THE DISTRIBUTION OF A TRUNCATED LINEAR DIFFERENCE BETWEEN INDEPENDENT CHI-SQUARE VARIATES

DAVID A. HARVILLE

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1. Introduction

Quadratic estimators of the variance components associated with random-effects models are in common use. If the experimental design is a balanced completely-nested one, if the usual independence and normality assumptions are appropriate, and if the estimators are those derived by analysis-of-variance techniques, then each of the estimators, save that of the residual variance, is distributed as a linear difference between two independent chi-square variates. More generally, any quadratic form in multivariate-normal data and in particular any quadratic variance-component estimator based on such data is either distributed as a linear difference between two independent chi-squares or, as Press [2] has shown, its distribution can be represented as a mixture of such distributions.

In cases where a quadratic estimator of a variance component can take on negative values, it is common practice to replace negative estimates by zero. If the data are multivariate-normal, the distribution of the modified estimator is identical to that of a truncated (at zero) linear difference between independent chi-squares or is a mixture of such truncated distributions.

It will be our purpose to investigate the distribution of a truncated linear difference between independent chi-square variates. Rather than assume that the truncation is from below and at zero, we will consider the more general situation where the truncation is either from above or below or both and at arbitrary values. In particular, we will obtain expressions for the probability density function, distribution function, moments, moment generating function, and characteristic function of such a distribution. In the case of the probability density function, distribution function, and moments, emphasis will be on obtaining representations that appear to be convenient computational forms. Extensive use will be made of known results on hypergeometric functions.

Previously, Press [2] obtained the probability density function of a linear difference between independent chi-squares in terms of a confluent hypergeometric function of the second kind. Wang [5] found a simpler expression for the probability density function for the special case where the degrees of freedom associated with both chi-squares are even. She also gave, for that same special case, convenient expressions for the moments of that distribution, after its truncation from below at zero.

2. Notation and preliminary results

Define

$$X=\alpha W_1-\beta W_2 ,$$

where α and β are positive constants, and W_1 and W_2 are independent chi-square variates having m and n degrees of freedom, respectively. Truncation of X from above and/or below yields the random variable

$$Y=\begin{cases} y_1 & \text{if } X>y_1 , \\ X & \text{if } y_1\geqq X\geqq y_0 , \\ y_0 & \text{if } X<y_0 , \end{cases}$$

where y_0 and y_1 are known constants with $-\infty\leqq y_0<y_1\leqq\infty$.

A chi-square variate with r degrees of freedom has kth moment $2^k(r/2)_k$, where $(a)_0=1$ and $(a)_k=a(a+1)\cdots(a+k-1)$, $k=1, 2,\cdots$; moment generating function

$$M(\theta)=(1-2\theta)^{-r/2} , \qquad \theta<1/2 ;$$

and characteristic function

$$C(\theta)=(1-2i\theta)^{-r/2} .$$

It follows that X has kth moment

$$(2\alpha)^k\sum_{j=0}^{k}\binom{k}{j}(-\zeta)^j(n/2)_j(m/2)_{k-j} , \tag{1}$$

where $\zeta=\beta/\alpha$; moment generating function

$$M_X(\theta)=(1-2\alpha\theta)^{-m/2}(1+2\beta\theta)^{-n/2} , \qquad (-2\beta)^{-1}<\theta<(2\alpha)^{-1} ; \tag{2}$$

and characteristic function

$$C_X(\theta)=(1-2\alpha i\theta)^{-m/2}(1+2\beta i\theta)^{-n/2} .$$

Take

$$U(c, d; x)=[1/\Gamma(c)]\int_0^\infty e^{-xt}t^{c-1}(1+t)^{d-c-1}dt\ , \qquad c>0,\ x>0\ ,$$

to be a confluent hypergeometric function of the second kind, and put

$$(3)\qquad g(r, s, a, b; t)=\frac{t^{(r+s-2)/2}e^{-t/(2a)}U[s/2, (r+s)/2; (a+b)t/(2ab)]}{2^{(r+s)/2}a^{r/2}b^{s/2}\Gamma(r/2)}$$

for $r=1, 2, \cdots$; $s=1, 2, \cdots$; $a>0$, $b>0$; and $t>0$.

From [2], we have that the probability density function of X is

$$(4)\qquad f_X(x)=\begin{cases} g(m, n, \alpha, \beta; x) & \text{if } x>0\ ,\\ g(n, m, \beta, \alpha; -x) & \text{if } x<0\ ,\\ \dfrac{\zeta^{m/2}\Gamma(m/2+n/2-1)}{2^{(m+n)/2}\beta[(\zeta+1)/2]^{(m+n-2)/2}\Gamma(m/2)\Gamma(n/2)} & \text{if } x=0\ .\end{cases}$$

It will prove convenient to express the distribution functions of X and Y and the moments, moment generating function, and characteristic function of Y in terms of the function

$$(5)\qquad G(r, s, a, b; k, z, u)=\begin{cases}\displaystyle\int_0^u t^k e^{zt} g(r, s, a, b; t)dt & \text{for } u>0\ ,\\ 0 & \text{for } u=0\ ,\end{cases}$$

where $k=0, 1, 2, \cdots$.

Note that, for $v<0$,

$$(6)\qquad \int_v^0 t^k e^{zt} g(s, r, b, a; -t)dt=(-1)^k G(s, r, b, a; k, -z, -v)\ .$$

The distribution function of X is

$$F_X(x)=\begin{cases} G(m, n, \alpha, \beta; 0, 0, x)+G(n, m, \beta, \alpha; 0, 0, \infty) & \text{if } x\geqq 0\ ,\\ G(n, m, \beta, \alpha; 0, 0, \infty)-G(n, m, \beta, \alpha; 0, 0, -x) & \text{if } x\leqq 0\ .\end{cases}$$

Denote by $P_{r,s}(\cdot)$ the distribution function of an F random variable having r and s degrees of freedom. Note that

$$P_{m,n}[(n\beta)/(m\alpha)]=F_X(0)=G(n, m, \beta, \alpha; 0, 0, \infty)\ .$$

The distribution function of Y is

$$F_Y(y)=\begin{cases} 1 & \text{if } y\geqq y_1\ ,\\ F_X(y) & \text{if } y_0\leqq y<y_1\ ,\\ 0 & \text{if } y<y_0\ .\end{cases}$$

If y_0 and y_1 are finite, $F_Y(\cdot)$ is discontinuous at those points. It makes a jump of $F_X(y_0)$ at y_0 and $1-F_X(y_1)$ at y_1. At other points, the probability density function of Y is

$$F'_Y(y)=\begin{cases} f_X(y) & \text{if } y_0<y<y_1\,, \\ 0 & \text{if } y<y_0 \text{ or } y>y_1\,. \end{cases}$$

Define

$$H(u_1, u_0; k, z)=\begin{cases} G(m, n, \alpha, \beta; k, z, u_1)-G(m, n, \alpha, \beta; k, z, u_0) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } 0\leqq u_0<u_1\,, \\ G(m, n, \alpha, \beta; k, z, u_1)+(-1)^k G(n, m, \beta, \alpha; k, -z, -u_0) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } u_0<0\leqq u_1\,, \\ (-1)^k[G(n, m, \beta, \alpha; k, -z, -u_0) \\ \qquad -G(n, m, \beta, \alpha; k, -z, -u_1)] \qquad \text{if } u_0<u_1<0\,; \end{cases}$$

and

$$H^*(u_1, u_0; k, z)=H(u_1, u_0; k, z)+u_0^k e^{zu_0}F_X(u_0)+u_1^k e^{zu_1}[1-F_X(u_1)] \quad \text{for } -\infty<u_0<u_1<\infty\,,$$

$$H^*(\infty, u_0; k, z)=H(\infty, u_0; k, z)+u_0^k e^{zu_0}F_X(u_0) \quad \text{for } -\infty<u_0\,,$$

$$H^*(u_1, -\infty; k, z)=H(u_1, -\infty; k, z)+u_1^k e^{zu_1}[1-F_X(u_1)] \quad \text{for } u_1<\infty\,,$$

and

$$H^*(\infty, -\infty; k, z)=H(\infty, -\infty; k, z)\,,$$

with $0^0=1$.

The kth moment of Y is $H^*(y_1, y_0; k, 0)$, and its moment generating function and characteristic function are

$$M_Y(\theta)=H^*(y_1, y_0; 0, \theta)\,, \quad \theta>(-2\beta)^{-1} \text{ if } y_0<0 \text{ and } \theta<(2\alpha)^{-1} \text{ if } y_1>0\,,$$

and

$$C_Y(\theta)=H^*(y_1, y_0; 0, i\theta)\,,$$

respectively.

It is clear that the problem of evaluating the probability density function of X or Y can be reduced to one of evaluating $g(r, s, a, b; t)$ for appropriate values of that function's parameters. Moreover, to evaluate the distribution function of X or Y and the moments and moment generating function of Y, it suffices to evaluate $G(r, s, a, b; k,$

$\theta, u)$ for appropriate values of r, s, a, and b and for $k=0, 1, 2, \cdots, \theta$ real, and $u>0$. For the most part, we will find it convenient to carry out the analysis for all such (k, θ, u) triplets, rather than to separately consider only those particular triplets that are relevant to the evaluation of the distribution function, the moments, and the moment generating function, respectively. Because, in (5), $t^k e^{zt}$ is 'absorbed' by $g(r, s, a, b; t)$, the more general approach is no more difficult. It has the advantage of avoiding repetition. Since the expressions for both the moment generating function and the moments are to be obtained directly, the derivatives at zero of expressions for the moment generating function can be compared with the corresponding expressions for the moments, so as to obtain a check on the correctness of the results.

Expressions for the characteristic function $C_Y(\theta)$ of Y will not be given explicitly; however, substitution of $i\theta$ for θ, in any of the expressions indicated for the moment generating function $M_Y(\theta)$, produces a valid representation for $C_Y(\theta)$, provided that representation is meaningful.

3. Evaluation of $g(r, s, a, b; x)$. *Useful relationships*

It is easy to show that, for $c>0$,

$$g(r, s, a, b; t)=cg(r, s, ca, cb; ct) . \tag{7}$$

The recurrence relations for confluent hypergeometric functions given by (13.4.15)–(13.4.20) of Slater [4] yield

$$\begin{aligned} g(r, s, a, b; t)=(b/a)(r-2)^{-1}\{[t(a^{-1}+b^{-1})+s-r+4] \\ \cdot g(r-2, s+2, a, b; t)+(b/a)(s+2) \\ \cdot g(r-4, s+4, a, b; t)\} , \end{aligned} \tag{8}$$

$$\begin{aligned} g(r, s, a, b; t)=b(a+b)^{-1}(r-2)^{-1}\{[t(a^{-1}+b^{-1})+s+r-4] \\ \cdot g(r-2, s, a, b; t)-(t/a)g(r-4, s, a, b; t)\} , \end{aligned} \tag{9}$$

$$\begin{aligned} g(r, s, a, b; t)=[a(r-2)]^{-1}[bsg(r-2, s+2, a, b; t) \\ +tg(r-2, s, a, b; t)] , \end{aligned} \tag{10}$$

$$g(r, s, a, b; t)=(a+b)^{-1}[bg(r-2, s, a, b; t)+ag(r, s-2, a, b; t)] , \tag{11}$$

$$\begin{aligned} g(r, s, a, b; t)=b(a+b)^{-1}(r-2)^{-1}\{[t(a^{-1}+b^{-1})+s] \\ \cdot g(r-2, s, a, b; t)+(bs/a)g(r-4, s+2, a, b; t)\} , \end{aligned} \tag{12}$$

and

$$\begin{aligned} g(r, s, a, b; t)=(b/a)(r-2)^{-1}\{[t(a^{-1}+b^{-1})+s]g(r-2, s+2, a, b; t) \\ -(t/a)g(r-4, s+2, a, b; t)\} . \end{aligned} \tag{13}$$

The probability density function of a chi-square variate with r degrees of freedom is

$$p_r(t)=\begin{cases} [2^{r/2}\Gamma(r/2)]^{-1}t^{(r/2)-1}e^{-t/2}, & 0<t<\infty, \\ 0, & \text{elsewhere}. \end{cases}$$

The domain of the function g can be extended by putting

$$g(r, 0, a, b; t)=(1/a)p_r(t/a)$$

for $r=1, 2, \cdots; a>0; b>0; t>0$. The relationships (7)–(13) remain valid. Thus, tables of the chi-square density may be of use in evaluating the densities of X and Y.

Case (i): *r an even integer.* For $c>0$ and $p=1, 2, \cdots$,

$$U(c, c+p; x)=[1/\Gamma(c)]\sum_{j=0}^{p-1}\binom{p-1}{j}\int_0^\infty e^{-xt}t^{c+j-1}dt = \sum_{j=0}^{p-1}\binom{p-1}{j}(c)_j x^{-c-j}. \tag{14}$$

Substitution of this expression into (3) gives

$$g(r, s, a, b; x)=[a2^{r/2}\Gamma(r/2)]^{-1}[a/(a+b)]^{s/2}e^{-x/(2a)} \cdot \sum_{j=0}^{r/2-1}\binom{r/2-1}{j}(s/2)_j[2b/(a+b)]^j(x/a)^{r/2-j-1}, \tag{15}$$

for $r=2, 4, 6, \cdots$.

Case (ii): *s an even integer.* Denote by $P_u(\cdot)$ the distribution function of a chi-square variate with u degrees of freedom. For $p=1, 2, \cdots$ and $2c=1, 2, \cdots$,

$$\begin{aligned} U(p, p+c; x) &= [1/\Gamma(p)]\int_1^\infty e^{-x(v-1)}(v-1)^{p-1}v^{c-1}dv \\ &= [1/\Gamma(p)]e^x\sum_{j=0}^{p-1}\binom{p-1}{j}(-1)^{p-j-1}\int_1^\infty e^{-xv}v^{c+j-1}dv \\ &= [1/\Gamma(p)]e^x\sum_{j=0}^{p-1}\binom{p-1}{j}(-1)^{p-j-1}x^{-c-j}\Gamma(c+j) \\ &\quad \cdot [1-P_{2c+2j}(2x)]. \end{aligned}$$

Substituting this expression into (3), we find

$$\begin{aligned} g(r, s, a, b; x) &= [b2^{s/2}\Gamma(s/2)]^{-1}[b/(a+b)]^{r/2}e^{x/(2b)} \\ &\quad \cdot \sum_{j=0}^{s/2-1}\binom{s/2-1}{j}(r/2)_j[2a/(a+b)]^j(-x/b)^{s/2-j-1} \\ &\quad \cdot \{1-P_{r+2j}[x(a+b)/(ab)]\}, \end{aligned}$$

for $s=2, 4, 6, \cdots$.

Case (iii): *Both r and s are odd integers.* Let

$$c_j(u, v)=\frac{(u)_j}{j!(v)_j}\,, \qquad j=1, 2, \cdots .$$

Take

$$\Phi(u, v; z)=\sum_{j=0}^{\infty} c_j(u, v)z^j$$

to be a confluent hypergeometric function of the first kind. This series is absolutely convergent for all values of u, v, and z, save $v=0, -1, -2, \cdots$. Its remainder after q terms will be denoted by the symbol

$$R_q(u, v; z)=\sum_{j=q}^{\infty} c_j(u, v)z^j .$$

The numerical evaluation of $U[s/2, (r+s)/2; (a+b)x/(2ab)]$ and consequently of $g(r, s, a, b; x)$ can be accomplished for $r=1, 3, 5, \cdots$, $s=1, 3, 5, \cdots$, by applying the well-known relationship

$$\text{(16)} \qquad U(v, 1+p; t)=[(-1)^p p!\Gamma(v-p)]^{-1}L(v, 1+p; t) + [(p-1)!/\Gamma(v)]\sum_{j=0}^{p-1} c_j(v-p, 1-p)t^{j-p}\,,$$

for $p=0, 1, 2, \cdots$; where, taking $\psi(t)=\Gamma'(t)/\Gamma(t)$ to be the logarithmic derivative of the gamma function and $\gamma=-\psi(1)=0.5772156649\cdots$ to be Euler's constant, $L(v, 1+p; t)$ is obtained by adding together the three convergent series

$$\text{(17)} \qquad (-\gamma-\ln t)\Phi(v, 1+p; t)\,,$$

$$\text{(18)} \qquad \sum_{j=0}^{\infty} c_j(v, 1+p)t^j[\psi(1+j+p)-\psi(v+j)]\,,$$

and

$$\text{(19)} \qquad \sum_{j=0}^{\infty} c_j(v, 1+p)t^j \sum_{u=1}^{j} u^{-1} .$$

Here, the meaningless sums $\sum_{j=0}^{-1}$ and $\sum_{u=1}^{0}$ are interpreted as zero.

The series (17), (18), and (19) can be approximated by their partial sums. We now show, for $0<v<1+p$ and $t>0$, how to estimate the errors incurred in these approximations. These results, together with (16) and tables of the gamma function, allow us to calculate $U(v, 1+p; t)$, for such v, p, and t values, with known accuracy.

Denote by $[v]$ the largest integer that is less than v. By assumption $0<v<1+p$ and $t>0$, so that

$$0<R_q(v,1+p;t)<R_q(1+[v],1+p;t)\ . \tag{20}$$

Furthermore, since $\psi'(y)>0$ and $\psi''(y)<0$ for $y>0$, we have

$$\begin{aligned} 0&<\sum_{j=q}^{\infty} c_j(v,1+p)t^j[\psi(1+j+p)-\psi(v+j)] \\ &<[\psi(1+q+p)-\psi(v+q)]R_q(v,1+p;t) && (21) \\ &<[\psi(1+q+p)-\psi(v+q)]R_q(1+[v],1+p;t)\ . && (22) \end{aligned}$$

Also, for $q\geqq 1$,

$$\begin{aligned} 0&<\sum_{j=q}^{\infty} c_j(v,1+p)t^j\sum_{u=1}^{j}u^{-1} \\ &\leqq\sum_{j=q}^{\infty} jc_j(v,1+p)t^j \\ &=vt(1+p)^{-1}R_{q-1}(1+v,2+p;t) \\ &\leqq vt(1+p)^{-1}R_{q-1}(2+[v],2+p;t)\ . && (23) \end{aligned}$$

The recurrence relationship

$$\begin{aligned} w(w+1)\Phi(y,w;t)-(w+1)(w+t)\Phi(y,w+1;t)& \\ +t(w-y+1)\Phi(y,w+2;t)&=0\ ; \end{aligned} \tag{24}$$

together with the formulas

$$\Phi(j,j;t)=e^t$$

and

$$\Phi(j,1+j;t)=(-1)^j j!t^{-j}[1-e^t e_{j-1}(-t)]\ ,\qquad j=1,2,\cdots,$$

where $e_q(t)=\sum_{j=0}^{q} t^j/j!$; can be used to construct convenient procedures for computing $R_q(1+[v],1+p;t)$ and $R_{q-1}(2+[v],2+p;t)$ and thus, in light of (20), (22), and (23), can be used for estimating errors resulting from the use of partial sums in place of the series (17), (18), and (19), and for determining the number of terms that need be included in each partial sum.

For many v, p, and t values, it may be advantageous to modify the outlined procedures for approximating the series (17), (18), and (19) with known accuracy. Existing tables, Kummer's transformations, recurrence relationships like (24), and/or other techniques from the theory of confluent hypergeometric functions (see e.g. [3] or [4]) can often be utilized to approximate $\Phi(v,1+p;t)$ and consequently the series (17) in more efficient fashion than the partial-sum approach described above. Our more efficient approximation for $\Phi(v,1+p;t)$ also yields $R_q(v,1+p;t)$ which, upon applying (21), can also be used to estimate the error resulting from approximating (18) by one of its partial sums. Note that this error estimate is more precise than the one based on $R_q(1+[v],$

$1+p; t)$. Similarly, it may be possible to approximate $\Phi(1+v, 2+p; t)$ and consequently $R_{q-1}(1+v, 2+p; t)$ to known accuracy with techniques not requiring the computation of $R_{q-1}(2+[v], 2+p; t)$; thus permitting more precise, and possibly easier, estimation of errors resulting from truncation of the series (19).

4. Evaluation of $G(r, s, a, b; k, \theta, x)$. *Useful relationships*

Using (7), we find

$$G(r, s, a, b; k, \theta, x)=c^{-k}G(r, s, ca, cb; k, \theta/c, cx) .$$

Upon substituting the expression (8) for g into the definition (5), we obtain

$$\begin{aligned} &G(r, s, a, b; k, \theta, x)\\ &\quad=(b/a)(r-2)^{-1}\{(a^{-1}+b^{-1})G(r-2, s+2, a, b; k+1, \theta, x)\\ &\qquad+(s-r+4)G(r-2, s+2, a, b; k, \theta, x)\\ &\qquad+(b/a)(s+2)G(r-4, s+4, a, b; k, \theta, x)\} .\end{aligned}$$

Recurrence relations for G, corresponding to (9)-(13), can easily be obtained in the same way.

The domain of G can be extended by defining $G(r, 0, a, b; k, \theta, x)$ in terms of $g(r, 0, a, b; \cdot)$ through application of (5). Our recurrence relations for G remain valid. Thus, tables of the chi-square distribution function may prove useful in evaluating the distribution functions of X and Y and the moments of Y.

By making use of (1), (2), (4), and (6), we obtain

$$\begin{aligned} G(r, s, a, b; k, 0, \infty)=&(-1)^{k+1}G(s, r, b, a; k, 0, \infty)\\ &+(2a)^k\sum_{j=0}^{k}\binom{k}{j}(-b/a)^j(s/2)_j(r/2)_{k-j} \end{aligned}\tag{25}$$

and

$$\begin{aligned} G(r, s, a, b; 0, \theta, \infty)=&(1-2a\theta)^{-r/2}(1+2b\theta)^{-s/2}\\ &-G(s, r, b, a; 0, -\theta, \infty) ,\\ &-1/(2b)<\theta<1/(2a) .\end{aligned}\tag{26}$$

Case (i): *r an even integer*. Using (15), we find

$$\begin{aligned} &G(r, s, a, b; k, \theta, x)\\ &\quad=[a2^{r/2}\Gamma(r/2)]^{-1}[a/(a+b)]^{s/2}\\ &\qquad\cdot\sum_{j=0}^{r/2-1}\binom{r/2-1}{j}(s/2)_j[2b/(a+b)]^j a^{-r/2+j+1} \end{aligned}$$

$$\cdot \int_0^x t^{r/2+k-j-1} \exp[-t(1-2a\theta)/(2a)]dt$$

$$= [2^{r/2}\Gamma(r/2)]^{-1}[a/(a+b)]^{s/2}a^k$$

$$\cdot \sum_{j=0}^{r/2-1} \binom{r/2-1}{j} (s/2)_j[2b/(a+b)]^j(1-2a\theta)^{-r/2-k+j}$$

$$\cdot \int_0^{x(1-2a\theta)/a} y^{r/2+k-j-1}e^{-y/2}dy$$

$$= (2a)^k[a/(a+b)]^{s/2} \sum_{j=0}^{r/2-1} \frac{(s/2)_j(r/2-j)_k}{j!}\left(\frac{b}{a+b}\right)^j \tag{27}$$

$$\cdot (1-2a\theta)^{-r/2-k+j}P_{r+2k-2j}[x(1-2a\theta)/a] ;$$

for $r=2, 4, 6, \cdots$; $\theta<1/(2a)$; $0\leqq x\leqq\infty$.

In particular,

$$G(r, s, a, b; k, \theta, \infty) = (2a)^k[a/(a+b)]^{s/2} \sum_{j=0}^{r/2-1} \frac{(s/2)_j(r/2-j)_k}{j!} \tag{28}$$

$$\cdot \left(\frac{b}{a+b}\right)^j (1-2a\theta)^{-r/2-k+j} ;$$

for $r=2, 4, 6, \cdots$; $\theta<1/(2a)$.

Applying the well-known relationship

$$P_{2v}(2x) = [\Gamma(v+1)]^{-1}x^v\Phi(v, 1+v; -x)$$

to (27), we obtain, for $0\leqq x<\infty$, the alternate representation

$$G(r, s, a, b; k, \theta, x)$$

$$= x^k[x/(2a)]^{r/2}[a/(a+b)]^{s/2}$$

$$\cdot \sum_{j=0}^{r/2-1} \frac{(s/2)_j}{j!(r/2-j-1)!(r/2+k-j)}\left(\frac{2ab}{x(a+b)}\right)^j$$

$$\cdot \Phi[r/2+k-j, r/2+k-j+1; -x(1-2a\theta)/(2a)] ;$$

$$r=2, 4, 6, \cdots; \ \theta<1/(2a) .$$

Case (ii): *s an even integer.* By working with the joint probability distribution of the random variables W_1 and X, we obtain the representation

$$f_X(t) = (1/\beta) \int_{\max(0, t/\alpha)}^{\infty} p_m(u)p_n[(\alpha u-t)/\beta]du \tag{29}$$

for the probability density function of X.

Using (29), together with (4) and (6), we find; for $s=2, 4, 6, \cdots$; $-1/(2b)<\theta<1/(2a)$; $0<x<\infty$;

$$G(r, s, a, b; k, \theta, x)$$

$$= (-1)^{k+1}G(s, r, b, a; k, -\theta, \infty)$$

$$+(1/b)\int_{-\infty}^{x} dtt^k e^{\theta t}\int_{\max(0,t/a)}^{\infty} p_r(u)p_s[(au-t)/b]du$$
$$=(-1)^{k+1}G(s, r, b, a; k, -\theta, \infty)$$
$$+a^k(a/b)\int_{-\infty}^{x/a} dvv^k e^{a\theta v}\int_{\max(0,v)}^{\infty} p_r(u)p_s[a(u-v)/b]du$$
$$=(-1)^{k+1}G(s, r, b, a; k, -\theta, \infty) \tag{30}$$
$$+G_1(r, s, a, b; k, \theta, x)+G_2(r, s, a, b; k, \theta, x)$$

where

$$G_1(r, s, a, b; k, \theta, x)$$
$$=a^k(a/b)\int_0^{x/a} dup_r(u)\int_{-\infty}^{u} v^k e^{a\theta v}p_s[a(u-v)/b]dv$$
$$=a^k\int_0^{x/a} dup_r(u)\int_0^{\infty}(u-by/a)^k e^{a\theta(u-by/a)}p_s(y)dy$$
$$=a^k\int_0^{x/a} dup_r(u)e^{a\theta u}\sum_{j=0}^{k}\binom{k}{j}u^j(-b/a)^{k-j}\int_0^{\infty} y^{k-j}e^{-b\theta y}p_s(y)dy$$
$$=(-2b)^k(1-2a\theta)^{-r/2}(1+2b\theta)^{-s/2-k}$$
$$\cdot\sum_{j=0}^{k}\binom{k}{j}(r/2)_j(s/2)_{k-j}(-a/b)^j[(1+2b\theta)/(1-2a\theta)]^j$$
$$\cdot P_{r+2j}[x(1-2a\theta)/a]$$

and

$$G_2(r, s, a, b; k, \theta, x)$$
$$=a^k(a/b)\int_{x/a}^{\infty} dup_r(u)\int_{-\infty}^{x/a} v^k e^{a\theta v}p_s[a(u-v)/b]dv$$
$$=a^k(a/b)^{s/2}[2^{s/2}\Gamma(s/2)]^{-1}\int_{x/a}^{\infty} dup_r(u)e^{-au/(2b)}$$
$$\cdot\sum_{j=0}^{s/2-1}\binom{s/2-1}{j}u^{s/2-j-1}(-1)^j\int_{-\infty}^{x/a} v^{k+j}e^{av(1+2b\theta)/(2b)}dv$$
$$=(-2b)^k[b/(a+b)]^{r/2}(1+2b\theta)^{-s/2-k}e^{x(1+2b\theta)/(2b)}$$
$$\cdot\sum_{j=0}^{s/2-1}\frac{(r/2)_j(s/2-j)_k}{j!}\left(\frac{a}{a+b}\right)^j(1+2b\theta)^j$$
$$\cdot e_{s/2+k-j-1}[-x(1+2b\theta)/(2b)]\{1-P_{r+2j}[x(a+b)/(ab)]\}\,.$$

Note that $G(s, r, b, a; k, -\theta, \infty)$, which appears on the right-hand side of (30), can be evaluated from (28). Expression (28), together with (25) and (26), can also be used to evaluate $G(r, s, a, b; k, 0, \infty)$ and $G(r, s, a, b; 0, \theta, \infty)$ for $s=2, 4, 6, \cdots$.

Case (iii): *General case.* Take

$$F(a, b, c; z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt\,,$$
$$\operatorname{Re} c>\operatorname{Re} b>0,\ |\arg(1-z)|<\pi\,,$$

to be the analytic continuation of the hypergeometric function.

For the Laplace transform of $y^{v-1}U(c, d; y)$, we have

$$\int_0^\infty e^{-zy}y^{v-1}U(c, d; y)dy$$
$$=\left\{\frac{\Gamma(v)\Gamma(1+v-d)}{\Gamma(1+c+v-d)}\right\}F(v, 1+v-d, 1+c+v-d; 1-z)$$

providing $\operatorname{Re} v>0$, $\operatorname{Re} d<\operatorname{Re} v+1$, $\operatorname{Re} z>0$ (Equation (3.2.51) of Slater [3] and its extension by analytic continuation); so that

$$G(r, s, a, b; k, \theta, \infty)$$
$$=[2^{(r+s)/2}a^{r/2}b^{s/2}\Gamma(r/2)]^{-1}[2ab/(a+b)]^{r/2+s/2+k}$$
$$\cdot\int_0^\infty y^{r/2+s/2+k-1}\exp[-yb(1-2a\theta)/(a+b)]$$
$$\cdot\ U(s/2, r/2+s/2; y)dy$$
$$=(2a)^k\left(\frac{a}{a+b}\right)^{s/2}\left(\frac{b}{a+b}\right)^{r/2+k}\frac{\Gamma(k+1)\Gamma(r/2+s/2+k)}{\Gamma(r/2)\Gamma(s/2+k+1)}$$
$$\cdot\ F[r/2+s/2+k, k+1, s/2+k+1; a(1+2b\theta)/(a+b)]$$

for $\theta<1/(2a)$.

By making use of linear transformation formulas for the hypergeometric function ((9.5.2) and (9.5.3) in Lebedev's book [1]), we obtain alternate representations

$$G(r, s, a, b; k, \theta, \infty) \tag{31}$$
$$=(2a)^k\left(\frac{a}{a+b}\right)^{s/2}\left(\frac{b}{a+b}\right)^{r/2-1}\frac{\Gamma(k+1)\Gamma(r/2+s/2+k)}{\Gamma(r/2)\Gamma(s/2+k+1)}(1-2a\theta)^{-k-1}$$
$$\cdot\ F[1-r/2, k+1, s/2+k+1; -(a/b)(1+2b\theta)/(1-2a\theta)]\,,$$
$$\theta<1/(2a)\,,$$

and

$$G(r, s, a, b; k, \theta, \infty) \tag{32}$$
$$=(2a)^k\left(\frac{a}{a+b}\right)^{s/2}\frac{\Gamma(k+1)\Gamma(r/2+s/2+k)}{\Gamma(r/2)\Gamma(s/2+k+1)}(1-2a\theta)^{-r/2-k}$$
$$\cdot\ F[1-r/2, s/2, s/2+k+1; a(1+2b\theta)/(a+b)]\,,$$
$$\theta<1/(2a)\,.$$

In particular, the representation (31) yields

(33) $$G(r, s, a, b; k, 0, \infty) = (2a)^k \left(\frac{a}{a+b}\right)^{s/2} \left(\frac{b}{a+b}\right)^{r/2-1} \frac{\Gamma(k+1)\Gamma(r/2+s/2+k)}{\Gamma(r/2)\Gamma(s/2+k+1)} \cdot F(1-r/2, k+1, s/2+k+1; -a/b) .$$

This expression is an especially fruitful computational form.

For $r=2, 4, 6, \cdots$, it reduces to

(34) $$G(r, s, a, b; k, 0, \infty) = (2a)^k \left(\frac{a}{a+b}\right)^{r/2+s/2-1} \cdot \sum_{j=0}^{r/2-1} \frac{(1-r/2-s/2-k)_j (r/2-j)_k}{j!} (-b/a)^j ,$$

providing an alternative to (28) for computing purposes.

For $a/b \leqq 1$ and $r=1, 3, 5, \cdots$, terms $(r-1)/2, (r+1)/2, (r+3)/2, \cdots$ of the right-hand side of

(35) $$F(1-r/2, k+1, s/2+k+1; -a/b) = \sum_{j=0}^{\infty} \frac{(1-r/2)_j (k+1)_j}{j!(s/2+k+1)_j} (-a/b)^j$$

comprise a convergent alternating series, whose remainder is smaller in absolute value than the first neglected term and has the same sign; so that, by making use of the series representation (35), $G(r, s, a, b; k, 0, \infty)$ can be calculated to any desired accuracy.

In evaluating $G(r, s, a, b; k, 0, \infty)$ from (25) for $s=2, 4, 6, \cdots$, formula (34) can be used as an alternative to (28) for computing $G(s, r, b, a; k, 0, \infty)$. Similarly, if $a/b \geqq 1$ and $s=1, 3, 5, \cdots$, $G(s, r, b, a; k, 0, \infty)$, which appears on the right-hand side of the expression (25) for $G(r, s, a, b; k, 0, \infty)$, can be evaluated by making use of (31) and the series representation (35).

From (32), we obtain

$$G(r, s, a, b; k, 0, \infty) = (2a)^k (r/2+s/2)_k \sum_{j=0}^{k} \binom{k}{j} [(s/2)_j/(r/2+s/2)_j] (-1)^j (1+b/a)^j \cdot \{1-P_{r, s+2j}[(b/a)(s+2j)/r]\} ,$$

so that this function can also be computed from tables of the F distribution function or the incomplete beta function.

For $0 < x < \infty$, we find

$$2^{(r+s)/2} a^{r/2} b^{s/2} \Gamma(r/2)\Gamma(s/2) G(r, s, a, b; k, \theta, x) = \int_0^\infty du\, u^{s/2-1}(1+u)^{r/2-1} \int_0^x t^{r/2+s/2+k-1} \cdot \exp\{-t[u(1+a/b)+1-2a\theta]/(2a)\} dt$$

$$=(r/2+s/2+k)^{-1}x^{r/2+s/2+k}\exp[-x(1-2a\theta)/(2a)]$$
$$\cdot\int_0^\infty u^{s/2-1}(1+u)^{r/2-1}\exp[-ux(a+b)/(2ab)]$$
$$\cdot\Phi\{1,\ r/2+s/2+k+1;\ x[u(1+a/b)+1-2a\theta]/(2a)\}\,du$$
$$=(r/2+s/2+k)^{-1}x^{r/2+s/2+k}\exp[-x(1-2a\theta)/(2a)]$$
$$\cdot\sum_{p=0}^{\infty}\frac{[x/(2a)]^p}{(r/2+s/2+k+1)_p}\sum_{j=0}^{p}\binom{p}{j}(1+a/b)^j(1-2a\theta)^{p-j}$$
$$\cdot\Gamma(s/2+j)U[s/2+j,\ r/2+s/2+j;\ x(a+b)/(2ab)]$$
$$(36)\qquad =x^{r/2+s/2+k}\exp[-x(1-2a\theta)/(2a)]$$
$$\cdot\sum_{j=0}^{\infty}\frac{\Gamma(s/2+j)}{(r/2+s/2+k)_{j+1}}[x(a+b)/(2ab)]^j$$
$$\cdot\ U[s/2+j,\ r/2+s/2+j;\ x(a+b)/(2ab)]$$
$$\cdot\ \Phi[j+1,\ r/2+s/2+k+j+1;\ x(1-2a\theta)/(2a)]\ ,$$
$$\theta<1/(2a)\ .$$

Using Kummer's transformation, we obtain an alternate representation

$$2^{(r+s)/2}a^{r/2}b^{s/2}\Gamma(r/2)\Gamma(s/2)G(r,\ s,\ a,\ b;\ k,\ \theta,\ x)$$
$$=x^{r/2+s/2+k}\sum_{j=0}^{\infty}\frac{\Gamma(s/2+j)}{(r/2+s/2+k)_{j+1}}[x(a+b)/(2ab)]^j$$
$$\cdot\ U[s/2+j,\ r/2+s/2+j;\ x(a+b)/(2ab)]$$
$$\cdot\ \Phi[r/2+s/2+k,\ r/2+s/2+k+j+1;\ -x(1-2a\theta)/(2a)]\ ,$$
$$0<x<\infty,\ \theta<1/(2a)\ .$$

We now consider; for $r=1, 3, 5,\cdots$; $s=1, 3, 5,\cdots$; estimation of the error incurred in approximating the series appearing on the right-hand side of (36) by a partial sum.

We will need the simple inequalities

$$0<\Gamma(c)U(c,\ d;\ t)<\Gamma(c-\delta)U(c-\delta,\ d;\ t)\ ,\qquad t>0,\ c>c-\delta>0\ ,$$

and

$$0<\Phi(c,\ d;\ t)<e^t\ ,\qquad t>0,\ d>c>0\ ,$$

and formula (14).

Putting $r^*=r+1$ and $s^*=s-1$, we find that, for $r=1, 3, 5,\cdots$, $s=1, 3, 5,\cdots$, and for any positive integer v,

$$0<\sum_{j=v}^{\infty}[\Gamma(s/2+j)/(r/2+s/2+k)_{j+1}][x(a+b)/(2ab)]^j$$
$$\cdot\ U[s/2+j,\ r/2+s/2+j;\ x(a+b)/(2ab)]$$
$$\cdot\ \Phi[j+1,\ r/2+s/2+k+j+1;\ x(1-2a\theta)/(2a)]$$

$$(37)\qquad <\sum_{j=v}^{\infty}[\Gamma(s^*/2+j)/(r^*/2+s^*/2+k)_{j+1}][x(a+b)/(2ab)]^j$$
$$\cdot\, U[s^*/2+j,\ r^*/2+s^*/2+j\,;\ x(a+b)/(2ab)]$$
$$\cdot\, \Phi[j+1,\ r^*/2+s^*/2+k+j+1\,;\ x(1-2a\theta)/(2a)]$$
$$<e^{x(1-2a\theta)/(2a)}\sum_{j=v}^{\infty}[\Gamma(s^*/2+j)/(r^*/2+s^*/2+k)_{j+1}][x(a+b)/(2ab)]^j$$
$$\cdot\, U[s^*/2+j,\ r^*/2+s^*/2+j\,;\ x(a+b)/(2ab)]$$
$$=[x(a+b)/(2ab)]^{-s^*/2}e^{x(1-2a\theta)/(2a)}\sum_{j=v}^{\infty}[(r^*/2+s^*/2+k)_{j+1}]^{-1}$$
$$\cdot\sum_{p=0}^{r^*/2-1}\binom{r^*/2-1}{p}\Gamma(s^*/2+p+j)[x(a+b)/(2ab)]^{-p}$$
$$(38)\qquad =[x(a+b)/(2ab)]^{-s^*/2}e^{x(1-2a\theta)/(2a)}\sum_{p=0}^{r^*/2-1}\binom{r^*/2-1}{p}[x(a+b)/(2ab)]^{-p}$$
$$\cdot\left\{\frac{\Gamma(s^*/2+p+1)}{(r^*/2+s^*/2+k)(r^*/2+k-p)}-\sum_{j=1}^{v-1}\frac{\Gamma(s^*/2+p+j)}{(r^*/2+s^*/2+k)_{j+1}}\right\},$$
$$0<x<\infty,\quad \theta<1/(2a)\,.$$

For $r=1, 3, 5, \cdots$, $s=3, 5, 7, \cdots$, we note that the error bound (37) equals

$$(39)\qquad 2^{(r+s)/2}a^{r^*/2}b^{s^*/2}\Gamma(r^*/2)\Gamma(s^*/2)x^{-r/2-s/2-k}$$
$$\cdot\, e^{x(1-2a\theta)/(2a)}G(r^*,\ s^*,\ a,\ b\,;\ k,\ \theta,\ x)$$
$$-\sum_{j=0}^{v-1}[\Gamma(s^*/2+j)/(r/2+s/2+k)_{j+1}][x(a+b)/(2ab)]^j$$
$$\cdot\, U[s^*/2+j,\ r/2+s/2+j\,;\ x(a+b)/(2ab)]$$
$$\cdot\, \Phi[j+1,\ r/2+s/2+k+j+1\,;\ x(1-2a\theta)/(2a)]\,.$$

Here, $G(r^*, s^*, a, b\,; k, \theta, x)$ can be computed, for example, from (27); $U[s^*/2+j,\ r/2+s/2+j\,;\ x(a+b)/(2ab)]$ can be calculated from (14); and $\Phi[j+1,\ r/2+s/2+k+j+1\,;\ x(1-2a\theta)/(2a)]$ can be evaluated by using the recurrence relationship (24) together with its two accompanying formulas.

The error estimate (38) is less precise than (39); however it should prove easier to compute. Also, it can be calculated for $s=1$.

To use (36) to approximate $G(r, s, a, b\,; k, \theta, x)$ for $r=1, 3, 5, \cdots$, $s=1, 3, 5, \cdots$, by replacing the infinite series with a partial sum, we must of course be able to compute the terms of the series. The evaluation of $\Phi[j+1,\ r/2+s/2+k+j+1\,;\ x(1-2a\theta)/(2a)]$ can be accomplished as described above. A method for approximating $U[s/2+j,\ r/2+s/2+j\,;\ x(a+b)/(2ab)]$ to any desired accuracy can be found in Section 3 under Case (iii).

AEROSPACE RESEARCH LABORATORIES, WRIGHT-PATTERSON AFB, OHIO

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