in U_{11}, and define $u_0=1$. Let $Z(j)$, whose elements are in Z, be an $s^{n-1}\times n$ matrix corresponding to $M_j=(AB^{\alpha_2}\cdots K^{\alpha_n})_j$, where at least two of $\alpha_2,\cdots,\alpha_n$ are not zero, then in each column of $Z(0)$, each level number occurs an equal number of times, say μ times; all s^2 treatment combinations corresponding to any two factors, chosen from n factors, occur an equal number of times, say ν times, in $Z(0)$.

Then, using a property of $X^{(s)}$, the following relations hold in the matrix U_{11}:

$$\boldsymbol{u}_0\cdot\boldsymbol{u}_h(j) = \mu\sum_{i=0}^{s-1}\xi_{ij} = 0 \qquad \text{for } j=1,\cdots,s-1;\ h=1,2,\cdots,n$$

$$\boldsymbol{u}_h(j)\cdot\boldsymbol{u}_h(g) = \mu\sum_{i=0}^{s-1}\xi_{ij}\xi_{ig} = 0$$
$$\text{for } j\neq g;\ j,g=0,1,\cdots,s-1 \text{ and } h=1,2,\cdots,n\ .$$

$$\boldsymbol{u}_h(j)\cdot\boldsymbol{u}_k(g) = \nu\sum_{i=1}^{s-1}\sum_{m=1}^{s-1}\xi_{ij}\xi_{mg} = 0$$
$$\text{for } h\neq k;\ j,g=0,1,\cdots,s-1 \text{ and } h,k=1,2,\cdots,n$$

and the theorem is proved.

THEOREM 4.3. *Let $X_{0\cdot}^* = [X_{00}^*, X_{01}^*]$ be a $2^{n-1}\times 2^n$ matrix corresponding to $Z(0)$ with defining contrast $M_0 \doteq (AB^{\alpha_2}\cdots K^{\alpha_n})_0$, $\alpha_h=0$ or 1 for $h=2,\cdots,n$, in a 2^n-factorial, then $X_{0\cdot}^*$ can be rearranged as*

$$(4.9)\qquad X_{0\cdot} = [X_{00}^*,\ \pm X_{00}^*]\ ,$$

where the parameter order corresponding to column order in $X_{0\cdot}$ is M, $K,\cdots,BC\cdots K$; W, $KW,\cdots,BC\cdots KW$, where $W=AB^{\alpha_2}\cdots K^{\alpha_n}$.

PROOF. Using the notation (4.7), the column vectors in $X_{0\cdot}^*$, corresponding to M, $K,\cdots,BC\cdots K$; W, $KW,\cdots,BC\cdots KW$ are expressed as $\boldsymbol{x}_{00}(0,0,\cdots,0)$, $\boldsymbol{x}_{00}(0,0,\cdots,1),\cdots,\boldsymbol{x}_{00}(0,1,\cdots,1)$; $\boldsymbol{x}_{01}(1,\alpha_2,\cdots,\alpha_n)$, $\boldsymbol{x}_{00}(0,0,\cdots,1) : \boldsymbol{x}_{01}(1,\alpha_2,\cdots,\alpha_n),\cdots,\boldsymbol{x}_{00}(0,1,\cdots,1) : \boldsymbol{x}_{01}(1,\alpha_2,\cdots,\alpha_n)$, respectively.

From the defining contrast $M_0 \doteq (AB^{\alpha_2}\cdots K^{\alpha_n})_0$,

$$\boldsymbol{x}_{01}(1, \alpha_2, \cdots, \alpha_n) = \pm \boldsymbol{x}_{00}(0, 0, \cdots, 0) , \tag{4.10}$$

where the sign $+$ or $-$ is dependent upon whether $1+\sum_{h=2}^{n} \alpha_h$ is an even or odd number in this 2^n-factorial system. Then,

(4.11)

$$\boldsymbol{x}_{00}(0, 0, \cdots, 1) : \boldsymbol{x}_{01}(1, \alpha_2, \cdots, \alpha_n) = \boldsymbol{x}_{00}(0, 0, \cdots, 1) : [\pm \boldsymbol{x}_{00}(0, 0, \cdots, 0)]$$
$$= \pm \boldsymbol{x}_{00}(0, 0, \cdots, 1)$$
$$\cdots$$
$$\boldsymbol{x}_{00}(0, 1, \cdots, 1) : \boldsymbol{x}_{01}(1, \alpha_2, \cdots, \alpha_n) = \pm \boldsymbol{x}_{00}(0, 1, \cdots, 1) .$$

Using the results in (4.10) and (4.11), we see that this completes the proof of the theorem.

5. Construction of saturated fractional replicates

We shall consider mostly the method of constructing saturated main effect plans in an s^n-factorial. Although we could always construct various saturated non-orthogonal plans for any given parameter set, the general steps of the construction method may not be too instructive. The following steps, however, will be common in constructing any fractional replicate for the specified parameters (also, see Banerjee and Federer [3] and Paik and Federer [5], [6] in this connection). Special cases will be illustrated in the following examples.

Step 1. Given the design matrix and parameter and observation vectors, $X\boldsymbol{B} = \mathrm{E}\,(\boldsymbol{y})$ in any fashion and not necessarily that of the previous section, we now rearrange the parameter matrix such that the p parameters, $p<N$, are arranged to have the p parameters of interest first and $N-p$ parameters not of interest last to obtain $\boldsymbol{B}^*$ rearranged as $[\boldsymbol{B}_p', \boldsymbol{B}_{N-p}']'$. This also rearranges the columns of X such that

$$X^*\boldsymbol{B}^* = \mathrm{E}\,(y) \tag{5.1}$$

or

$$[X_1, X_2]\begin{bmatrix} \boldsymbol{B}_p \\ \boldsymbol{B}_{N-p} \end{bmatrix} = \mathrm{E}\,(y) , \tag{5.2}$$

where $X^* = [X_1, X_2]$, X_1 is an $N\times p$ matrix, and X_2 is an $N\times(N-p)$ matrix.

Step 2. Search through rows of X_1 until there is an X_{11}, $p\times p$, which is non-singular.

Step 3. Corresponding to the rows in X_{11} will be rows in X_1 and treatments in Z. Rearrange the treatments in Z into $[Z_p', Z_{N-p}']'$, where Z_p corresponds to the rows in X_{11} from X_1. The treatment combinations Z_p yield a saturated design for the parameters in $\boldsymbol{B}_p$. This obtained set is one of the possible sets. All possible sets are found by identifying all X_{11} which have an inverse.

Example 5.1. Saturated main effect plans in a 2^4-factorial.

If we consider a 2^4-factorial design matrix $X^{(2^4)}$ with the defining contrast $M \doteq ABCD$, then the aliasing scheme is as follows:

$$M \doteq ABCD\,, \quad A \doteq BCD\,, \quad B \doteq ACD\,, \quad C \doteq ABD\,, \quad D \doteq ABC\,,$$
$$AB \doteq CD\,, \quad AC \doteq BD\,, \quad BC \doteq AD\,.$$

After rearranging the rows and columns taking into consideration the above aliasing scheme and after using Theorems 4.2 and 4.3, we obtain the following matrix X^*:

$$X^* = \begin{bmatrix} X_{00}^* & X_{00}^* \\ X_{00}^* & -X_{00}^* \end{bmatrix} \tag{5.3}$$

where $X_{00}^* = X^{(2^3)}$, and in this case, the treatment order is

(5.4)
0000, 1001, 1010, 0011, 1100, 0101, 0110, 1111;
1000, 0001, 0010, 1011, 0100, 1101, 1110, and 0111,

and the parameter order is

(5.5)
M, D, C, CD, B, BD, BC, BCD;
$ABCD$, ABC, ABD, AB, ACD, AC, AD, and A.

Now consider the saturated main effect plans in a 2^4-factorial. Let the treatments be arranged such as (5.4) and using the 7th, 6th, and 4th columns in X_{00}^* corresponding to effect BC, BD, and CD, and let U_{12} be an 8×3 matrix corresponding to parameters BC, BD, CD in X_{00}^*, then we may easily find three independent rows in the matrix U_{12} and obtain the saturated main effect plans in a 2^4-factorial.

Let $(n_1, n_2, n_3, n_4, n_5)$, where n_i is a treatment order number in (5.4), be one of the plans constructed by the above procedure, then by recalling Theorems 4.1 and 4.3 we know the following treatment combinations are also saturated main effect plans in a 2^4-factorial, i.e., for treatment 8 being 1000 we obtain

$$(n_1+8,\ n_2+8,\ n_3+8,\ n_4+8,\ n_5+8)\,. \tag{5.6}$$

Finally, it will be worthwhile to note that all plans (64 plans) in this example belong to the sets generated by the following generators:

$$
\begin{array}{cccc}
0000 & 0000 & 0000 & 0000 \\
0011 & 0011 & 0101 & 0101 \\
0101 & 1001 & 1001 & 1001 \\
0110 & 1010 & 1100 & 1111 \\
1001 & 0101 & 0011 & 0011\,.
\end{array} \tag{5.7}
$$

In these cases, $|X_{11}'X_{11}|=1024=32^2$.

Example 5.2. Saturated main effect plans in a 3^3-factorial.

In a 3^3-factorial, after rearranging the row order for the defining contrast $M \doteq ABC^2$, we obtain the following matrix:

$$
X^* = \begin{bmatrix} X_{00}^* & X_{01}^* & X_{02}^* \\ X_{00}^* & X_{11}^* & X_{12}^* \\ X_{00}^* & X_{21}^* & X_{22}^* \end{bmatrix}, \tag{5.8}
$$

where each X_{ij}^* is a 9×9 square matrix, $X_{00}^*=X^{(3^2)}$, and treatment order is

$$
\begin{array}{l}
000,\ 101,\ 202,\ 210,\ 011,\ 112,\ 120,\ 221,\ 022; \\
100,\ 201,\ 002,\ 010,\ 111,\ 212,\ 220,\ 021,\ 122; \\
200,\ 001,\ 102,\ 110,\ 211,\ 012,\ 020,\ 121,\ 222,
\end{array} \tag{5.9}
$$

and the parameter order is

$$
\begin{array}{l}
M,\ C,\ C^2,\ B,\ BC,\ BC^2\ B^2,\ B^2C,\ B^2C^2; \\
A,\ AC,\ AC^2,\ AB,\ ABC,\ ABC^2,\ AB^2,\ AB^2C,\ AB^2C^2; \\
A^2,\ A^2C,\ A^2C^2,\ A^2B,\ A^2BC,\ A^2BC^2,\ A^2B^2,\ A^2B^2C, \text{ and } \\
A^2B^2C^2\,.
\end{array} \tag{5.10}
$$

If we rearrange the column order in X^* to correspond to the following parameter order:

$$
M,\ A,\ A^2,\ B,\ B^2,\ C,\ C^2,\ BC,\ BC^2,\cdots,
$$

and let the first 9×9 submatrix of the rearranged matrix be A_{00}, and if we use the symbols $\boldsymbol{M}$, $\boldsymbol{A}$, $\boldsymbol{A}^2$, $\boldsymbol{B}$, $\boldsymbol{B}^2$ $\boldsymbol{C}$, $\boldsymbol{C}^2$, $\boldsymbol{BC}$, $\boldsymbol{BC}^2$ as the symbol of each corresponding column vector in A_{00}, respectively, then, from Theorem 4.2, the column vectors $\boldsymbol{M}$, $\boldsymbol{A}$, $\boldsymbol{A}^2$, $\boldsymbol{B}$, $\boldsymbol{B}^2$, $\boldsymbol{C}$, and $\boldsymbol{C}^2$ are orthogonal to each other and also $\boldsymbol{M}$, $\boldsymbol{B}$, $\boldsymbol{B}^2$, $\boldsymbol{C}$, $\boldsymbol{C}^2$, $\boldsymbol{BC}$, and $\boldsymbol{BC}^2$, are orthogonal to each other. Using the Schmidt method of orthogonalizing the column vectors, we can make $\boldsymbol{BC}$ and $\boldsymbol{BC}^2$ orthogonal vectors to the first 7 column vectors. Let such new vectors of $\boldsymbol{BC}$ and $\boldsymbol{BC}^2$ be z_1 and z_2 respectively; then, if we find a non-singular 2×2 matrix from the 9×2 matrix, $[z_1, z_2]$, we can construct a corresponding information matrix X_{11} for saturated main effect plans.

Let $(n_1, n_2, n_3, n_4, n_5, n_6, n_7)$, where n_i is the treatment order number in (5.9), be one of the plans constructed from the above procedure, then the following sets of treatment combinations are also saturated main effect plans in a 3^3-factorial, i.e.,

$$(n_1+9,\ n_2+9,\ n_3+9,\ n_4+9,\ n_5+9,\ n_6+9,\ n_7+9)$$

and

$$(n_1+18,\ n_2+18,\ n_3+18,\ n_4+18,\ n_5+18,\ n_6+18,\ n_7+18)\ . \tag{5.11}$$

In this example, all above plans (81 plans) belong to the sets generated by the following generators:

000	000	000
011	011	011
022	022	022
101	101	101
112	112	120
120	202	210
202	210	221

In these cases, $|X_{11}'X_{11}|=419904=3^2(2^3\cdot 3^3)^2$.

Remarks. (i) In the case of saturated main effect plans in a 2^4-factorial, every $|X_{11}'X_{11}|$ has one of the four values, i.e., 2304, 1024, 256 or 0. The set generated by a plan (0000, 0111, 1011, 1101, 1110) has the maximum value 2304. Note that

$$2304=(3\cdot 2^4)^2=48^2\ , \qquad 1024=(2\cdot 2^4)^2=32^2\ ,$$
$$256=(1\cdot 2^4)^2=16^2\ , \quad \text{and} \quad 0=(0\cdot 2^4)^2\ .$$

Also, note that there are 16(1) plans for which $|X_{11}'X_{11}|=2304$, 16(20) plans for which $|X_{11}'X_{11}|=1024$, 16(167) plans for which $|X_{11}'X_{11}|=256$, and 16(85) plans for which $|X_{11}'X_{11}|=0$. (These plans have been completely enumerated by Paik and Federer [9]; plans for the 2^2 and 2^3 factorials were also enumerated.)

(ii) In the case of saturated main effect plans in a 3^3-factorial, every $|X_{11}'X_{11}|$ has one of the five values, i.e., 746496, 419904, 186624, 46656, or 0. The sets generated by the following 9 plans have the maximum value 746496.

000	000	000	000	000	000	000	000	000
021	012	012	011	011	012	011	022	022
101	102	021	101	102	101	101	202	202
112	110	102	112	110	110	110	220	220
120	121	110	120	201	211	122	211	011
202	201	211	210	121	021	212	121	101
210	220	220	222	222	222	221	112	110

It is of interest to note that for $2^3(3^3)=216$ that

$$746496=[4(216)]^2 \quad 419904=[3(216)]^2 \quad 186624=[2(216)]^2$$
$$46656=[1(216)]^2, \text{ and } 0=[0(216)]^2.$$

For the cases $s=2$ or 3 and from the property of $X^{(s)}$, one is led to consider the values of the determinants of $X_{11}=[s(s-1)(s-2)\cdots 1]^n\cdot[n(s-1)-i]$ for $i=s-1, s, s+1, \cdots, n(s-1)$ for saturated main effect plans from an s^n-factorial with $n(s-1)+1$ observations, where the number of plans having $|X_{11}|$ equal to a specific value could be zero as in the 2^2 case. It is not difficult to find exceptions to the above. Hence, the question of the possible values of the determinant of X_{11} remains an open question, even for $s=2$.

The complete characterization of all X_{11} poses some interesting and difficult combinatorial problems. Partial characterizations in addition to those presented in this paper, have been made by Raktoe and Federer [10] and by Werner [13]. Raktoe and Federer [10] have obtained a lower bound on the number of singular X_{11} for saturated main effect plans in s^n-factorials. Werner [13] has obtained the frequency distribution of plus ones in all X_{11} from saturated main effect plans for the 2^n-factorial.

6. Alias schemes in some fractional replicates

This section is concerned with some alias schemes in some fractional replications. Ehrenfeld and Zacks [4] and Paik and Federer [7] presented randomized procedures to obtain an unbiased estimator of $\boldsymbol{B}_p$ in place of $\boldsymbol{B}_p^*=\boldsymbol{B}_p+X_{11}^{-1}X_{12}\boldsymbol{B}_{N-p}$ which estimates a sum of parameters. However, a randomized procedure may not be always applicable as, for example, in the missing data situation where the data are not missing at random, in situations wherein certain treatments are inadmissible, or in sequential selection of observations. In such cases, we may want to know the pattern of $X_{11}^{-1}X_{12}$ in irregular fractional replicates as this gives the aliasing scheme.

6.1. *Alias schemes in saturated fractional replicates for the 2^4-factorial*

In Example 5.1 (saturated main effect plans in a 2^4-factorial), suppose that the following partitioned matrix of X is obtained after rearranging the columns in X^* (the row order was arranged subject to $M \doteq ABCD$) in (5.3).

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}=\begin{bmatrix} X_{11} & X_{1211} & X_{1212} \\ X_{2111} & X_{2211} & X_{2212} \\ X_{2121} & X_{2221} & X_{2222} \end{bmatrix}, \tag{6.1}$$

where the parameter order corresponding in X is as follows: *M, A, B, C, D, CD, BD, BC*; *ABCD, BCD, ACD, ABD, ABC, AB, AC, AD*, and X_{11} is a $p\times p$ ($p<8$) non-singular matrix, X'_{2111} and X_{1211} are each $p\times(8-p)$ matrices, X_{2211} is an $(8-p)\times(8-p)$ matrix, X_{2121} and X'_{1212} are $8\times p$ matrices, and X_{2221} and X'_{2212} are $8\times(8-p)$ matrices.

We know from (5.3) that

$$\begin{bmatrix} X_{11} & X_{1211} \\ X_{2111} & X_{2211} \end{bmatrix}=\begin{bmatrix} X_{1212} \\ X_{2212} \end{bmatrix}\sim X^*_{00},$$

so that

$$X_{1212}=[X_{11}, X_{1211}] .$$

Then, X_{12} can be partitioned as follows:

$$X_{12}=[X_{1211}, X_{11}, X_{1211}] . \tag{6.2}$$

Hence,

$$X_{11}^{-1}X_{12}=[X_{11}^{-1}X_{1211}, I, X_{11}^{-1}X_{1211}] . \tag{6.3}$$

It may be easily verified that, in all plans in Example 5.1 (there are 64 plans), $X_{11}^{-1}X_{12}$ has the following form:

$$[\pm X_{11}^{-1}X_{1211}, \pm I, \pm X_{11}^{-1}X_{1211}] . \tag{6.4}$$

Example 6.1.

1000
1011
1101
1110
0001

This plan may be obtained by the following procedure from the first plan in (5.7);

$$\begin{bmatrix} 0000 \\ 0011 \\ 0101 \\ 0110 \\ 1001 \end{bmatrix} + \begin{bmatrix} 1000 \\ 1000 \\ 1000 \\ 1000 \\ 1000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 1011 \\ 1101 \\ 1110 \\ 0001 \end{bmatrix}, \bmod 2 .$$

In this case, we obtain the following solution:

$$\hat{\boldsymbol{B}}_p + \left[\begin{array}{ccccc:ccc:ccc} -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \hat{\boldsymbol{B}}_{N-p} = X_{11}^{-1}\boldsymbol{y}_p ,$$

where $\boldsymbol{B}'_p=(M, A, B, C, D)$ and $\boldsymbol{B}'_{N-p}=(ABCD, BCD, ACD, ABD, ABC; AB, AC, BC; CD, BD, AD)$. Note that the alias scheme is semi-invariant under the procedure $Z_p+J_p(i_1, i_2, \cdots, i_n)$ described in Section 3.

Similar results may be obtained from the plans which are constructed by the method of Example 5.1 with defining contrasts $M \doteq ABC$, $M \doteq ABD$, $M \doteq ACD$, $M \doteq BCD$. (In all these cases, $|X'_{11}X_{11}|=1024$). None of the saturated main effect plans except the above plans with $|X'_{11}X_{11}| =1024$ in the 2^4-factorial has the form in (6.4).

For the 2^n-factorial system, saturated fractions with $|X'_{11}X_{11}|$ a maximum, do not always have the best aliasing structure for $X_{11}^{-1}X_{12}$ given that complete confounding of effects has better properties than having an effect in $\boldsymbol{B}_p$ partially confounded with all the parameters in $\boldsymbol{B}_{N-p}$. The case of $p>8$ for a 2^4-factorial is considered next.

In a 2^4-factorial, suppose that $M \doteq ABCD$, $\boldsymbol{B}'_p=(M, A, B, C, D, AB, AC, AD, BC)$, and the matrix X in (6.1) is partitioned as follows:

$$X=\begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} = \left[\begin{array}{cc:c} X_{00} & X_{0012} & X_{121} \\ X_{0021} & X_{0022} & X_{122} \\ \hdashline X_{221} & X_{222} & X_{22} \end{array}\right] \tag{6.5}$$

where the parameter order corresponding to the columns in X is M, A, B, C, D, AB, AC, AD, BC; $ABCD$, BCD, ACD, ABD, ABC, CD, BD, and X_{11} is a $p\times p$ non-singular matrix, X_{22} is an $(N-p)\times(N-p)$ non-singular matrix, and $X_{00}\sim X_{00}^*$ in (5.3). Then, the treatment designation of observations in the order corresponding to the matrix X is 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111, and 1101, and the remaining 7 treatments in some order.

Since

$$X^{-1}=X'=\begin{bmatrix} X'_{11} & X'_{21} \\ X'_{12} & X'_{22} \end{bmatrix}, \qquad X'_{11}X_{12}=-X'_{21}X_{22}$$

and

$$X'_{12}X_{12}=I_{(N-p)\times(N-p)}-X'_{22}X_{22} .$$

Since

$$X_{11}^{-1}=X'_{11}-X'_{21}X'^{-1}_{22}X'_{12} ,$$

then

$$\begin{aligned} X_{11}^{-1}X_{12}&=X'_{11}X_{12}-X'_{21}X'^{-1}_{22}X'_{12}X_{12} \\ &=-X'_{21}X_{22}-X'_{21}X'^{-1}_{22}(I-X'_{22}X_{22}) \\ &=-X'_{21}X'^{-1}_{22} . \end{aligned}$$

Also, since the matrix X_{21} may be partitioned as $X_{21}=[-X_{22}\ \ -X_{222}\ \ X_{222}]$,

$$X_{11}^{-1}X_{12}=-\begin{bmatrix} -X'_{22} \\ -X'_{222} \\ X'_{222} \end{bmatrix} X'^{-1}_{22}=\begin{bmatrix} I_{(N-p)\times(N-p)} \\ X'_{222}X'^{-1}_{22} \\ -X'_{222}X'^{-1}_{22} \end{bmatrix} . \tag{6.6}$$

Example 6.2. ($p=9$) Suppose that the treatment combinations corresponding to X_{11} in (6.5) are 0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111, and 1101. Then

$$X_{11}^{-1}X_{12}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline -1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix},$$

where $\boldsymbol{B}'_p=(M, A, B, C, D, AB, AC, AD, BC)$ and $\boldsymbol{B}'_{N-p}=(ABCD, BCD, ACD, ABD, ABC, CD, BD)$.

If $p=12$, we may find a fractional plan in a 2^4-factorial such that

$$X'_{222}X'^{-1}_{22}=-I_{4\times4} ,$$

for example, in the case of (0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1100, 1101, 1110, 1111),

$$X_{11}^{-1}X_{12}=\begin{bmatrix} I_{4\times 4} \\ -I_{4\times 4} \\ I_{4\times 4} \end{bmatrix}$$

where $\boldsymbol{B}_p'=(M, D, C, CD, B, BD, BC, BCD, A, AD, AC, ACD)$ and $\boldsymbol{B}_{N-p}'=(AB, ABD, ABC, ABCD)$.

6.2. *Some unsaturated main effect fractional replicates*

In the saturated main effect plans in an s^n-factorial, $s>2$, we are unable to find the pattern of $X_{11}^{-1}X_{12}$ similar to the cases in a 2^n-factorial, because, in general, $X_{ij}^* \neq cX_{i'j'}^*$, where c is some constant, for $i\neq i'$ and $j\neq j'$ in an s^n-factorial $(s>2)$. However, we shall present an application of a method similar to the case in Example 6.1 for some unsaturated main effect fractional replicates in an s^n-factorial $(s>2)$.

Consider the following split-plot type design in a 3^3-factorial:

000	100	200
001	101	201
010	110	210 ;
012	112	212
022	122	222

then, from (2.3), and using the partitioning method in 6.1, we may obtain the following three equations:

$$\begin{aligned} &\hat{\boldsymbol{B}}_5+[X_{11}^{-1}X_{1211},\ -I,\ -X_{11}^{-1}X_{1211},\ I,\ X_{11}^{-1}X_{1211}]\hat{\boldsymbol{B}}_{22}=X_{11}^{-1}\boldsymbol{y}_{5,1}\ ,\\ (6.7)\quad &\hat{\boldsymbol{B}}_5+[X_{11}^{-1}X_{1211},\ (0)I,\ (0)X_{11}^{-1}X_{1211},\ -2I,\ -2X_{11}^{-1}X_{1211}]\hat{\boldsymbol{B}}_{22}=X_{11}^{-1}\boldsymbol{y}_{5,2}\ ,\\ &\hat{\boldsymbol{B}}_5+[X_{11}^{-1}X_{1211},\ I,\ X_{11}^{-1}X_{1211},\ I,\ X_{11}^{-1}X_{1211}]\hat{\boldsymbol{B}}_{21}=X_{11}^{-1}\boldsymbol{y}_{5,3}\ , \end{aligned}$$

where $\boldsymbol{B}_5=(M, C, C^2, B, B^2)'$, $\boldsymbol{B}_{22}=(\boldsymbol{B}_4', \boldsymbol{B}_{5,a}', \boldsymbol{B}_{4,a}', \boldsymbol{B}_{5,a^2}', \boldsymbol{B}_{4,a^2}')'$, where $\boldsymbol{B}_4=(BC, BC^2, B^2C, B^2C^2)'$, $\boldsymbol{B}_{5,a}=(A, AC, AC^2, AB, AB^2)'$, $\boldsymbol{B}_{4,a}=(ABC, ABC^2, AB^2C, AB^2C^2)'$, $\boldsymbol{B}_{5,a^2}=(A^2, A^2C, A^2C^2, A^2B, A^2B^2)'$, and $\boldsymbol{B}_{4,a^2}=(A^2BC, A^2BC^2, A^2B^2C, A^2B^2C^2)'$, and X_{11} is a 5×5 matrix, X_{1211} is a 5×4 matrix, I is a 5×5 identity matrix, and $\boldsymbol{y}_{5,1}$, $\boldsymbol{y}_{5,2}$, and $\boldsymbol{y}_{5,3}$ are observation vectors.

From (6.7), we obtain:

$$\hat{\boldsymbol{B}}_5+X_{11}^{-1}X_{1211}\hat{\boldsymbol{B}}_4=\frac{1}{3}X_{11}^{-1}(\boldsymbol{y}_{5,1}+\boldsymbol{y}_{5,2}+\boldsymbol{y}_{5,3})$$

$$\hat{\boldsymbol{B}}_{5,a}+X_{11}^{-1}X_{1211}\hat{\boldsymbol{B}}_{4,a}=\frac{1}{2}X_{11}^{-1}(\boldsymbol{y}_{5,3}-\boldsymbol{y}_{5,1})$$

$$\hat{\boldsymbol{B}}_{5,a^2}+X_{11}^{-1}X_{1211}\hat{\boldsymbol{B}}_{4,a^2}=\frac{1}{6}X_{11}^{-1}(\boldsymbol{y}_{5,1}-2\boldsymbol{y}_{5,2}+\boldsymbol{y}_{5,3})\ .$$

6.3. *An aliasing structure property*

In the above, an aliasing structure property was mentioned in connection with the examples. The goodness of the aliasing structure property will be defined by the number of effects that are partially or completely confounded with each other. In the absence of any knowledge concerning the relative magnitude of the aliased effects, the fewer the number of effects confounded with each other the more desirable is the aliasing structure property, that is, the more nearly the aliasing structure is to complete confounding of effects the more desirable it is. Likewise, the greater the number of effects partially confounded with each other the more undesirable is the plan. The fewer the number of effects that are confounded with any specified effect, the larger will be the number of effects that can be estimated free from the given effect.

Now, in order to completely describe the aliasing structure property, it is necessary to have an ordering of patterned matrices from a diagonal matrix to nonzero submatrices on the diagonal with zeros elsewhere, to submatrices which form diagonal matrices and on down to a matrix with no zero elements. Perhaps some classification of the aliasing matrix $X_{11}^{-1}X_{12}$ could be made on the number or proportion of zero elements in the matrix. When this problem is resolved, the aliasing property structure with its criterion for goodness will be completely described. There appears to be little mathematical theory on structuring matrices available at present. The work on the "consecutive-ones" property in matrices is interesting in this connection.

If one knows (or is willing to assume) that the magnitude of the parameters in $\boldsymbol{B}_{N-p}$ are likely to be small relative to those in $\boldsymbol{B}_p$, then the aliasing structure property is somewhat irrelevant. However, this property was introduced to complete the statistical and mathematical theory for situations wherein it is applicable, i.e., in the sequential selection of observations in multi-factor experiments without prior knowledge concerning the magnitude of the various parameters. Also, in the sequential selection of combinations resulting in regular fractional replicates for which the determinants of X_{11} is maximum, a subset of this regular fractional replicate will have the most desirable aliasing structure property as defined herein and the regular fraction will be optimal both for the aliasing structure property and in a minimum variance sense. In super-saturated screening designs, the designs with the least desirable aliasing structure property may be selected. For all these situations it is desirable to further the knowledge of the combinatorial properties of all possible fractions and to describe properties of the various fractions.

It should be pointed out that the aliasing structure property described above may be more appropriate in many experimental situations than is the minimum variance (maximum value of the determinant of $X_{11}'X_{11}$) property. Hence a fractional replicate may result in a maximum value of $|X_{11}'X_{11}|$ but may have an undesirable aliasing structure. In this case, a plan for which $|X_{11}'X_{11}|$ is not maximum would be selected in preference to one for which $|X_{11}'X_{11}|$ was maximum.

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KOREA UNIVERSITY
CORNELL UNIVERSITY

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