SOME PROPERTIES OF AN ESTIMATOR FOR THE VARIANCE OF A NORMAL DISTRIBUTION

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1. Introduction and summary

Let $X_1, X_2, \cdots, X_n$ be a random sample from a normal distribution and $\bar{X}$ be a sample mean. It is well known that

$$(1) \qquad r^2 = \frac{1}{n+1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

is the multiple of the sample variance with the minimum mean-squared error (m.s.e.) in estimating variance of a normal distribution (for example, see Kendall and Stuart [3]).

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2, and let $\beta_2 = \mu_4/\sigma^4$ be a coefficient of kurtosis where μ_4 is the fourth central moment. Under the situation that β_2 is known as an a priori information, Singh et al. [5] proposed the estimator

$$Y^* = \frac{n}{n^2 - 2n + 3 + \beta_2(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

(given by (5)) with the minimum m.s.e. in estimating variances. Searls [4] and the author [1] discussed the estimator for mean μ using a known value of the coefficient of variation as an a priori information. The author [2] gave the estimation procedure unified [1], [4] and [5].

In this paper we discuss some properties of the estimator r^2 for σ^2. We show that r^2 is more efficient than the sample unbiased variance in the sense of the m.s.e. loss criterion for any distributions with the coefficient of kurtosis not less than 2. Further, we compare the estimators for the variance in terms of another loss criterion. Finally, we give the most efficient estimator in some class of the estimators in the sense of this loss criterion.

2. Estimator r^2 and relative efficiency

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with unknown mean μ and unknown variance σ^2. Consider the estimator

$$Y = w \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{2}$$

for the variance σ^2, where $\bar{X} = \sum_{i=1}^{n} X_i/n$ and w is a positive constant and depends possibly on the parameter. We determine the constant w to minimize the mean-squared error (MSE) of Y. We have

$$\begin{aligned} \text{MSE}(Y) &= \text{E}(Y-\sigma^2)^2 \\ &= w^2(n-1)^2\left[\frac{\mu_4}{n} + \frac{(3-n)\sigma^4}{n(n-1)}\right] + \sigma^4[1-w(n-1)]^2 . \end{aligned} \tag{3}$$

Let w^* be a constant that is chosen so that MSE (Y) has a minimum. From (3), we have

$$w^* = \frac{n}{n^2 - 2n + 3 + \beta_2(n-1)} . \tag{4}$$

If the population coefficient of kurtosis β_2 is known as an a priori information, then we can use the estimator

$$Y^* = \frac{n}{n^2 - 2n + 3 + \beta_2(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2 \tag{5}$$

of the variance σ^2.

It is well known that the value w of (2) should be $1/(n+1)$ when the normality of the population is assumed, because of $\beta_2 = 3$. Here we are in position to compute the relative efficiency (REF) of r^2 given by (1) relative to the sample variance

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

which is the usual estimator for the variance, in the sense of the reciprocal of the ratio of the m.s.e.;

$$\text{REF}(r^2; s^2) = \frac{\text{MSE}(s^2)}{\text{MSE}(r^2)} . \tag{6}$$

It is well known that s^2 is the unbiased estimator of the variance σ^2 and

$$\text{MSE}(s^2) = \text{Var}(s^2) = \frac{1}{n}\left(\mu_4 - \frac{n-3}{n-1}\sigma^4\right) . \tag{7}$$

Under the assumption of the normality of the population, from (7) we have

(8) $$\text{MSE}\,(s^2)=\frac{2\sigma^4}{n-1}\,.$$

Similarly, from (3) we have

(9) $$\text{MSE}\,(r^2)=\frac{2\sigma^4}{n+1}\,.$$

Therefore, from (8) and (9) we have

(10) $$\text{REF}\,(r^2;s^2)=\frac{n+1}{n-1}\downarrow 1 \qquad \text{as } n\to\infty\,.$$

In Table 1, we give the value of (10) for different values n.

Table 1

n	5	8	10	13	15	18	20	$\cdots$	∞
REF $(r^2;s^2)$	1.500	1.286	1.222	1.167	1.143	1.118	1.105	$\downarrow$	1
$\frac{2(n-2)}{(n-1)}$ in (19)	1.500	1.714	1.777	1.833	1.857	1.882	1.894	$\uparrow$	2

The above Table 1 shows that if the sample size n is small, the estimator r^2 is more efficient than s^2. Hence the estimator r^2 is effective in small sample cases, at most about 20.

3. Estimators of variance in normal distribution

It is well known that the sample variance s^2 is the unbiased estimator of the variance. Even if the population is normally distributed and the sample sizes are small, it may be usual to utilize the sample variance s^2 in estimating variance σ^2. Let $X_1, X_2, \cdots, X_n$ be a random sample from the normal distribution $N(\mu, \sigma^2)$. Under the assumption of the normality we gave the estimator r^2 given by (1). Next, it is well known that the maximum likelihood estimators for the mean μ and the variance σ^2 are given by

(11) $$\bar{X} \text{ and } t^2=\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\,,$$

respectively. Here, consider the three estimators r^2, s^2 and t^2 for the variance σ^2 in the normal distribution $N(\mu, \sigma^2)$. We give the graphs of the density functions of r^2, s^2 and t^2. We will set $\sigma^2=1$ for simplification. The statistic $z=\sum_{i=1}^{n}(X_i-\bar{X})^2$ has a chi-square distribution with $n-1$ degrees of freedom. If we make the changes of variables, we

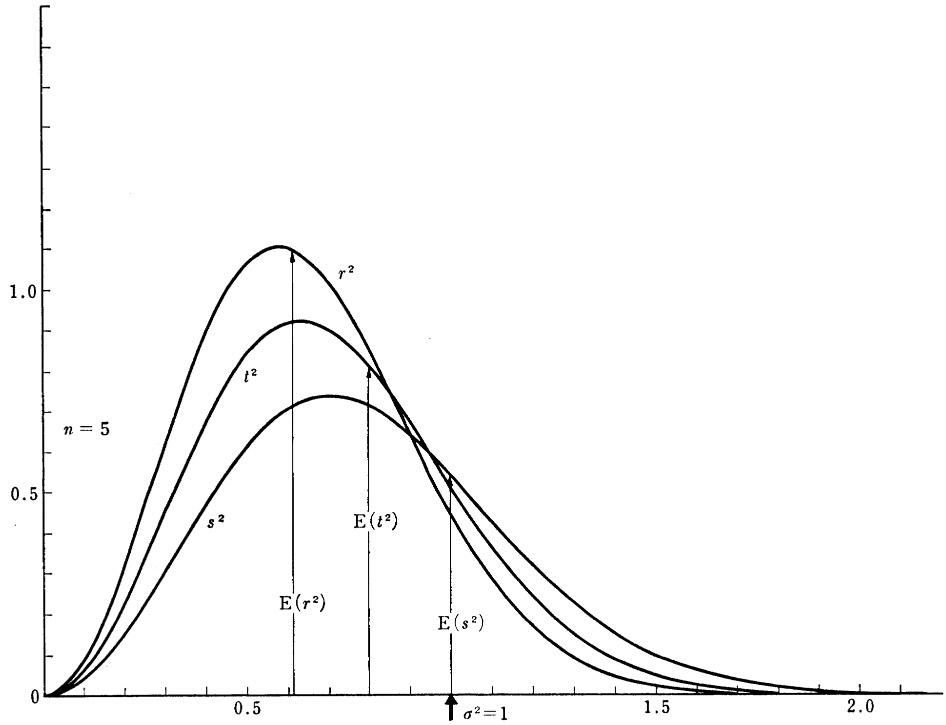
1.0
0.5
0
0.5
1.5
2.0
n = 5
r²
t²
s²
E(r²)
E(t²)
E(s²)
σ²=1

Fig. 1

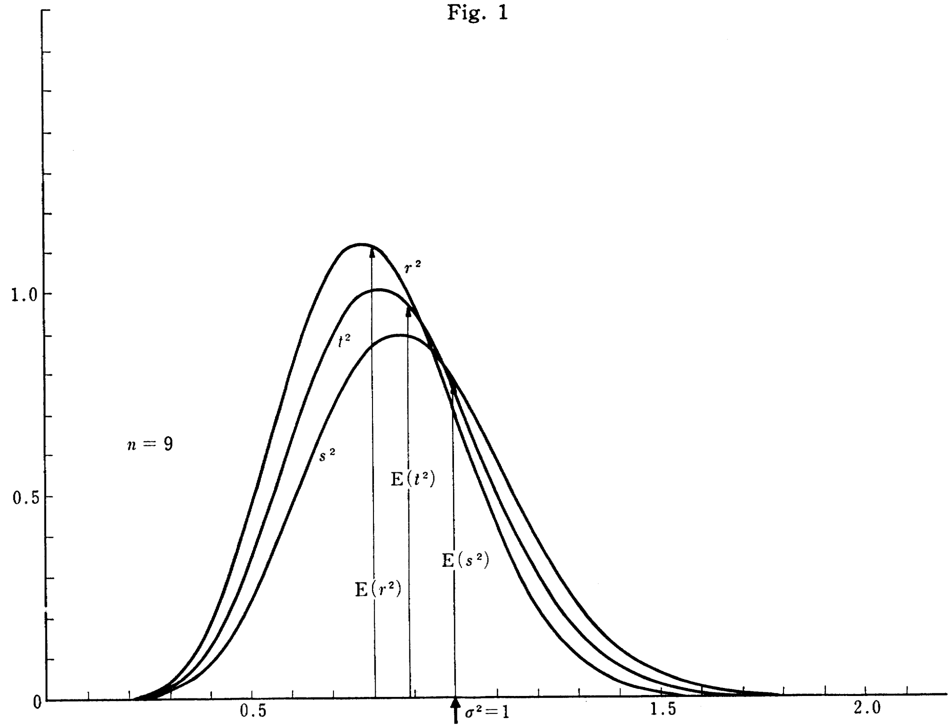
1.0
0.5
0
0.5
1.5
2.0
n = 9
r²
t²
s²
E(r²)
E(t²)
E(s²)
σ²=1

Fig. 2

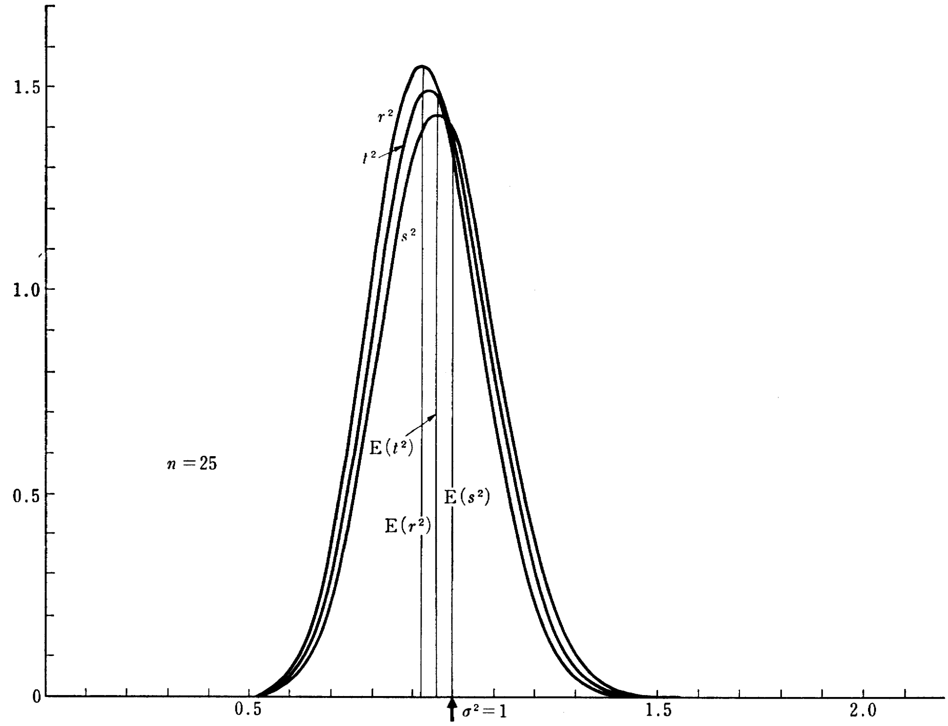

Fig. 3

obtain the density functions of r^2, s^2 and t^2, respectively. We give here the graphs of the density functions in Figs. 1, 2 and 3 for $n=5$, 9 and 25, respectively, because our interest is in small sample cases.

When we use the estimator r^2 for the variance σ^2, from Table 1, the largest gains are obtained for small sample sizes, at most about 20. Such sample sizes may be all that are available. From the graphs we may use the estimator r^2 or s^2 when we would like to estimate the variance σ^2.

4. Extent of distributions for which r^2 is more efficient than s^2 in the sense of m.s.e. criterion

What we have to discuss is the extent to which we are likely to be justified if we apply this so called "normal distribution" in circumstances where the underlying distributions are not in fact normal. When we apply the "normal kurtosis 3" in spite of a true coefficient of kurtosis 3α $(\alpha>0)$ of some distribution, we would like to have the extent of the distributions for which the estimator r^2 is more efficient than the usual estimator s^2 for the variance σ^2 in the sense of the m.s.e. criterion.

We assume that the true coefficient of kurtosis of some distribution is equal to 3α ($\alpha>0$). Then we have

$$\mu_4=3\alpha\sigma^4 . \tag{12}$$

From (7) and (12)

$$\text{MSE}\,(s^2)=\frac{1}{n}\left(3\alpha-\frac{n-3}{n-1}\right)\sigma^4 . \tag{13}$$

We apply the normal distribution in circumstances where the underlying distributions are not in fact normal, so that we use the estimator r^2 for the variance σ^2. Hence, from (3), the m.s.e. of r^2 is give by

$$\text{MSE}\,(r^2)=w^{*2}(n-1)^2\left[\frac{\mu_4}{n}+\frac{(3-n)\sigma^4}{n(n-1)}\right]+\sigma^4[1-w^*(n-1)]^2 \tag{14}$$

where

$$w^*=\frac{1}{n+1} . \tag{15}$$

Therefore, from (12), (14) and (15), we have

$$\text{MSE}\,(r^2)=\frac{(n-1)^2}{(n+1)^2}\left[\frac{3\alpha}{n}+\frac{3-n}{n(n-1)}\right]\sigma^4+\sigma^4\left[1-\frac{n-1}{n+1}\right]^2 . \tag{16}$$

We would like to have the range of α that r^2 is more efficient than the usual estimator s^2. Its range is equivalent to the solution that satisfies the following inequality;

$$\text{REF}\,(r^2\,;\,s^2)=\frac{\text{MSE}\,(s^2)}{\text{MSE}\,(r^2)}\geqq 1 . \tag{17}$$

The inequality (17), from (13) and (16), is equivalent to

$$\frac{1}{n}\left(3\alpha-\frac{n-3}{n-1}\right)\sigma^4\geqq\frac{(n-1)^2}{(n+1)^2}\left[\frac{3\alpha}{n}+\frac{3-n}{n(n-1)}\right]\sigma^4+\sigma^4\left[1-\frac{n-1}{n+1}\right]^2 . \tag{18}$$

Solving (18) with respect to α, we have

$$3\alpha\geqq\frac{2(n-2)}{n-1}\uparrow 2 \qquad \text{as } n\to\infty , \tag{19}$$

for all $n\geqq 2$. In Table 1, we give the values of $2(n-2)/(n-1)$ for different values n. Since the range $3\alpha\geqq 2$ is the smallest set in the ranges (19) for various n, it is enough to consider the distributions with the smallest range $3\alpha\geqq 2$. Summarizing, we have the following result.

THEOREM. *For any distributions with the coefficient of kurtosis not* *less than 2, the estimator r^2 of the variance is more efficient than the sample unbiased variance s^2 in the sense of the m.s.e. criterion.*

We give the population coefficient of kurtosis of several distribu-

Table 2

Name	Density and Domain	Restrictions on Parameters	Kurtosis β_2
Normal* $N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$ $-\infty<x<\infty$	$-\infty<\mu<\infty$ $0<\sigma<\infty$	3
Log-normal*	$\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{y}\exp\left[-\frac{1}{2\sigma^2}(\log y-\mu)^2\right]$ $0<y<\infty$	$-\infty<\mu<\infty$ $0<\sigma<\infty$	$g^4+2g^3+3g^2-3$ where $g=e^{\sigma^2}$
Uniform	$\frac{1}{\mu}$ $\alpha-\frac{\mu}{2}\leqq x\leqq\alpha+\frac{\mu}{2}$	$-\infty<\alpha<\infty$ $0<\mu<\infty$	1.8 [1]
Exponential*	$\frac{1}{\beta}\exp\left[-\frac{x-\alpha}{\beta}\right]$ $\alpha\leqq x<\infty$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	9
Double-exponential*	$\frac{1}{2\beta}\exp\left[-\frac{\|x-\alpha\|}{\beta}\right]$ $-\infty<x<\infty$	$-\infty<\alpha<\infty$ $0<\beta<\infty$	6
Gamma*	$\frac{1}{\Gamma(\mu)}x^{\mu-1}e^{-x}$ $0\leqq x<\infty$	$0<\mu<\infty$	$3+\frac{6}{\mu}$
t-distribution*	$\frac{\Gamma((k+1)/2)}{\Gamma(k/2)\sqrt{\pi k}}\left(1+\frac{t^2}{k}\right)^{-(k+1)/2}$ $-\infty<t<\infty$	$0<k<\infty$	$\frac{3(k-2)}{k-4}$ $(k>4)$
Weibull* [3]	$\frac{m}{\alpha}w^{m-1}\exp\left[-\frac{w^m}{\alpha}\right]$ $0\leqq w<\infty$	$0<\alpha<\infty$ $0<m<\infty$	[2]
Binomial*	$\binom{n}{x}p^x(1-p)^{n-x}$ $x=0, 1, \cdots, n$	$0<p<1$	$3+\frac{1-6p+6p^2}{np(1-p)}$
Poisson*	$\frac{\mu^x e^{-\mu}}{x!}$ $x=0, 1, 2, \cdots, \infty$	$0<\mu<\infty$	$3+\frac{1}{\mu}$
Geometric*	$p(1-p)^x$ $x=0, 1, 2, \cdots, \infty$	$0<p<1$	$9+\frac{p^2}{1-p}$
Negative binomial*	$\binom{x-1}{k-1}p^k(1-p)^{x-k}$ $x=k, k+1, \cdots, \infty$	$0<p<1$	$3+\frac{p^2+6(1-p)}{n(1-p)}$

The distributions with * have the value of the coefficient of kurtosis which is always more than 2.

1) If $n\leqq 11$, then the estimator r^2 is more precise than s^2.

2) $\beta_2=\frac{\Gamma(4/m+1)-4\Gamma(3/m+1)\Gamma(1/m+1)+6\Gamma(2/m+1)\Gamma^2(1/m+1)-3\Gamma^4(1/m+1)}{[\Gamma(2/m+1)-\Gamma^2(1/m+1)]^2}$.

3) It is the conclusion by the numerical calculations that the values of the coefficient of kurtosis of Weibull distribution are more than 2 for each m.

tions at Table 2. We may see that most distributions have the population coefficient of kurtosis not less than 2.

Note. From above discussions we can regard the population coefficient of kurtosis β_2 as a measurement of departure from the normality.

5. Comparison of three estimators in terms of another loss criterion

In the previous sections, we have compared the three estimators for the variance σ^2 by the m.s.e. criterion. It was shown that r^2 in (1) was more efficient than s^2 for any distributions with the coefficient of kurtosis β_2 not less than 2. But the m.s.e. criterion is not always the best criterion. Especially, for the scale parameter it has a following defect; The square loss is defined by

$$L_1(Y) \equiv (Y-\sigma^2)^2 , \tag{20}$$

where Y is given by (2). Then $L_1(Y)$ is the symmetric function with respect to $Y=\sigma^2$ and hence

$$L_1(0)=L_1(2\sigma^2) .$$

Therefore we have

$$\text{MSE}\,(Y)|_{Y=0}=\text{MSE}\,(Y)|_{Y=2\sigma^2} .$$

(See Fig. 4). But in practical sense this relation is unnatural. We may

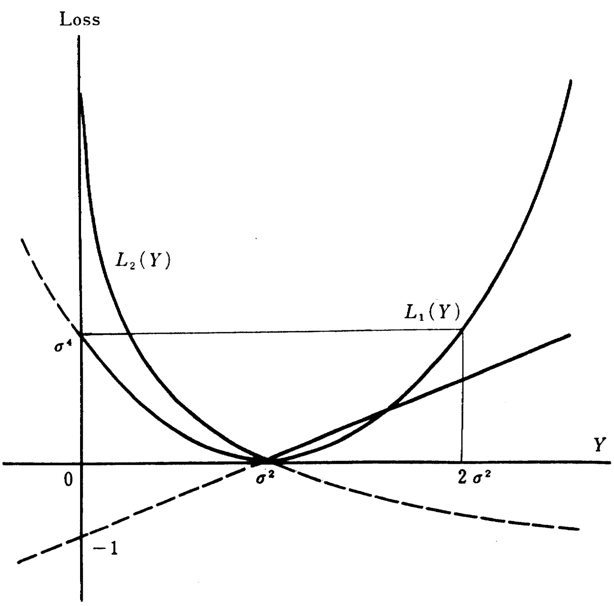

Fig. 4

assert that $\mathrm{MSE}(Y)|_{Y=0}$ is far larger than $\mathrm{MSE}(Y)|_{Y=2\sigma^2}$.

In this section we compare the three estimators r^2, s^2 and t^2 in terms of another loss criterion

$$L_2(Y)=\mathrm{Max}\left(\frac{Y}{\sigma^2}-1,\ \frac{\sigma^2}{Y}-1\right). \tag{21}$$

This loss function is not the symmetric function with respect to $Y=\sigma^2$; for an arbitrary $\varepsilon>0$, if $Y=\sigma^2+\varepsilon$, then $L_2(Y)=\varepsilon/\sigma^2$ and if $Y=\sigma^2-\varepsilon$, then $L_2(Y)=\varepsilon/(\sigma^2-\varepsilon)$. We calculate the expected loss

$$I=\mathrm{E}\left[L_2(Y)\right]=\mathrm{E}\left[\mathrm{Max}\left(\frac{Y}{\sigma^2}-1,\ \frac{\sigma^2}{Y}-1\right)\right]. \tag{22}$$

It is well known that $\chi^2_{n-1}=\sum_{i=1}^{n}(X_i-\bar{X})^2/\sigma^2$ has the chi-square distribution function F_{n-1} with $n-1$ degrees of freedom under the normal population and

$$Y=w\sigma^2\chi^2_{n-1}. \tag{23}$$

We have the following equivalent relations;

$$\begin{aligned} Y\gtreqqless\sigma^2 &\Longleftrightarrow \frac{Y}{\sigma^2}\gtreqqless 1 \\ &\Longleftrightarrow w\chi^2_{n-1}\gtreqqless 1 \\ &\Longleftrightarrow \chi^2_{n-1}\gtreqqless\frac{1}{w}. \end{aligned} \tag{24}$$

Hence, from (22), (23) and (24), we have

$$\begin{aligned} I &= \int_0^\infty \mathrm{Max}\left\{w\chi^2_{n-1}-1,\ \frac{1}{w\chi^2_{n-1}}-1\right\}dF_{n-1}(\chi^2_{n-1}) \\ &= \int_0^{1/w}\frac{1}{wt}\,\frac{1}{2\Gamma((n-1)/2)}\left(\frac{t}{2}\right)^{(n-1)/2-1}e^{-t/2}dt \\ &\quad +\int_{1/w}^\infty wt\,\frac{1}{2\Gamma((n-1)/2)}\left(\frac{t}{2}\right)^{(n-1)/2-1}e^{-t/2}dt-1 \\ &= \frac{1}{2w\Gamma((n-1)/2)}\int_0^{1/(2w)} z^{(n-5)/2}e^{-z}dz+\frac{2w}{\Gamma((n-1)/2)} \\ &\quad \cdot\int_{1/(2w)}^\infty z^{(n-1)/2}e^{-z}dz-1. \end{aligned} \tag{25}$$

Finally, for each n and $w=1/(n-1)$, $1/n$ and $1/(n+1)$ we give the values of the incomplete gamma functions in (25) by the numerical calculations and the values of I are presented in Table 3.

From Table 3, it is expected that

Table 3

	s^2	t^2	r^2	Y^{**}
n \ w	$\frac{1}{n-1}$	$\frac{1}{n}$	$\frac{1}{n+1}$	w^{**}
4	2.4502	3.2301	4.1228	1.9366
5	1.4060	1.7298	2.1328	1.2426
6	1.0403	1.2266	1.4719	0.9594
7	0.8485	0.9722	1.1420	0.7999
8	0.7282	0.8174	0.9438	0.6956
9	0.6447	0.7126	0.8113	0.6212
10	0.5828	0.6365	0.7163	0.5650
11	0.5347	0.5785	0.6446	0.5207
12	0.4961	0.5326	0.5885	0.4848
13	0.4642	0.4953	0.5433	0.4549
14	0.4375	0.4642	0.5060	0.4296
15	0.4146	0.4379	0.4747	0.4078
16	0.3947	0.4152	0.4480	0.3888
17	0.3773	0.3956	0.4249	0.3721
18	0.3618	0.3782	0.4047	0.3573
19	0.3480	0.3628	0.3869	0.3440
20	0.3356	0.3491	0.3710	0.3320
21	0.3244	0.3366	0.3568	0.3211

(26) $$\mathrm{E}\,[L_2(r^2)] > \mathrm{E}\,[L_2(t^2)] > \mathrm{E}\,[L_2(s^2)]\,,$$

for all $n \geqq 4$, and we give the proof of this at Section 7. Hence the estimator s^2 is more efficient than the others in the sense of loss criterion L_2. The estimator r^2 that is robust in the sense of the m.s.e. criterion is not more efficient than s^2 in the sense of loss criterion L_2.

6. The most efficient estimator in the sense of L_2

We gave the well-defined loss criterion L_2 for the scale parameter in the previous section. Further we compared the three estimators r^2, s^2 and t^2 for the variance in terms of the loss criterion L_2, so that the error of s^2 is less than the others. But the estimator s^2 is not the most efficient estimator in the sense of L_2 in the class of the estimators $\left\{w \sum_{i=1}^{n} (X_i - \bar{X})^2\right\}$ for the variance σ^2. In this section we study the most efficient estimator in the sense of L_2 in the class of the estimators.

In order to obtain the most efficient estimator in the sense of L_2, we would like to find a constant w that minimizes $I = I(w)$ given by (25) with respect to w.

Putting

$$x=\frac{1}{2w} \qquad (x>0) \tag{27}$$

in $I=I(w)$ given by (25), we have

$$\begin{aligned} I=I(x)&=\frac{x}{\Gamma((n-1)/2)}\int_0^x z^{(n-5)/2}e^{-z}dz+\frac{1}{\Gamma((n-1)/2)x} \\ &\quad\cdot\int_x^\infty z^{(n-1)/2}e^{-z}dz-1 \\ &=\frac{x}{\Gamma((n-1)/2)}\int_0^x z^{(n-5)/2}e^{-z}dz+\frac{\Gamma((n+1)/2)}{\Gamma((n-1)/2)x} \\ &\quad-\frac{1}{\Gamma((n-1)/2)x}\int_0^x z^{(n-1)/2}e^{-z}dz-1 \,. \end{aligned} \tag{28}$$

It is easily seen that $I=I(x)$ has a minimum and

$$\begin{aligned} \frac{dI(x)}{dx}&=\frac{1}{\Gamma((n-1)/2)}\int_0^x z^{(n-5)/2}e^{-z}dz-\frac{\Gamma((n+1)/2)}{\Gamma((n-1)/2)x^2} \\ &\quad+\frac{1}{\Gamma((n-1)/2)x^2}\int_0^x z^{(n-1)/2}e^{-z}dz \,. \end{aligned} \tag{29}$$

The equation

$$\frac{dI(x)}{dx}=0$$

is equivalent to

$$x^2\int_0^x z^{(n-5)/2}e^{-z}dz+\int_0^x z^{(n-1)/2}e^{-z}dz-\Gamma\left(\frac{n+1}{2}\right)=0 \,. \tag{30}$$

In order to find x satisfying this equation (30), we put the left hand of (30) as $f(x)$;

$$f(x)=x^2\int_0^x z^{(n-5)/2}e^{-z}dz+\int_0^x z^{(n-1)/2}e^{-z}dz-\Gamma\left(\frac{n+1}{2}\right) .$$

Then we can obtain the zero point $x(n)$ of $f(x)$ by the numerical calculations. From (27) we can obtain the constant $w^{**}\equiv 1/2x(n)$ to minimize the expected loss $I=I(w)$ given by (25) for each n. Therefore we can obtain the most efficient estimator, in the sense of L_2,

$$Y^{**}=w^{**}\sum_{i=1}^{n}(X_i-\bar{X})^2 \tag{31}$$

for the variance σ^2. The values of w^{**} and $1/(n-1)$ are presented in Table 4.

Finally, the expected loss of this estimator Y^{**}, $E[L_2(Y^{**})]$, are presented in Table 3. From Table 3, we can see that the expected loss of Y^{**} are considerably smaller than the others for small sample sizes.

Note. From Table 4, we may conclude that the most efficient estimator Y^{**} in the sense of L_2 is approximated by

$$\frac{1}{n-2}\sum_{i=1}^{n}(X_i-\bar{X})^2 . \tag{32}$$

Table 4

n	w^{**}	$1/(n-1)$	n	w^{**}	$1/(n-1)$	n	w^{**}	$1/(n-1)$
4	0.55801	0.33333	34	0.03103	0.03030	64	0.01606	0.01587
5	0.34239	0.25000	35	0.03010	0.02941	65	0.01581	0.01563
6	0.25080	0.20000	36	0.02922	0.02857	66	0.01556	0.01538
7	0.19882	0.16667	37	0.02839	0.02778	67	0.01532	0.01515
8	0.16502	0.14286	38	0.02760	0.02703	68	0.01509	0.01493
9	0.14119	0.12500	39	0.02686	0.02632	69	0.01487	0.01471
10	0.12345	0.11111	40	0.02616	0.02564	70	0.01465	0.01449
11	0.10971	0.10000	41	0.02549	0.02500	71	0.01444	0.01429
12	0.09874	0.09091	42	0.02485	0.02439	72	0.01423	0.01408
13	0.08978	0.08333	43	0.02425	0.02381	73	0.01403	0.01389
14	0.08233	0.07692	44	0.02368	0.02326	74	0.01384	0.01370
15	0.07602	0.07143	45	0.02313	0.02273	75	0.01365	0.01351
16	0.07061	0.06667	46	0.02260	0.02222	76	0.01347	0.01333
17	0.06593	0.06250	47	0.02210	0.02174	77	0.01329	0.01316
18	0.06183	0.05882	48	0.02163	0.02128	78	0.01311	0.01299
19	0.05821	0.05556	49	0.02117	0.02083	79	0.01294	0.01282
20	0.05500	0.05263	50	0.02073	0.02041	80	0.01278	0.01266
21	0.05212	0.05000	51	0.02031	0.02000	81	0.01262	0.01250
22	0.04953	0.04762	52	0.01990	0.01961	82	0.01246	0.01235
23	0.04718	0.04545	53	0.01951	0.01923	83	0.01231	0.01220
24	0.04505	0.04348	54	0.01914	0.01887	84	0.01216	0.01205
25	0.04310	0.04167	55	0.01878	0.01852	85	0.01201	0.01190
26	0.04131	0.04000	56	0.01843	0.01818	86	0.01187	0.01176
27	0.03967	0.03846	57	0.01810	0.01786	87	0.01173	0.01163
28	0.03815	0.03704	58	0.01778	0.01754	88	0.01159	0.01149
29	0.03675	0.03571	59	0.01747	0.01724	89	0.01146	0.01136
30	0.03544	0.03448	60	0.01717	0.01695	90	0.01133	0.01124
31	0.03423	0.03333	61	0.01688	0.01667	91	0.01120	0.01111
32	0.03309	0.03226	62	0.01660	0.01639			
33	0.03203	0.03125	63	0.01633	0.01613			

7. Proof of the inequalities (26)

PROOF OF (26). In the previous section we showed that $I=I(x)$ has a minimum at only $x=x(n)$ and $dI(x)/dx$ given by (29) is the increasing function with respect to x. Hence to show the inequalities (26) it is sufficient to show

$$\frac{1}{n-1}<x(n) , \tag{32}$$

because of $1/(n+1)<1/n<1/(n-1)$ for each $n\geqq 2$. Since $f(x)$ is the increasing function and $f(x(n))=0$, to show (32) it is sufficient to show the following inequality;

$$f\left(\frac{1}{n-1}\right)=\frac{1}{(n-1)^2}\int_0^{1/(n-1)} z^{(n-5)/2}e^{-z}dz+\int_0^{1/(n-1)} z^{(n-1)/2}e^{-z}dz -\Gamma\left(\frac{n+1}{2}\right)<0 . \tag{33}$$

Each term of the right hand of (33) is evaluated as follows;

$$\text{The first term of (33)}<\frac{1}{(n-1)^2}\int_0^{1/(n-1)} z^{(n-5)/2}dz =\frac{2}{n-3}\left(\frac{1}{n-1}\right)^{(n+1)/2} .$$

$$\text{The second term of (33)}<\frac{2}{n+1}\left(\frac{1}{n-1}\right)^{(n+1)/2} .$$

Hence it is easily shown that $f(1/(n-1))<0$ for $n=4$. For $n\geqq 5$, from the above evaluations, we have as follows;

The first term of (33)<1.

The second term of (33)<1.

The third term of (33)$=\Gamma((n+1)/2)\geqq 6$.

Therefore we have $f(1/(n-1))<0$ for all $n\geqq 4$. The proof of (26) is complete.

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