A FEW REMARKS ABOUT SOME RECENT ARTICLES ON THE EXACT DISTRIBUTIONS OF MULTIVARIATE TEST CRITERIA : I

A. M. MATHAI

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1. Introduction and summary

In recent years, a large number of articles on the exact and asymptotic distributions of multivariate test criteria have appeared in various journals. These include the distributions of, one two three to several latent roots of Wishart matrices, products ratios and other functions of latent roots of Wishart matrices and that too for the real as well as for the complex variable cases. Due to the multiplicity of articles on parallel topics it appears that some misleading implications have crept in some of these articles. Articles on the exact distributions of multivariate test criteria, where the properties of Meijer's G-function and hypergeometric function are used, are examined in this article. Some of the misleading implications and shortcomings are pointed out and a general technique to solve the problems under consideration is also given in this article.

Meijer's G-function is defined in ([3], p. 207) as follows.

$$(1.1)\quad G_{p,q}^{m,n}\left(z\left|\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_C \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}z^{-s}ds\ ,\qquad i=(-1)^{1/2}\ .$$

For convenience s is replaced by $(-s)$ in (1.1). C is a contour separating the poles of $\Gamma(b_j+s)$, $j=1,2,\cdots,m$ and $\Gamma(1-a_j-s)$, $j=1,2,\cdots,n$ and the various types of contours and the various existence conditions are discussed in ([3], p. 207) and hence the details are omitted here. a_j's and b_j's are complex numbers such that

$$(1.2)\quad 1-a_j+r\neq -b_k-v\ ,\qquad j=1,2,\cdots,n;\ k=1,2,\cdots,m;\quad r,v=0,1,\cdots,0\leqq n\leqq p,\ 1\leqq m\leqq q\ .$$

An empty product is interpreted as unity. From the structure of the

Gammas in the integrand of (1.1) it is apparent that if a statistical test criterion or a one-to-one function of the criterion is structurally products and ratios of independent Gamma or Beta variates then the sth moment of such a criterion can give the type of Gamma product appearing in (1.1) and hence its density can be written as a G-function. Also if a test criterion is structurally products and ratios of rational powers of independent Gamma and Beta variates the density of the criterion can be written as a G-function after a little transformation of the variable and simplification of the Gammas with the help of Gauss-Legendre multiplication formula, namely,

$$\Gamma(mz)=(2\pi)^{(1-m)/2}m^{mz-1/2}\prod_{j=0}^{m-1}\Gamma(z+j/m)\ . \tag{1.3}$$

These observations follow trivially from the definition of the G-function itself. These are also pointed out in [5].

Consul [2] discussed at length the distributions of some multivariate test criteria and wrote their densities in G-functions. As mentioned earlier these follow trivially from the definition of a G-function. Also formulae (4.7), (5.2), (6.6) and (7.1) of [7], (2.14) and (2.19) of [8], Theorems 3.1 and 4.1 of [9] follow trivially from the definition of a G-function. From (1.1) it is evident that a G-function is only a contour integral representation and a statement to the effect that the density of a particular test criterion is a G-function does not carry any meaning beyond giving the moment expression for the test criterion. The problem of getting the distribution is not solved unless the G-function is put into computable forms or in terms of elementary Special Functions. Pillai, Al-Ani and Jouris [7] have stated on p. 2036 that a general G-function can be written in terms of generalized hypergeometric functions and they have given the formula ([7], p. 2036 (3.3)) for this purpose. Their implication in ([7], (3.3)) is not correct because ([7], (3.3)) is given in ([3], p. 208) which in turn is the reduction formula when the poles of the integrand are simple whereas the problems under consideration in [7] all have poles of higher orders except in very particular cases such as the case for $p=2$ given there. When $p=2$ the distributions are trivially available from the distribution of a product of two independent Beta variates. The statement regarding the reduction of a G-function into hypergeometric functions is again repeated on p. 764 of [9].

The problems discussed in [7] are rediscussed in [8] for the complex Gaussian case. In ([5], Chapter v) a general technique is suggested to derive the distributions of the test criteria considered in [7]. In this article we will discuss the exact distributions of the test criteria considered in [8] and [9]. Since there is no significant difference mathematically between the real and complex cases only an outline is given here.

2. The exact distributions of a collection of test criteria

Pillai and Jouris [8] considered the problems of testing equality of two covariance matrices, multivariate analysis of variance model and the canonical correlation in the complex Gaussian case. Let $X:(p\times n_1)$ and $Y:(p\times n_2)$ have complex Gaussian distributions $N_c(0, \Sigma_1)$ and $N_c(0, \Sigma_2)$ respectively with $n_1, n_2 \geqq p$. Let,

$$W_1=\prod_{i=1}^{p}(1-w_i)\ , \qquad w_i=f_i/(1+f_i) \tag{2.1}$$

where $0<f_1<\cdots<f_p<\infty$ are the latent roots of $(X\bar{X}')(Y\bar{Y}')^{-1}$ where for example $\bar{X}'$ denotes the conjugate transpose of X. The hth moment of W_1 is given in [8] as,

$$\mathrm{E}\,(W_1^h)=|\Lambda|^{-n_1}\frac{\tilde{\Gamma}_p(n)\tilde{\Gamma}_p(n_2+h)}{\tilde{\Gamma}_p(n_2)\tilde{\Gamma}_p(n+h)}\,{}_2\tilde{F}_1(n, n_1; n+h; I_p-\Lambda^{-1}) \tag{2.2}$$

where $n=n_1+n_2$, ${}_2\tilde{F}_1(\cdot)$ is a hypergeometric function with matrix argument, Λ is a diagonal matrix with the diagonal elements as the latent roots of $\Sigma_1\Sigma_2^{-1}$, and for example,

$$\tilde{\Gamma}_p(a)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a-i+1)\ . \tag{2.3}$$

The ${}_2\tilde{F}_1(\cdot)$ in (2.2) has the usual series representation as in the real variable case with the Gammas replaced by $\tilde{\Gamma}(\cdot)$ and the Zonal polynomial $C_K(\cdot)$ replaced by $\tilde{C}_K(\cdot)$ where $\tilde{C}_K(\cdot)$ denotes a Zonal polynomial of a hermitian matrix and K is the partition of the non-negative integer k into not more than p parts $k_1, k_2, \cdots, k_p$ such that $k_1\geqq k_2\geqq\cdots\geqq k_p\geqq 0$, $k_1+k_2+\cdots+k_p=k$. Evidently an integral representation of the density function $f_1(w_1)$ of W_1 is available by expanding ${}_2\tilde{F}_1(\cdot)$ in (2.2) and taking the inverse Mellin transform of (2.2). That is,

$$f_1(w_1)=\frac{\tilde{\Gamma}_p(n)|\Lambda|^{-n_1}}{\tilde{\Gamma}_p(n_2)}\sum_{k=0}^{\infty}\sum_{K}\frac{[n]_K[n_1]_K}{k!}\tilde{C}_K(I_p-\Lambda^{-1})w_1^{n_2-p} \times G_{p,p}^{p,0}\left(w_1\middle|\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\right) \tag{2.4}$$

where for example

$$[a]_K=\prod_{i=1}^{p}(a-i+1)_{k_i}\ , \qquad (a)_r=a(a+1)\cdots(a+r-1); \tag{2.5}$$
$$n=n_1+n_2,\ a_i=n_1+k_{p-i+1}+b_i,\ b_i=i-1,\ i=1, 2,\cdots, p\ .$$

As remarked earlier, (2.4) is only an integral representation and the distribution is obtained only if the G-function in (2.4) is put into tractable forms.

Now consider the multivariate analysis of variance problem. Let $X:(p\times n_1)\sim N_c(\mu,\Sigma)$ and $Y:(p\times n)\sim N_c(0,\Sigma)$ be independent with $n_1, n \geq p$. Let W_2 be as defined in (2.1) and let Ω be a diagonal matrix with the diagonal elements as the latent roots of $\mu\bar{\mu}'\Sigma^{-1}$. The hth moment of W_2 and the density function $f_2(w_2)$ of W_2 are given in [8] as follows.

$$(2.6)\qquad f_2(w_2)=\exp(-\mathrm{tr}\,\Omega)\frac{\tilde{\Gamma}_p(n_1+n)}{\tilde{\Gamma}_p(n)}\sum_{k=0}^{\infty}\sum_{K}\frac{[n_1+n]_K}{k!}\tilde{C}_K(\Omega)w_2^{n-p}\times G_{p,p}^{p,0}\left(w_2\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\right)$$

where $a_1,\cdots,a_p$ and $b_1,\cdots,b_p$ are as defined in (2.5). Now consider the problem of canonical correlation. Let,

$$\begin{bmatrix}X:p\times n\\ Y:q\times n\end{bmatrix}\sim N_c\left[0,\begin{pmatrix}\Sigma_{11}, & \Sigma_{12}\\ \bar{\Sigma}_{12}', & \Sigma_{22}\end{pmatrix}\right]$$

$n\geq p+q$ and $q\geq p$. Let $0<r_1^2<\cdots<r_p^2$ be the latent roots of $(X\bar{Y}')\cdot(Y\bar{Y}')^{-1}(Y\bar{X}')(X\bar{X}')^{-1}$ and let P^2 be a diagonal matrix with diagonal elements as the latent roots of $\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}_{12}'$. Then the density of,

$$(2.7)\qquad W_3=\prod_{i=1}^{p}(1-r_i^2)$$

is available from (2.4) by replacing (n_1, n_2, A) by $(n, n-q, (I_p-P^2)^{-1})$ and a_i by $q+k_{p-i+1}+b_i$. The G-functions appearing in the densities of W_1, W_2 and W_3 are available from

$$(2.8)\qquad f(x)=G_{p,p}^{p,0}\left(x\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\right)\qquad \text{where } b_i=i-1,\ a_i=m+k_{p-i+1}+b_i\quad i=1,2,\cdots,p;\ m\geq p\,.$$

When m is replaced by n_1, n_1 and q we get the G-functions appearing in the densities of W_1, W_2 and W_3 respectively. The problem of deriving the distributions of W_1, W_2 and W_3, which is considered in [8], is solved only if (2.8) is put into tractable forms. This can be achieved by the following procedure. $f(x)$ can also be written in the following form.

$$(2.9)\qquad f(x)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\Delta(s)x^{-s}ds$$

where

$$(2.10)\qquad \Delta(s)=\{\Gamma(s)\Gamma(s+1)\cdots\Gamma(s+p-1)/[\Gamma(s+m+k_p)$$

$$\times\Gamma(s+m+k_{p-1}+1)\cdots(s+m+k_1+p-1)]\}$$
$$=\prod_{j\in\beta}(s+j)^{-\beta_j}$$

where

$$(2.11)\quad \beta_j=\begin{cases} j+1\,, & j=0,1,\cdots,p-2\,,\\ p\,, & j=p-1,p,\cdots,m+k_p-1\,,\\ p-i\,, & j=m+k_{p-i+1}+i-1,\cdots,m+k_{p-i}+i-1,\\ & \qquad i=1,2,\cdots,p-1\end{cases}$$

and

$$(2.12)\qquad \beta=\{0,1,\cdots,m+k_1+p-2\}\,.$$

The contour in (2.9) is one of the choices of the contours available for the G-function. In this problem (2.8) and (2.9) are the same and further (2.9) exists and it can be evaluated as the sum of the residues at the poles of the integrand in (2.9). Since these results are all known in the theory of G-functions the details are omitted here. The simplification in (2.10) is achieved by cancelling all the common factors. Now (2.9) can be evaluated either by partial fraction technique or by the method of Calculus of residues. Evidently (2.9) is a finite sum involving terms of the type $x^{\alpha_1}(-\log x)^{\alpha_2}$, $0\leqq\alpha_1\leqq m+k_1+p-2$, $0\leqq\alpha_2\leqq p-1$. The method of residues is explained in ([4], [5]) and hence the results are given here without the details. That is,

$$(2.13)\qquad f(x)=\sum_{j\in\beta}\frac{x^j}{(\beta_j-1)!}\sum_{v=0}^{\beta_j-1}\binom{\beta_j-1}{v}(-\log x)^{\beta_j-1-v}$$
$$\times\left\{\left[\sum_{v_1=0}^{v-1}\binom{v-1}{v_1}A_j^{(v-1-v_1)}\sum_{v_2=0}^{v_1-1}\binom{v_1-1}{v_2}A_j^{(v_1-1-v_2)}\cdots\right]B_j\right\}$$

where

$$(2.14)\quad B_j=\prod_{\substack{i\in\beta\\ i\neq j}}(-j+i)^{-\beta_i}\quad\text{and}\quad A_j^{(r)}=(-1)^{r+1}r!\sum_{\substack{i\in\beta\\ i\neq j}}[\beta_i/(-j+i)^{r+1}]\,,\qquad r\geqq 0\,.$$

For various values of p explicit expressions are available from (2.13). For $p=2$. (2.13) reduces to the following form.

$$(2.15)\qquad f(x)=B_0+\sum_{j=1}^{m+k_2-1}B_j[A_j+(-\log x)]x^j+\sum_{j=m+k_2}^{m+k_1}B_jx^j$$

where B_j and $A_j=A_j^{(0)}$ are given in (2.14). Using a reduction formula for $G_{2,2}^{2,0}\left(x\middle|\begin{matrix}a_1,a_2\\ b_1,b_2\end{matrix}\right)$ in terms of hypergeometric function, Pillai and Jouris [8] obtained a particular case for $p=2$. It should be remarked that their formula ([8], (2.15)) is not true in general unless further conditions are put on the parameters. Also there is a real danger in writing the particular cases in terms of hypergeometric functions without stating the nature of the series which is put into a hypergeometric function because a hypergeometric function does not always have a simple power series expansion which can be seen from ([3], p. 202(2), p. 63(18), (19), p. 74(2), (3)). However their formula ([8], (2.16)) is valid which also follows trivially from the distribution of a product of two independent Beta variates.

3. The sphericity test

Consider the problem of testing the hypothesis, $H_0 : \Sigma=\sigma^2 I_p$, where $\sigma^2>0$ is unknown and I_p is an identity matrix, in a multivariate normal case $N(\mu, \Sigma)$ against the alternative $\Sigma\neq\sigma^2 I_p$. The likelihood ratio criterion is of the form,

$$W=|S|/[(\mathrm{tr}\, S)/p]^p \tag{3.1}$$

which is given in ([1], [9]). The exact distribution of W in the null case is given by Mathai and Rathie [6] and in the non-null case the hth moment of W is given in [9] as follows.

$$\mathrm{E}\,(W^h)=\frac{p^{ph}|\Sigma|^{-n/2}}{\Gamma_p(n/2)}\sum_{k=0}^{\infty}\sum_{K}\frac{C_K(M)2^k\Gamma_p(n/2+h, K)\Gamma(np/2+k)}{k!\Gamma(np/2+ph+k)} \tag{3.2}$$

in the real normal case where $M=(I-\Sigma^{-1})/2$ and in the complex normal case,

$$\mathrm{E}\,(W^h)=\frac{p^{ph}|\Sigma|^{-n}}{\tilde{\Gamma}_p(n)}\sum_{k=0}^{\infty}\sum_{K}\frac{\tilde{C}_K(\tilde{M}_1)\Gamma(np+k)\tilde{\Gamma}_p(n+h, K)}{k!\Gamma(np+k+ph)} \tag{3.3}$$

where $\tilde{M}_1=I_p-\tilde{\Sigma}^{-1}$ and for example,

$$\Gamma_p(a, K)=\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(a+k_i-(i-1)/2) \tag{3.4}$$

and

$$\tilde{\Gamma}_p(a, K)=\pi^{p(p-1)/2}\prod_{i=1}^{p}\Gamma(a+k_i-i+1)\,. \tag{3.5}$$

Also Pillai and Nagarsenker [9] have written the density of W in terms of a G-function. Remarks about their Theorems 3.1 and 4.1 and their particular case $p=2$ are already made in Sections 1 and 2 of this article. If $\Gamma(np+ph+k)$ is rewritten with the help of Gauss-Legendre multiplication formula (1.3) then the density $g(w)$ of W in (3.3) can be written as follows.

$$(3.6)\qquad g(w)=\frac{\pi^{p(p-1)/2}|\Sigma|^{-n}(2\pi)^{(p-1)/2}}{\tilde{\Gamma}_p(n)}\sum_{k=0}^{\infty}\sum_{K}\frac{\tilde{C}_K(\tilde{M}_1)\Gamma(np+k)p^{1/2-pn-k}w^{n-p}}{k!}\times G^{p,0}_{p,p}\left(w\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\right)$$

where $a_j=k/p+(j-1)/p+p-1$, $b_j=k_j-j+p$, $j=1,2,\cdots,p$. Pillai and Nagarsenker ([9], p. 764) assert to the effect that (3.6) can be written in terms of hypergeometric functions. This assertion is not correct which is already pointed out in Sections 1 and 2. The density of W is obtained only if the G-function in (3.6) is put into tractable forms. In this section the G-function in (3.6) will be put into a computable series. Let,

$$(3.7)\qquad h(w)=G^{p,0}_{p,p}\left(w\,\middle|\,\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_p\end{matrix}\right)$$

with a_j's and b_j's as in (3.6) then $h(w)$ can be written as,

$$(3.8)\qquad h(w)=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\Delta(s)w^{-s}ds$$

where

$$(3.9)\qquad \Delta(s)=\Gamma(s+k_1+p-1)\Gamma(s+k_2+p-2)\cdots\Gamma(s+k_p)/[\Gamma(s+k/p+p-1)\Gamma(s+k/p+p-1+1/p)\cdots\times\Gamma(s+k/p+(p-1)/p+p-1)]$$

by using the same notations as in Section 2. Now (3.8) can be evaluated by the method of residues. (3.9) can be simplified by cancelling out all the common factors. Let,

$$(3.10)\qquad k=mp+r\qquad\text{for some fixed } m \text{ and } r,\ m=0,1,\cdots;\quad 0\leqq r\leqq p-1\,.$$

Since $k_1\geqq k_2\geqq\cdots\geqq k_p\geqq 0$ and $k_1+\cdots+k_p=k$, $k_1+p-1\geqq m+p-1\geqq k_p$ so that $\Gamma(s+m+p)$ can be cancelled with $\Gamma(s+k_p)$. That is, $\Delta(s)$ can be simplified to the following form.

$$(3.11)\qquad \Delta(s)=\left[\prod_{i=1}^{p-1}\Gamma(s+k_i+p-i)\right]\Big/\left\{\left[\prod_{\substack{i=1\\ i\neq p-r+1}}^{p}\Gamma(s+m+p-1+(r+i-1)/p\right]\cdot[(s+m+p-1)(s+m+p-2)\cdots(s+k_p)]\right\}$$

$$=\Gamma^{p-1}(s+k_1+p-1)\Big/\left\{\prod_{i\in\alpha}(s+i)^{\alpha_i}\left[\prod_{\substack{i=1\\ i\neq p-r+1}}^{p}\Gamma(s+m+p-1+(r+i-1)/p)\right]\right\}$$

where

$$\alpha=\{k_p, k_p+1, \cdots, k_1+p-2\}$$

and

$$(3.12)\qquad \alpha_i=\begin{cases} 1\,, & i=k_p, k_p+1, \cdots, k_{p-1}\,, \\ 2\,, & i=k_{p-1}+1, k_{p-1}+2, \cdots, k_{p-2}+1\,, \\ \vdots & \vdots \\ t-1\,, & i=k_{p-t}+t, k_{p-t}+t+1, \cdots, k_{p-t}+t+u\,, \\ t\,, & i=k_{p-t}+t+u+1, \cdots, k_{p-(t+1)}+t\,, \\ t-1\,, & i=k_{p-t-1}+t+1, \cdots, k_{p-t-2}+t+1\,, \\ \cdots & \\ p-2\,, & i=k_2+p-2, k_2+p-1, \cdots, k_1+p-2 \end{cases}$$

where it is assumed that $m+p-1=t+k_{p-t}+u$ for some fixed t and u $(0\leqq u\leqq k_{p-t-1}-k_{p-t})$. The poles of the integrand in (3.8) are available by equating to zero the various factors of $\prod_{j\in\beta}(s+j)^{\beta_j}$ where $\beta_j=\alpha_j$ for the $j\in\alpha$ and $\beta_j=p-1$ for $j=k_1+p-1, k_1+p, \cdots$ and

$$\beta=\{k_p, k_p+1, \cdots, k_1+p-1, \cdots\}\,.$$

Now $h(w)$ can be written by using the procedure discussed in Section 2. That is, $h(w)$ is the same as (2.13) with x replaced by w and with B_j and A_j as given below. For $j<k_1+p-1$,

$$(3.13)\qquad B_j=\Gamma^{p-1}(-j+k_1+p-1)\Big/\Big\{\prod_{\substack{i\in\alpha\\ i\neq j}}(-j+i)^{\alpha_i}\prod_{\substack{i=1\\ i\neq p-r+1}}^{p}\Gamma(-j+m+p-1+(r+i-1)/p)\Big\}\,;$$

$$(3.14)\qquad A_j=(p-1)\Psi(-j+k_1+p-1)-\sum_{\substack{i\in\alpha\\ i\neq j}}[\alpha_i/(-j+i)] - \sum_{\substack{i=1\\ i\neq p-r+1}}^{p}\Psi(-j+m+p-1+(r+i-1)/p)\,;$$

$$(3.15)\qquad A_j^{(q)},\ q\geqq 1,=(-1)^{q+1}q!\Big\{(p-1)\zeta(q+1, -j+k_1+p-1) +\sum_{\substack{i\in\alpha\\ i\neq j}}[\alpha_i/(-j+i)^{q+1}]-\sum_{\substack{i=1\\ i\neq p-r+1}}^{p}\zeta(q+1, -j+m+p-1+(r+i-1)/p)\Big\}\,;$$

and for $j\geqq k_1+p-1$,

$$(3.16)\qquad B_j=1\Big/\Big\{[(-1)(-2)\cdots(-j+k_1+p-1)]^{p-1}$$

$$\times \prod_{i\in\alpha} (-j+i)^{\alpha_i} \prod_{\substack{i=1\\ i\neq p-r+1}}^{p} \Gamma(-j+m+p-1+(r+i-1)/p)\Big\} ;$$

$$\text{(3.17)}\quad A_j=(p-1)\Psi(1)-(p-1)[1/(-1)+1/(-2)+\cdots+1/(-j+k_1+p-1)] -\sum_{i\in\alpha}[\alpha_i/(-j+i)]-\sum_{\substack{i=1\\ i\neq p-r+1}}^{p} \Psi(-j+m+p-1+(r+i-1)/p) ;$$

$$\text{(3.18)}\quad A_j^{(q)},\ q\geqq 1,=(-1)^{q+1}q!\Big\{(p-1)\zeta(q+1,1)+(p-1)[1/(-1)^{q+1} +1/(-2)^{q+1}+\cdots+1/(-j+k_1+p-1)^{q+1}] +\sum_{i\in\alpha}[\alpha_i/(-j+i)^{q+1}]-\sum_{\substack{i=1\\ i\neq p-r+1}}^{p} \zeta(q+1,\ -j+m+p-1 +(r+i-1)/p)\Big\} ;$$

where the Psi and the generalized Zeta function are defined as follows.

$$\text{(3.19)}\qquad \Psi(z)=-\gamma+(z-1)\sum_{n=0}^{\infty}[(n+1)(n+z)]^{-1}$$

where γ is the Euler's constant: $\gamma=0.577\cdots$

$$\text{(3.20)}\qquad \zeta(z, v)=\sum_{n=0}^{\infty}(v+n)^{-z}, \qquad v\neq 0, -1,\cdots,\ R(z)>1$$

where $R(\cdot)$ denotes the real part of $(\cdot)$. Particular cases can be easily written down from (2.13) and (3.12) to (3.18) and the cumulative distribution function is available by term by term integration. The real case, that is the density function corresponding to (3.2) can also be worked out in a similar fashion. In this case two cases p-even and p-odd are to be considered separately. Since the technique is the same the discussion is omitted.

McGILL UNIVERSITY, MONTREAL, CANADA

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