THE ASYMPTOTIC REPRESENTATION OF THE HODGES-LEHMANN ESTIMATOR BASED ON WILCOXON TWO-SAMPLE STATISTIC

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(Received Sept. 7, 1973; revised Jan. 30, 1974)

1. Introduction

Our work is an extension of Bahadur's representation of sample quantiles: we shall show the same asymptotic relation between Wilcoxon two-sample statistic and the Hodges-Lehmann (H-L) estimator based on it as that between the sample distribution and the sample quantile which is obtained by Bahadur [1] and, thereafter, refined and extended by some authors (for example, see Kiefer [6], [7], and Sen [10]. Earlier than Bahadur, Okamoto [9] obtains a similar result which is, however, represented "in probability"). In the case of the Wilcoxon one-sample test, Geertsema [2] obtained the similar representation as in (5) of Theorem below. But our results in the two-sample case are more sharpened by using Lemmas 2 and 3 below. These results illustrate that in particular cases there are more closed relations than those between a general estimating function and the estimator based on it (which are studied by Huber [4], and Inagaki [5], for example). See Hodges and Lehmann [3] and Van Eeden [12] for discussions about H-L estimators.

2. Theorem

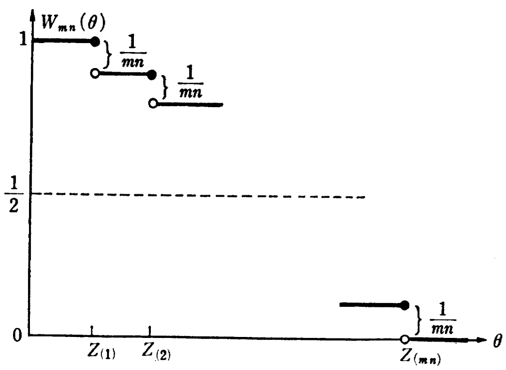
Let $X_1,\cdots,X_m,\cdots;\ Y_1,\cdots,Y_n,\cdots$ be independent random variables such that $X_1,\cdots,X_m,\cdots$ are identically distributed according to a probability distribution $F(x)$ and that $Y_1,\cdots,Y_n,\cdots$ are according to $F(x-\theta_0)$ where θ_0 is a fixed but unknown real number. For $m+n$ $(=N$, say) observations $X_1,\cdots,X_m;\ Y_1,\cdots,Y_n$ and any real number θ, put

$$(1)\qquad W_{m,n}(\theta)=\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\delta(Y_k-X_j-\theta)$$

where $\delta(x)=1$, if $x\geqq 0$, and $=0$, if $x<0$. Then $W_{m,n}(0)$ is Wilcoxon two-sample statistic. Let $F_m^X(x)$ denote the sample distribution function of $X_1,\cdots,X_m$. We may rewrite $W_{m,n}(\theta)$ as

$$(2)\qquad W_{m,n}(\theta)=\frac{1}{n}\sum_{k=1}^{n}F_m^X(Y_k-\theta)\ .$$

It is easy to see that $W_{m,n}(\theta)$ is a non-increasing function of θ with $W_{m,n}(-\infty)=1$ and $W_{m,n}(\infty)=0$, and further, that



$$\mathrm{E}\,W_{m,n}(\theta)=\int_{-\infty}^{\infty}F(y-\theta+\theta_0)\cdot dF(y)=\mu(\theta)\ ,\qquad \text{say}\ ,$$

and then, especially, $\mu(\theta_0)=1/2$. Put $Z_{j,k}=Y_k-X_j$, $j=1,\cdots,m$; $k=1,\cdots,n$, and let their order statistics be $Z_{(1)}<Z_{(2)}<\cdots<Z_{(mn)}$. Then it follows that the H-L estimator of θ_0 based on Wilcoxon statistic is

$$(3)\qquad \hat{\theta}_{m,n}=\text{median}\,\{Z_{j,k};\ j=1,\cdots,m,\ k=1,\cdots,n\}\ .$$

That is, it hods that

$$(4)\qquad W_{m,n}(\hat{\theta}_{mn}+)\leqq 1/2\leqq W_{mn}(\hat{\theta}_{mn})\ .$$

Then we have an asymptotic representation of the H-L estimator $\hat{\theta}_{mn}$:

THEOREM. *Assume*: (a). *The distribution function $F(x)$ has the first and second derivatives, $F'(x)=f(x)$ and $F''(x)=f'(x)$ (say), which are bounded for $-\infty<x<\infty$.* (b) *For X sample size m and Y sample size n, as $N=m+n$ (say)$\to\infty$, $m/N\to\lambda$ $(0<\lambda<1)$. Put*

$$(5)\qquad \hat{\theta}_{m,n}=\theta_0+\Gamma^{-1}[W_{m,n}(\theta_0)-1/2]+R_{m,n}$$

where $\Gamma=\int_{-\infty}^{\infty}\{f(x)\}^2dx$. Then it holds that

$$(6)\qquad R_{m,n}=O(N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4})\ ,\qquad \textit{as } N\to\infty\ ,$$

with probability one. Further it holds that

$$(7)\qquad \begin{aligned}&\limsup_{N\to\infty}\,[N^{1/2}(\hat{\theta}_{mn}-\theta_0)/(2\log\log N)^{1/2}]=[12\lambda(1-\lambda)\Gamma^2]^{-1/2}\ ,\\ &\liminf_{N\to\infty}\,[N^{1/2}(\hat{\theta}_{mn}-\theta_0)/(2\log\log N)^{1/2}]=-[12\lambda(1-\lambda)\Gamma^2]^{-1/2}\end{aligned}$$

with probability one.

By Theorem and Lemma 2 below, it is easy to prove that the

H-L estimator $\hat{\theta}_{mn}$ is asymptotically normally distributed:

$$N^{1/2}(\hat{\theta}_{mn}-\theta_0)\rightarrow N(0, [12\lambda(1-\lambda)\Gamma^2]^{-1}) , \qquad \text{in law, as } N\rightarrow\infty . \tag{8}$$

3. Some lemmas

Let c_1 and c_2 be positive constants to be chosen later, and let $\{a_N\}$, $\{b_N\}$ and $\{\gamma_N\}$ be sequences of positive constants such that, as $N\rightarrow\infty$,

$$\begin{aligned} &a_N\sim c_1N^{-1/2}(\log\log N)^{1/2} , \\ &b_N\sim N^{1/4} , \\ &\gamma_N\sim c_2N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4} . \end{aligned} \tag{9}$$

Consider the interval with the central point y:

$$I_N(y)=(-a_N+y,\, y+a_N) \tag{10}$$

and its division points:

$$\eta_{r,N}(y)=y+a_Nb_N^{-1}\cdot r , \qquad \text{for integers } |r|\leqq b_N . \tag{11}$$

For sample size m and N such as in Assumption (b) of Theorem, put

$$\begin{aligned} &G_m(x, y)=[F_m^X(x)-F_m^X(y)]-[F(x)-F(y)] , \\ &H_N(y)=\sup\{|G_m(x, y)|;\ x\in I_N(y)\} . \end{aligned} \tag{12}$$

Then, since F_m^X and F are non-decreasing, it follows that for $x\in[\eta_{rN}(y), \eta_{r+1,N}(y)]$

$$\begin{aligned} G_m(x, y)&\leqq[F_m^X(\eta_{r+1,N}(y))-F_m^X(y)]-[F(\eta_{rN}(y))-F(y)] \\ &=G_m(\eta_{r+1,N}(y), y)+[F(\eta_{r+1,N}(y))-F(\eta_{rN}(y))] \end{aligned}$$

and, similarly, that

$$G_m(x, y)\geqq G_m(\eta_{rN}(y), y)-[F(\eta_{r+1,N}(y))-F(\eta_{rN}(y))] .$$

Hence it follows that

$$\begin{aligned} H_N(y)&\leqq\max\{|G_m(\eta_{rN}(y), y)|;\ -b_N\leqq r\leqq b_N\} \\ &\quad+\max\{F(\eta_{r+1,N}(y))-F(\eta_{rN}(y));\ -b_N\leqq r\leqq b_N\} \\ &=H_N^*(y)+\beta_N(y) , \qquad \text{say} . \end{aligned} \tag{13}$$

Suppose that $|f(x)|\leqq M$, for $-\infty<x<\infty$, then we have from (11) that $F(\eta_{r+1,N}(y))-F(\eta_{rN}(y))\leqq M\cdot a_Nb_N^{-1}$, and hence, that

$$\beta_N(y)\leqq M\cdot a_Nb_N^{-1} \tag{14}$$

where the right hand is independent of y.

In general, let $U_1, \cdots, U_m$ be independent and identically distributed (i.i.d.) r.v.'s with mean 0 and variance σ^2 and be bounded by constant c. Then Bernstein's inequality holds (see Uspensky [12], pp. 204–206):

$$\text{(15)} \qquad \mathrm{P}\{|U_1+\cdots+U_m|\geqq t\} \leqq 2\exp\left\{-t^2\Big/\left(2m\sigma^2+\frac{2}{3}ct\right)\right\}.$$

Denote by $B(m, p)$ the number of successes in m Bernoulli trials with the probability of success, p. Then Bernstein's inequality implies that

$$\text{(16)} \qquad \mathrm{P}\{|B(m, p)-mp|\geqq t\} \leqq 2\exp\left\{-t^2\Big/\left(2mp(1-p)+\frac{2}{3}t\right)\right\}.$$

Now, since the probability distribution of $|G_m(\eta_{rN}(y), y)|$ is the same as that of $m^{-1}|B(m, p_{rN})-m\cdot p_{rN}|$ where $p_{rN}=|F(\eta_{rN}(y))-F(y)|\leqq M\cdot a_N$ (from (11)), it follows from (16) that

$$\text{(17)} \qquad \mathrm{P}\{|G_m(\eta_{rN}(y), y)|\geqq\gamma_N\} \leqq 2\exp\left\{-(m\gamma_N)^2\Big/\left(2mM\cdot a_N+\frac{2}{3}m\gamma_N\right)\right\}.$$

Therefore we have from (13) and (17) that

$$\text{(18)} \qquad \begin{aligned} \mathrm{P}\{H_N^*(y)\geqq\gamma_N\} &\leqq \sum_{-b_N\leqq r\leqq b_N} \mathrm{P}\{|G_m(\eta_{rN}(y), y)|\geqq\gamma_N\} \\ &\leqq 4b_N \exp\left\{-m\gamma_N^2\Big/\left(2M\cdot a_N+\frac{2}{3}\gamma_N\right)\right\}=\rho_N, \qquad \text{(say)}, \end{aligned}$$

where ρ_N is independent of y. The above-mentioned are the essential points of Lemma 1 due to Bahadur [1] but added that bounds in (14) and (18), $M\cdot a_N b_N^{-1}$ and ρ_N, are independent of y $(-\infty<y<\infty)$.

LEMMA 1. *Under the same assumptions as in Theorem, it holds that*

$$\text{(19)} \qquad \begin{aligned} K_N &= \sup\{|[W_{mn}(\theta)-W_{mn}(\theta_0)]+\Gamma(\theta-\theta_0)|;\ \theta\in I_N(\theta_0)\}, \qquad \textit{(say)}, \\ &= O(N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4}), \qquad \textit{as } N\to\infty, \end{aligned}$$

with probability one.

PROOF. According to Assumption (a), suppose that $|f(x)|$ and $|f'(x)|\leqq M$, for $-\infty<x<\infty$. From (2) and (12) it follows that

$$\text{(20)} \qquad \begin{aligned} &[W_{mn}(\theta)-W_{mn}(\theta_0)]+\Gamma(\theta-\theta_0) \\ &=\frac{1}{n}\sum_{k=1}^{n}\{[F_m^X(Y_k-\theta)-F_m^X(Y_k-\theta_0)]-[F(Y_k-\theta)-F(Y_k-\theta_0)]\} \\ &\quad+\left\{\frac{1}{n}\sum_{k=1}^{n}[F(Y_k-\theta)-F(Y_k-\theta_0)]+\int_{-\infty}^{\infty}(f(x))^2dx\cdot(\theta-\theta_0)\right\} \\ &=\frac{1}{n}\sum_{k=1}^{n}G_m(Y_k-\theta, Y_k-\theta_0)+\left\{\left[-\frac{1}{n}\sum_{k=1}^{n}f(Y_k-\theta_0)\right.\right. \end{aligned}$$

$$+\int_{-\infty}^{\infty}(f(x))^2dx\Big]\cdot(\theta-\theta_0)+\frac{1}{n}\sum_{k=1}^{n}f'(\tilde{Y}_k)(\theta-\theta_0)^2/2\Big\}\ ,$$

where $\tilde{Y}_k=Y_k-\omega_k\theta-(1-\omega_k)\theta_0$, $0\leqq\omega_k\leqq 1$. It follows from (11)–(15), (19), (20), and Assumption (a) that

$$K_N\leqq\left\{\frac{1}{n}\sum_{k=1}^{n}H_N^*(Y_k-\theta_0)+M\cdot a_N b_N^{-1}\right\}$$
$$+\left\{\left|\frac{1}{n}\sum_{k=1}^{n}f(Y_k-\theta_0)-\int_{-\infty}^{\infty}(f(x))^2dx\right|\cdot a_N+M\cdot a_N^2/2\right\}\ . \tag{21}$$

We have from (10) that

$$\begin{aligned}&M\cdot a_N b_N^{-1}\sim M\cdot c_1 N^{-3/4}(\log\log N)^{1/2}\ ,\\&M\cdot a_N^2/2\sim(M\cdot c_1^2/2)N^{-1}\log\log N\ .\end{aligned} \tag{22}$$

It follows from (10) and Assumption (b) and by the law of the iterated logarithm for bounded r.v.'s (see Loève [8], p. 260) that

$$a_N\left|\frac{1}{n}\sum_{k=1}^{n}f(Y_k-\theta_0)-\int_{-\infty}^{\infty}(f(x))^2dx\right|=O(N^{-1}\log\log N)\ , \tag{23}$$

as $N\to\infty$, with probability one. On the other hand, from (18) we have that

$$\begin{aligned}\mathrm{P}\left\{\frac{1}{n}\sum_{k=1}^{n}H_N^*(Y_k-\theta_0)\geqq\gamma_N\right\}&\leqq\sum_{k=1}^{n}\mathrm{P}\left\{H_N^*(Y_k-\theta_0)\geqq\gamma_N\right\}\\&\leqq\sum_{k=1}^{n}\mathrm{E}_Y\left\{\mathrm{P}\left[H_N^*(Y_k-\theta_0)\geqq\gamma_N\,|\,Y\right]\right\}\\&\leqq n\cdot\rho_N\ .\end{aligned} \tag{24}$$

From Assumption (b) it is easy to see that

$$\lim_{N\to\infty}\left[\log n\cdot\rho_N/\log N\right]=\frac{5}{4}-(\lambda\cdot c_2^2)/(2M\cdot c_1)\ . \tag{25}$$

Choosing c_2 to be sufficiently large for e_1, we can make the limit in (25) <-1 and so, we have by Borel-Cantelli lemma that

$$\frac{1}{n}\sum_{k=1}^{n}H_N^*(Y_k-\theta_0)=O(N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4})\ ,\qquad\text{as } N\to\infty\ , \tag{26}$$

with probability one. From (21), (22), (23), and (26), the conclusion (19) of this lemma is obtained.

Consider Wilcoxon statistic and its projection:

$$W_{mn}(\theta_0)=\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\delta(Y_k-\theta_0-X_j)\ ,$$

and

$$W_{mn}^*(\theta_0)=\frac{1}{n}\sum_{k=1}^{n}F(Y_k-\theta_0)-\frac{1}{m}\sum_{j=1}^{m}F(X_j)+\frac{1}{2} \qquad \text{(say)} .$$

Suppose that the distribution function F is continuous and let

$$X_j'=F(X_j) \qquad \text{and} \qquad Y_k'=F(Y_k-\theta_0) ,$$
$$j=1,\cdots,m,\cdots;\ k=1,\cdots,n,\cdots .$$

Then $X_1',\cdots,X_m',\cdots;\ Y_1',\cdots,Y_n',\cdots$ are i.i.d. according to Uniform distribution $U(0,1)$. Since $\delta(Y_k-\theta_0-X_j)=\delta(Y_k'-X_j')$, $j=1,2,\cdots;\ k=1,2,\cdots$ with probability one, we may as well discuss about

$$W_{mn}=\frac{1}{mn}\sum_{j=1}^{m}\sum_{k=1}^{n}\delta(Y_k'-X_j') ,$$

and

$$W_{mn}^*=\frac{1}{n}\sum_{k=1}^{n}Y_k'-\frac{1}{m}\sum_{j=1}^{m}X_j'+\frac{1}{2} , \qquad \text{(say)} ,$$

as $W_{mn}(\theta_0)$ and $W_{mn}^*(\theta_0)$.

LEMMA 2. *Under Assumption* (b) *of Theorem, it holds that for any* α $(0<\alpha<1)$

(27) $$|W_{mn}-W_{mn}^*|=O(N^{-1/2-\alpha/4}) , \qquad \text{as } N\to\infty ,$$

with probability one.

PROOF. For $y=(y_1,\cdots,y_n)$, let

$$Z_j(y)=\frac{1}{n}\sum_{k=1}^{n}\left[\delta(y_k-X_j')-y_k+X_j'-\frac{1}{2}\right] , \qquad j=1,\cdots,m .$$

Then $Z_1(y),\cdots,Z_m(y)$ are i.i.d. r.v.'s with mean 0 and variance

(28) $$\sigma_y^2=\mathrm{E}\,Z_j(y)^2=\frac{1}{n^2}\left\{\frac{1}{12}+2\sum_{k<k'}\left[\frac{1}{12}+\frac{y_{(k)}-y_{(k')}}{2}+\frac{(y_{(k)}-y_{(k')})^2}{2}\right]\right\} ,$$

where $y_{(k)}<y_{(k')}$ are the smaller one and the bigger one between y_k and $y_{k'}$, respectively. Since $|\delta(y-x)-y+x-1/2|\leqq 1/2$, for $0\leqq x,\ y\leqq 1$, which implies that $|Z_j(y)|\leqq 1/2$, we have by Bernstein's inequality (15) that

(29) $$\mathrm{P}\left\{\left|\frac{1}{m}\sum_{j=1}^{m}Z_j(y)\right|\geqq t\right\}\leqq 2\exp\left\{-m^2t^2\Big/\left(2m\sigma_y^2+\frac{1}{3}mt\right)\right\} .$$

Next if we substitute $Y=(Y_1',\cdots,Y_n')$ for y into σ_y^2, we can see that

$$\mathrm{E}\,\sigma_Y^2=\frac{1}{n^2}\left\{\frac{n}{12}+2\sum_{k<k'}\mathrm{E}\left[\frac{1}{12}+\frac{Y_{(k)}-Y_{(k')}}{2}+\frac{(Y_{(k)}-Y_{(k')})^2}{2}\right]\right\}=\frac{1}{12n}\ ,$$

and

$$\mathrm{E}\left(\sigma_Y^2-\frac{1}{12n}\right)^2=n(n-1)/360n^4\ .$$

By the fact that $W_{mn}-W_{mn}^*=(1/m)\sum_{j=1}^{m}Z_j(Y)$, it follows from (29) that

$$\text{(30)}\qquad \mathrm{P}\{|W_{mn}^*-W_{mn}|\geqq t\}=\mathrm{E}_Y\left\{\mathrm{P}\left[\left|\frac{1}{m}\sum_{j=1}^{m}Z_j(Y)\right|\geqq t\,|\,Y\right]\right\}$$
$$\leqq \mathrm{E}_Y\{2\exp[-mt^2/(2\sigma_Y^2+t/3)]\}\ .$$

Since $2\exp[-mt^2/(2\sigma_Y^2+t/3)]\leqq 2$ and

$$\mathrm{E}\,(\sigma_Y^2)^2=n(n-1)/360n^4+(1/12n)^2=(7n-2)/720n^3\ ,$$

it follows by Chebyshev's inequality that

$$\text{(31)}\qquad \mathrm{E}_Y\{2\exp[-mt^2/(2\sigma_Y^2+t/3)]\}$$
$$\leqq 2\exp[-mt^2/(2\cdot\varepsilon+t/3)]+2\,\mathrm{P}\{\sigma_Y^2\geqq\varepsilon\}$$
$$\leqq 2\exp[-mt^2/(2\cdot\varepsilon+t/3)]+2\cdot(7n-2)/(720n^3\cdot\varepsilon^2)\ .$$

For α $(0<\alpha<1)$, choose α' such that $\alpha<\alpha'<1$ and let $t=m^{-1/2-\alpha/4}$ and $\varepsilon=n^{-\alpha'/2}$ in (31). Then we have from (30) and (31) that

$$\mathrm{P}\{|W_{mn}-W_{mn}^*|\geqq m^{-1/2-\alpha/4}\}$$
$$\leqq 2\exp\{-m^{-\alpha/2}/(2n^{-\alpha'/2}+m^{-1/2-\alpha/4}/3)\}+2\cdot n^{\alpha'}(7n-2)/720n^3$$
$$=\delta_N\ ,\qquad \text{say}\ .$$

It is easy to see that

$$\lim_{N\to\infty}\{\log\delta_N/\log N\}=\lim_{N\to\infty}\{\log[2n^{\alpha'}(7n-2)/720n^3]/\log N\}$$
$$=-2+\alpha'<-1\ .$$

Thus, by Borel-Cantelli lemma the conclusion (27) of this lemma is proved.

Suppose Assumption (b) of Theorem, then it holds by the law of the iterated logarithm for the sum of independent r.v.'s, W_{mn}^*, that

$$\limsup_{N\to\infty}[N^{1/2}(W_{mn}^*-1/2)/(2\log\log N)^{1/2}]=[12\lambda(1-\lambda)]^{-1/2}\ ,$$

and

$$\liminf_{N\to\infty}[N^{1/2}(W_{mn}^*-1/2)/(2\log\log N)^{1/2}]=-[12\lambda(1-\lambda)]^{-1/2}\ ,$$

with probability one. Therefore by Lemma 2 we have the following lemma:

LEMMA 3. *Under Assumption* (b) *of Theorem, the law of the iterated logarithm for Wilcoxon statistic, W_{mn}, holds*:

$$\limsup_{N\to\infty} [N^{1/2}(W_{mn}-1/2)/(2\log\log N)^{1/2}]=[12\lambda(1-\lambda)]^{-1/2}\,, \tag{32}$$
$$\liminf_{N\to\infty} [N^{1/2}(W_{mn}-1/2)/(2\log\log N)^{1/2}]=-[12\lambda(1-\lambda)]^{-1/2}\,,$$

with probability one.

LEMMA 4. *Under the same assumptions as those of Theorem, for constant c_1 chosen suitably, it holds that, with probability one, H-L estimator $\hat{\theta}_{mn}\in I_N(\theta_0)$, for all sufficiently large N.*

PROOF. Since $W_{mn}(\theta)$ is non-increasing in θ, it follows that

$$\inf\{|W_{mn}(\theta)-1/2|;\ \theta\notin I_N(\theta_0)\} =\min\{|W_{mn}(\theta_0-a_N)-1/2|,\ |W_{mn}(\theta_0+a_N)-1/2|\}\ . \tag{33}$$

Now,

$$|W_{mn}(\theta_0+a_N)-1/2|\geqq\Gamma\cdot a_N-|W_{mn}(\theta_0+a_N)-W_{mn}(\theta_0)+\Gamma\cdot a_N| -|W_{mn}(\theta_0)-1/2|\ .$$

Thus, it follows from (12), Lemmas 1 and 3 that, with probability one,

$$\begin{aligned}|W_{mn}(\theta_0+a_N)-1/2|\geqq&(\Gamma\cdot c_1-\varepsilon)\cdot N^{-1/2}(\log\log N)^{1/2}\\ &-(c_2+\varepsilon)\cdot N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4}\\ &-\{[6\lambda(1-\lambda)]^{-1/2}+\varepsilon\}\cdot N^{-1/2}(\log\log N)^{1/2}\,,\end{aligned}$$

for any $\varepsilon>0$ and all sufficiently large N, and hence, that, with probability one,

$$|W_{mn}(\theta_0+a_N)-1/2|\geqq\{\Gamma\cdot c_1-[6\lambda(1-\lambda)]^{-1/2}-\varepsilon'\}\cdot N^{-1/2}(\log\log N)^{1/2}\,,$$

for any $\varepsilon'>0$ and all sufficiently large N. Choose c_1 to be so large that $\{\Gamma\cdot c_1-[6\lambda(1-\lambda)]^{-1/2}-\varepsilon'\}=A$ (say)>0. Then we have that with probability one,

$$|W_{mn}(\theta_0+a_N)-1/2|\geqq A\cdot N^{-1/2}(\log\log N)^{1/2}\,, \tag{34}$$

for all sufficiently large N, and similarly, that, with probability one,

$$|W_{mn}(\theta_0-a_N)-1/2|\geqq A\cdot N^{-1/2}(\log\log N)^{1/2}\,, \tag{35}$$

for all sufficiently large N. Subsequently from (33), (34) and (35) it holds that with probability one,

(36) $$\inf\{|W_{mn}(\theta)-1/2|;\ \theta\notin I_N(\theta_0)\}\geqq A\cdot N^{-1/2}(\log\log N)^{1/2}$$

for all sufficiently large N.

On the other hand it holds from (4) that

(37) $$|W_{mn}(\hat{\theta}_{mn})-1/2|\leqq\frac{1}{mn}\,,\qquad \text{with probability one}\,,$$

and further, that $1/mn\sim 1/\lambda(1-\lambda)N^2$, as $N\to\infty$. Hence from (36) and (37) it holds that with probability one, $\hat{\theta}_{mn}\in I_N(\theta_0)$, for all sufficiently large N. The proof of this lemma is complete.

4. Proof of theorem

Choose constants c_1 and c_2 as in Section 3. From Lemma 4 and (20) it follows that with probability one,

$$|[W_{mn}(\hat{\theta}_{mn})-W_{mn}(\theta_0)]+\Gamma\cdot(\hat{\theta}_{mn}-\theta_0)|\leqq K_N\,,$$

for all sufficiently large N, and hence, from (37) and Lemma 1 that

$$|[1/2-W_{mn}(\theta_0)]+\Gamma\cdot(\hat{\theta}_{mn}-\theta_0)|=O(N^{-3/4}(\log N)^{1/2}(\log\log N)^{1/4})\,,$$

as $N\to\infty$, with probability one. That is, (6) in Theorem is proved. Furthermore from (6) and Lemma 3 we see that (7) holds.

Acknowledgement

The author is grateful to the referee for his comments.

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