ASYMPTOTIC FORMULAS FOR THE DISTRIBUTIONS OF THREE STATISTICS FOR MULTIVARIATE LINEAR HYPOTHESIS*

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1. Introduction and summary

Let S_e and S_h denote the independent $p\times p$ matrices with the central Wishart distribution $W_p(n, \Sigma)$ and the non-central Wishart distribution $W_p(q, \Sigma, \Omega)$, respectively. In the canonical form of multivariate linear hypothesis we can express the likelihood ratio (=L.R.) statistic by $W=-n_1 \log |S_e(S_e+S_h)^{-1}|$, Hotelling's statistic by $T_0^2=n \operatorname{tr} S_h S_e^{-1}$ and Pillai's criterion by $V=(n+q) \operatorname{tr} S_h(S_h+S_e)^{-1}$, where $n_1=n+(q-p-1)/2$. Then n and q mean the degrees of freedom for the error and the hypothesis, respectively. The null hypothesis is equivalent to $\Omega=0$.

In this paper we are concerned with the asymptotic expansions of the distributions of the three statistics mentioned above, under the assumption that q is a fixed constant and $\Omega=O(1)$. Expansions have been obtained in terms of central χ^2-distributions in the null case and non-central χ^2-distributions in the non-null case by several authors. In this paper we attempt to improve our asymptotic approximations by extending the asymptotic expansions and considering the modified statistics. The formulas necessary in derivation of further expansions are given in Section 5.

Asymptotic expansion of the null distribution of W was obtained up to order n^{-4} by Rao [18] and Box [2]. Posten and Bargmann [17] obtained the non-null distribution up to order n^{-2} when the non-centrality matrix is of rank two. Sugiura and Fujikoshi [22] derived the same asymptotic expansion without any assumption on the rank of Ω, based on the hypergeometric function of matrix argument. In Section 2 we give the term of order n^{-3} in the expanded form of the non-null distribution of W.

The distribution of T_0^2 in the null case was given by Ito [10] in the expanded form up to order n^{-2}. Siotani [20] and later Ito [11] derived the non-null distribution up to order n^{-1}. The non-null distribution up to order n^{-2} was obtained by Siotani [21], Fujikoshi [6], Hayakawa [7]

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and Lee [14]. The expansion of the percent point of T_0^2 was given by Ito [10] and Siotani [19] up to order n^{-2}. Davis [4] gave the percentile expansion of T_0^2 up to order n^{-3}. In Section 3 we treat the distribution of the modified T_0^2 statistic defined by $T=(n-p-1)\,\mathrm{tr}\, S_h S_e^{-1}$. This is due to the fact that the expectation of T is equal to the expectation by the limitting distribution. This consideration has been done by Sugiura [23] and Sugiura and Nagao [24]. From the numerical results we can roughly say that the accuracy of the approximation due to the present formula is better than that of the previous asymptotic formula.

The asymptotic expansions of the null and non-null distributions and the percentage points of V with respect to $n+q$ were obtained by the author [6]. On the other hand, Muirhead [15] and Davis [5] derived the expansions of the null distribution and percentage points of $\tilde{V}=n\,\mathrm{tr}\, S_h(S_h+S_e)^{-1}$ with respect to n and the expansion of the non-null distribution was obtained by Lee [14]. However, it may be noted that the expansion of V with respect to $n+q$ can be recommended by the similar reason as in the case of T_0^2. In Section 4 we derive asymptotic expansions of the null distribution and percentage points of V up to order n^{-3}. Consequently we can see that the simple and interesting relationship exists between the coefficients in asymptotic formulas for the null distributions of T and V.

2. Expansion of the non-null distribution of W

In this section we attempt to extend asymptotic formula for the non-null distribution of W by the direct extension of the method due to Sugiura and Fujikoshi [22]. The characteristic function of W has been expressed by

$$C(t)=\frac{\Gamma_p(n_1(1-2it)/2-(q-p-1)/4)\Gamma_p(n_1/2+(p+q+1)/4)}{\Gamma_p(n_1/2-(q-p-1)/4)\Gamma_p(n_1(1-2it)/2+(p+q+1)/4)} \cdot {}_1F_1(-itn_1;\, n_1(1-2it)/2+(p+q+1)/4;\, -\Omega)\,, \tag{2.1}$$

where

$$\Gamma_p(t)=\pi^{p(p-1)/4}\prod_{\alpha=1}^{p}\Gamma(t-(\alpha-1)/2)\,, \tag{2.2}$$

$${}_rF_s(a_1,\cdots,a_r;\, b_1,\cdots,b_s;\, Z)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(a_1)_\kappa\cdots(a_r)_\kappa}{(b_1)_\kappa\cdots(b_s)_\kappa}\frac{C_\kappa(Z)}{k!}\,, \tag{2.3}$$

where $a_1,\cdots,a_r$, $b_1,\cdots,b_s$ are real or complex constants,

$$(a)_\kappa=\prod_{\alpha=1}^{p}(a-(\alpha-1)/2)(a+1-(\alpha-1)/2)\cdots(a+k_\alpha-1-(\alpha-1)/2)\,, \tag{2.4}$$

and $C_\kappa(Z)$ is the zonal polynomial of the $p\times p$ symmetric matrix Z corresponding to the partition $\kappa=\{k_1,\cdots,k_p\}$ of the integer k such that $k_1+\cdots+k_p=k$ and $k_1\geqq\cdots\geqq k_p\geqq 0$. Applying the well-known asymptotic formula for the gamma function to each of the four gamma products, we get

$$\text{(2.5)}\quad \text{First factor}=(1-2it)^{-pq/2}[1+rn_1^{-2}((1-2it)^{-2}-1) + n_1^{-4}\{d((1-2it)^{-4}-1)-r^2((1-2it)^{-2}-1)\}]+O(n^{-6})\ ,$$

which is the asymptotic expansion of the characteristic function of W under the null hypothesis (c.f. Anderson [1], p. 208), where $r=pq(p^2+q^2-5)/48$, $d=r^2/2+pq\{3p^4+3q^4-50p^2-50q^2+10p^2q^2+159\}/1920$. Noting that $(n+b)_\kappa=\prod_{\alpha=1}^{p}\{\Gamma(n+b+k_\alpha-(\alpha-1)/2)/\Gamma(n+b-(\alpha-1)/2)\}$, we have

$$\text{(2.6)}\quad \log(n+b)_\kappa=k\log n+(2n)^{-1}A_1-(24n^2)^{-1}A_2+(24n^3)^{-1}A_3 -(960n^4)^{-1}A_4+O(n^{-5})\ ,$$

where

$$\text{(2.7)}\quad \begin{aligned} A_1&=a_1(\kappa)+2bk\ ,\\ A_2&=a_2(\kappa)+12ba_1(\kappa)+(12b^2-1)k\ ,\\ A_3&=a_3(\kappa)+2ba_2(\kappa)+(12b^2-1)a_1(\kappa)+2b(4b^2-1)k\ ,\\ A_4&=3a_4(\kappa)+120ba_3(\kappa)+10(12b^2-1)a_2(\kappa)+120b(4b^2-1)a_1(\kappa)\\ &\quad+[15(4b^2-1)^2-8]k\ , \end{aligned}$$

and $a_i(\kappa)$ $(i=1,2,3,4)$ are defined by (5.1) in Section 5. Exponentiation of the expression which is obtained by applying (2.6) to $\log[(-itn_1)_\kappa/(n_1(1-2it)/2+(p+q+1)/4)_\kappa]$ yields the following expansions of the second factor ${}_1F_1$ in (2.1):

$$\text{(2.8)}\quad \begin{aligned} &\sum_{k=0}^{\infty}\sum_{\kappa}[1-n_1^{-1}\{B_1a_1(\kappa)+B_2\}+n_1^{-2}\{B_3a_1(\kappa)^2+B_4a_1(\kappa)+B_5a_2(\kappa)+B_6\}\\ &\quad -n_1^{-3}\{B_7a_1(\kappa)^3+B_8a_1(\kappa)^2+B_9a_1(\kappa)a_2(\kappa)+B_{10}a_1(\kappa)+B_{11}a_2(\kappa)\\ &\quad +B_{12}a_3(\kappa)+B_{13}\}+n_1^{-4}\{B_{14}a_1(\kappa)^4+B_{15}a_1(\kappa)^3+B_{16}a_1(\kappa)^2+B_{17}a_1(\kappa)\\ &\quad +B_{18}a_1(\kappa)^2a_2(\kappa)+B_{19}a_1(\kappa)a_2(\kappa)+B_{20}a_2(\kappa)^2+B_{21}a_2(\kappa)\\ &\quad +B_{22}a_1(\kappa)a_3(\kappa)+B_{23}a_3(\kappa)+B_{24}a_4(\kappa)+B_{25}\}+O(n^{-5})]\\ &\quad \cdot C_\kappa(2it(1-2it)^{-1}\Omega)/k!\ , \end{aligned}$$

where the coefficients B_α $(\alpha=1,2,\cdots,25)$, whose concrete expressions are not given here for the complexity, depend on only k, p, q and it. By using the formulas for weighted sums of zonal polynomials given in Section 5, we can simplify (2.8) and consequently get the expansion of

$C(t)$ which may be inverted, using the fact that $(1-2it)^{-f/2}\exp\{2it\delta^2/(1-2it)\}$ is the characteristic function of the non-central χ^2-distribution with f degrees of freedom and non-centrality parameter δ^2, to yield the expansion of the non-null distribution of W. The final result, to order n^{-3} is

$$
\begin{aligned}
(2.9)\qquad \mathrm{P}(W<x)=&\mathrm{P}(\chi_f^2(\delta^2)<x)-n_1^{-1}\{-2s\omega_1\,\mathrm{P}(\chi_{f+2}^2(\delta^2)<x)\\
&+(2s\omega_1-\omega_2)\,\mathrm{P}(\chi_{f+4}^2(\delta^2)<x)+\omega_2\,\mathrm{P}(\chi_{f+6}^2(\delta^2)<x)\}\\
&+n_1^{-2}\Big[r\{\mathrm{P}(\chi_{f+4}^2(\delta^2)<x)-\mathrm{P}(\chi_f^2(\delta^2)<x)\}\\
&+\sum_{\alpha=2}^{6}H_\alpha\,\mathrm{P}(\chi_{f+2\alpha}^2(\delta^2)<x)\Big]\\
&-n_1^{-3}\sum_{\alpha=1}^{9}G_\alpha\,\mathrm{P}(\chi_{f+2\alpha}^2(\delta^2)<x)+O(n^{-4})\ ,
\end{aligned}
$$

where $f=pq$, $r=pq(p^2+q^2-5)/48$, $s=(p+q+1)/4$, $\delta^2=\mathrm{tr}\,\Omega$, $\omega_j=\mathrm{tr}\,\Omega^j$ and

$$
\begin{aligned}
(2.10)\qquad &H_2=-4s^2\omega_1+2s^2\omega_1^2+2s\omega_2\ ,\\
&H_3=4s^2\omega_1-(1+4s^2)\omega_1^2-(1+8s)\omega_2+2s\omega_1\omega_2+\frac{4}{3}\omega_3\ ,\\
&H_4=(1+2s^2)\omega_1^2+(1+6s)\omega_2-4s\omega_1\omega_2-4\omega_3+\frac{1}{2}\omega_2^2\ ,\\
&H_5=2s\omega_1\omega_2+\frac{8}{3}\omega_3-\omega_2^2\ ,\qquad H_6=\frac{1}{2}\omega_2^2\ ,
\end{aligned}
$$

$$
\begin{aligned}
(2.11)\qquad &G_1=2rs\omega_1\ ,\qquad G_2=-r(2s\omega_1-\omega_2)\ ,\\
&G_3=-2s(r+4s^2)\omega_1+2s(1+4s^2)\omega_1^2+(-r+2s+12s^2)\omega_2-\frac{4}{3}s^3\omega_1^3\\
&\qquad -4s^2\omega_1\omega_2-\frac{8}{3}s\omega_3\ ,\\
&G_4=2s(r+4s^2)\omega_1-(1+10s+16s^3)\omega_1^2-(3+r+10s+36s^2)\omega_2\\
&\qquad +2s(1+2s^2)\omega_1^3+2(2+s+12s^2)\omega_1\omega_2+4(1+6s)\omega_3\\
&\qquad -2s^2\omega_1^2\omega_2-2s\omega_2^2-\frac{8}{3}s\omega_1\omega_3-2\omega_4\ ,\\
&G_5=(1+8s+8s^3)\omega_1^2+(3+r+8s+24s^2)\omega_2-4s(1+s^2)\omega_1^3\\
&\qquad -4(3+s+9s^2)\omega_1\omega_2-12(1+4s)\omega_3+(1+6s^2)\omega_1^2\omega_2\\
&\qquad +(1+10s)\omega_2^2+\frac{32}{3}s\omega_1\omega_3+12\omega_4-\frac{4}{3}\omega_2\omega_3-s\omega_1\omega_2^2\ ,\\
&G_6=s\Big(2+\frac{4}{3}s^2\Big)\omega_1^3+2(4+s+8s^2)\omega_1\omega_2+8\Big(1+\frac{10}{3}s\Big)\omega_3
\end{aligned}
$$

$$-2(1+3s^2)\omega_1^2\omega_2-2(1+7s)\omega_2^2-\frac{40}{3}s\omega_1\omega_3-20\omega_4$$

$$+\frac{16}{3}\omega_2\omega_3+3s\omega_1\omega_2^2-\frac{1}{6}\omega_2^3\ ,$$

$$G_7=(1+2s^2)\omega_1^2\omega_2+(1+6s)\omega_2^2+\frac{16}{3}s\omega_1\omega_3+10\omega_4-\frac{20}{3}\omega_2\omega_3$$

$$-3s\omega_1\omega_2^2+\frac{1}{2}\omega_2^3\ ,$$

$$G_8=\frac{8}{3}\omega_2\omega_3+s\omega_1\omega_2^2-\frac{1}{2}\omega_2^3\ ,\qquad G_9=\frac{1}{6}\omega_2^3\ .$$

The symbol $\chi_f^2(\delta^2)$ means the non-central χ^2-variate with f degrees of freedom and non-centrality parameter δ^2.

From the above discussion the term of order n^{-4} may be also obtained by the straight-forward computations.

3. Expansions of the distributions of T and numerical comparison

Put $n_2=n-p-1$ and $T=n_2\operatorname{tr} S_hS_e^{-1}$. Then, from Hsu [9] we have

$$\text{(3.1)}\qquad \mathrm{E}\,[T]=pq+2\operatorname{tr}\Omega\ .$$

The limitting distribution of T is the non-central χ^2-distribution with f degrees of freedom and non-centrality parameter $\delta^2=\operatorname{tr}\Omega$. This shows that the expectation of T is equal to the expectation by the limitting distribution. Therefore asymptotic expansion of the distribution of T with respect to n_2 may be recommended. For the derivation of expansions we use the same method as in the author [6]. The characteristic function of T can be expressed as

$$\text{(3.2)}\qquad \frac{\Gamma_p(n_2/2+\gamma/2)}{(n_2/2)^{pq/2}\Gamma_p(n_2/2+(p+1)/2)}\cdot\frac{1}{\Gamma_p(q/2)}\int_{S>0}|I+2n_2^{-1}S|^{-(n_2+\gamma)/2}\cdot\operatorname{etr}(2itS)|S|^{(q-p-1)/2}{}_0F_1(q/2\,;2it\Omega S)dS\ ,$$

where $\gamma=p+q+1$. The first factor can be expanded as

$$\text{(3.3)}\qquad 1+(4n_2)^{-1}pq\gamma+(96n_2^2)^{-1}pq\{(3pq-8)\gamma^2+4\gamma+4(pq+2)\}+(384n_2^3)^{-1}pq\gamma\{(p^2q^2-8pq+16)\gamma^2+4(pq-4)\gamma+4(p^2q^2-2pq-8)\}+O(n^{-4})\ .$$

By the same line as in the author [6], we can write the second factor as follows:

$$\text{(3.4)}\qquad \mathcal{A}_\Omega[1+n_2^{-1}(-\gamma\operatorname{tr}S+\operatorname{tr}S^2)+(6n_2^2)^{-1}\{3\gamma^2(\operatorname{tr}S)^2+6\gamma\operatorname{tr}S^2$$

$$-6\gamma(\mathrm{tr}\, S)\,\mathrm{tr}\, S^2-8\,\mathrm{tr}\, S^3+3(\mathrm{tr}\, S^2)^2\}+(6n_2^3)^{-1}\{-\gamma^3(\mathrm{tr}\, S)^3$$
$$-6\gamma^2(\mathrm{tr}\, S)\,\mathrm{tr}\, S^2-8\gamma\,\mathrm{tr}\, S^3+3\gamma^2(\mathrm{tr}\, S)^2\,\mathrm{tr}\, S^2+6\gamma(\mathrm{tr}\, S^2)^2$$
$$+8\gamma(\mathrm{tr}\, S)\,\mathrm{tr}\, S^3+12\,\mathrm{tr}\, S^4-3\gamma\,\mathrm{tr}\, S(\mathrm{tr}\, S^2)^2-8(\mathrm{tr}\, S^2)\,\mathrm{tr}\, S^3$$
$$+(\mathrm{tr}\, S^2)^3\}+O(n^{-4})]\;,$$

where $\mathcal{A}_\Omega[\{\ \}]$ is defined by

$$(3.5)\quad 2^{p(p-1)/2}(2\pi i)^{-p(p+1)/2}\int_{\mathcal{R}(T)=X_0>0}\mathrm{etr}\,(T)|T|^{-q/2}$$
$$\cdot\left[\int_{S>0}\mathrm{etr}\,[-\{(1-2it)I-2it\Omega^{1/2}T^{-1}\Omega^{1/2}\}S]\,|S|^{(q-p-1)/2}\{\ \}dS\right]dT\;.$$

The formulas for $\mathcal{A}_\Omega[\{\ \}]$ which are necessary for evaluating the terms up to order n^{-2} have been obtained by the author [6]. Therefore we immediately have the asymptotic formula for the non-null distribution of T. To order n^{-2}:

$$(3.6)\quad \mathrm{P}\,(T<x)=\mathrm{P}\,(\chi_f^2(\delta^2)<x)+(4n_2)^{-1}\{f\gamma\,\mathrm{P}\,(\chi_f^2(\delta^2)<x)-2\gamma(f-2\omega_1)$$
$$\cdot\,\mathrm{P}\,(\chi_{f+2}^2(\delta^2)<x)+(f\gamma-8\gamma\omega_1+4\omega_2)\,\mathrm{P}\,(\chi_{f+4}^2(\delta^2)<x)$$
$$+4(\gamma\omega_1-2\omega_2)\,\mathrm{P}\,(\chi_{f+6}^2(\delta^2)<x)+4\omega_2\,\mathrm{P}\,(\chi_{f+8}^2(\delta^2)<x)\}$$
$$+(96n_2^2)^{-1}\sum_{\alpha=0}^{8}L_\alpha\,\mathrm{P}\,(\chi_{f+2\alpha}^2(\delta^2)<x)+O(n^{-3})\;,$$

where $f=pq$, $\gamma=p+q+1$, $\delta^2=\mathrm{tr}\,\Omega$, $\omega_j=\mathrm{tr}\,\Omega^j$ and L_α $(\alpha=0, 1,\cdots, 8)$ are given by (3.7).

$$(3.7)\quad L_0=fh_0\;,\qquad L_1=-h_1(f-2\omega_1)\;,$$
$$L_2=fh_2-96(f+2)\gamma^2\omega_1+48\gamma^2\omega_1^2+24(f+4)\gamma\omega_2\;,$$
$$L_3=-fh_3+48\{3(f+4)\gamma^2+2\gamma+2(f+2)\}\omega_1-192(\gamma^2+1)\omega_1^2$$
$$-96\{(f+8)\gamma+2\}\omega_2+96\gamma\omega_1\omega_2+128\omega_3\;,$$
$$L_4=fh_4-96\{(f+6)\gamma^2+2\gamma+2(f+2)\}\omega_1+96(3\gamma^2+7)\omega_1^2$$
$$+48\{3(f+12)\gamma+14\}\omega_2-384\gamma\omega_1\omega_2-768\omega_3+48\omega_2^2\;,$$
$$L_5=8h_4\omega_1-192(\gamma^2+4)\omega_1^2-96\{(f+16)\gamma+8\}\omega_2+576\gamma\omega_1\omega_2$$
$$+1536\omega_3-192\omega_2^2\;,$$
$$L_6=48(\gamma^2+6)\omega_1^2+24\{(f+20)\gamma+12\}\omega_2-384\gamma\omega_1\omega_2-1280\omega_3+288\omega_2^2\;,$$
$$L_7=96\gamma\omega_1\omega_2+384\omega_3-192\omega_2^2\;,\qquad L_8=48\omega_2^2\;,$$

with h_α $(\alpha=0, 1,\cdots, 4)$ defined by

$$(3.8)\quad h_0=(3f-8)\gamma^2+4\gamma+4(f+2)\;,$$
$$h_1=12f\gamma^2\;,\qquad h_2=6(3f+8)\gamma^2\;,$$

$$h_3=4\{(3f+16)\gamma^2+4\gamma+4(f+2)\}\ ,$$

$$h_4=3\{(f+8)\gamma^2+4\gamma+4(f+2)\}\ .$$

Next we derive the null distribution up to order n^{-3}. Note that $\mathcal{A}_0[\{\ \}]=(1-2it)^{-pq/2}L_{(1-2it)I}[\{\ \}]$, where $L_R[\{\ \}]$ is an abbreviated notation for

$$(3.9)\qquad |R|^{q/2}\{\Gamma_p(q/2)\}^{-1}\int_{S>0}\mathrm{etr}\,(-RS)|S|^{(q-p-1)/2}\{\ \}dS\ .$$

Some formulas for $L_R[\{\ \}]$ are given in Section 5. Therefore we can obtain the following asymptotic expansion of the null distribution up to order n^{-3}:

$$(3.10)\quad \begin{aligned}\mathrm{P}\,(T<x)&=\mathrm{P}\,(\chi_f^2<x)\\&+f\gamma(4n_2)^{-1}\{\mathrm{P}\,(\chi_f^2<x)-2\,\mathrm{P}\,(\chi_{f+2}^2<x)+\mathrm{P}\,(\chi_{f+4}^2<x)\}\\&+f(96n_2^2)^{-1}\sum_{\alpha=0}^{4}(-1)^\alpha h_\alpha\,\mathrm{P}\,(\chi_{f+2\alpha}^2<x)\\&+f(384n_2^3)^{-1}\sum_{\alpha=0}^{6}(-1)^\alpha g_\alpha\,\mathrm{P}\,(\chi_{f+2\alpha}^2<x)+O(n^{-4})\ ,\end{aligned}$$

where $f=pq$, the coefficients h_α ($\alpha=0,1,\cdots,4$) are given by (3.8) and the coefficients g_α ($\alpha=0,1,\cdots,6$) are defined by

$$(3.11)\quad \begin{aligned}&g_0=\gamma\{(f^2-8f+16)\gamma^2+4(f-4)\gamma+4(f^2-2f-8)\}\ ,\\&g_1=2f\gamma h_0\ ,\qquad g_2=f\gamma\{5(3f+8)\gamma^2+4\gamma+4(f+2)\}\ ,\\&g_3=4\gamma\{5(f^2+8f+16)\gamma^2+4(f+4)\gamma+4(f^2+6f+8)\}\ ,\\&g_4=5(3f^2+40f+144)\gamma^3+4(11f+108)\gamma^2+4(11f^2+130f+288)\gamma\\&\qquad+96(f+2)\ ,\\&g_5=2\{(3f^2+56f+288)\gamma^3+4(5f+72)\gamma^2+4(5f^2+82f+216)\gamma\\&\qquad+96(f+2)\}\ ,\\&g_6=(f^2+24f+160)\gamma^3+4(3f+56)\gamma^2+4(3f^2+62f+184)\gamma+96(f+2)\ .\end{aligned}$$

Asymptotic formula for T percentiles in terms of percentage points, u say, of χ_f^2 can be obtained from the general inverse expansion formula of Hill and Davis [8]. To order n^{-3}:

$$(3.12)\quad \begin{aligned}&u-(2n_2)^{-1}\gamma\{u-u^2/(f+2)\}+(24n_2^2)^{-1}\{u[7\gamma^2-2\gamma-2(f+2)]\\&-u^2[11\gamma^2+2\gamma+2(f+2)]/(f+2)+2u^3[2(f+5)\gamma^2-(f+2)\gamma\\&-(f+2)^2]/[(f+2)^2(f+4)]+6u^4(p-1)(p+2)(q-1)(q+2)/\\&\quad[(f+2)^2(f+4)(f+6)]\}-(48n_2^3)^{-1}\{3u\gamma[3\gamma^2-2\gamma-2(f+2)]\\&-u^2\gamma[17\gamma^2+2\gamma+2(f+2)]/(f+2)+2u^3\gamma[(5f+26)\gamma^2-(f-2)\gamma\end{aligned}$$

$$-(f-2)(f+2)]/[(f+2)^2(f+4)]-2u^4\gamma[(f^2+24f+68)\gamma^2$$
$$-(7f+22)(f+2)\gamma-(7f+22)(f+2)^2]/[(f+2)^3(f+4)(f+6)]$$
$$+4u^5(p-1)(p+2)(q-1)(q+2)[(f-28)\gamma+6(f+2)]/[(f+2)^3$$
$$\cdot(f+4)(f+6)(f+8)]-8u^6(p-1)(p+2)(q-1)(q+2)$$
$$\cdot[(f-10)\gamma+3(f+2)]/[(f+2)^3(f+4)(f+6)(f+8)(f+10)]\}$$
$$+O(n^{-4}) .$$

In the above derivation of asymptotic formulas (3.6) and (3.10) we can not examine the convergent bound of T. However, from Davis [4] and Hayakawa [7] it may be noted that the bound of the convergent for T is $0<T<n-p-1$.

The tables below are used to test the accuracies of the asymptotic approximations presented earlier and here.

Table 1. Comparison of approximations to the upper 5% points of $\mathrm{tr}\, S_h S_e^{-1}$ for $p=2$ and $n=53$

q	A1	A2	A3	Exact
3	.2605	.26032	.26032	.26031
7	.4968	.4964	.49608	.49605
13	.8273	.8266	.82466	.82447

In Tables 1 and 2, A1, A2 and A3 mean the values of the approximations due to the earlier formula up to order n^{-2}, the new formula (3.6) and the formula (3.12) up to order n^{-2}, respectively. The exact values are taken from Pillai and Jayachandran [16]. Our power was computed by using the exact significant points in [16] and λ_i are the characteristic roots of Ω. From two tables it may be seen that the new asymptotic approximation, A2 is better than the earlier asymptotic approximation, A1 and in the calculation of the percentage points the formula (3.12) can be recommended.

Table 2. Comparison of approximations to the power of $\mathrm{tr}\, S_h S_e^{-1}$ for $p=2$ and $\alpha=.05$

n	q	λ_1	λ_2	A1	A2	Exact
33	3	.125	.125	.0682	.0675	.0676
		0	.5	.0877	.0869	.0871
	7	0	.5	.0726	.0708	.0705
		.5	.5	.0972	.0941	.0944
83	5	0	.5	.07959	.07947	.07948
		.5	.5	.11498	.11484	.11487
	13	.5	.5	.08442	.08378	.08376
		0	1.5	.1051	.1043	.1044

4. Expansion of the null distribution of V

For Pillai's criterion $V=n_3 \operatorname{tr} S_h(S_h+S_e)^{-1}$ with $n_3=n+q$, we have $\mathrm{E}[V]=pq$ and that the limitting distribution of V is the χ^2-distribution with pq degrees of freedom, under the null hypothesis. Therefore, by the similar reason as in the case of T_0^2 we consider asymptotic expansion of the distribution of V with respect to n_3, as in the author [6]. From James [12] the characteristic function of V under the null hypothesis can be expressed as

$$(4.1)\qquad {}_1F_1(q/2; n_3/2; n_3 itI) = \{\Gamma_p(q/2)\}^{-1}\int_{S>0} \operatorname{etr}(-S)|S|^{(q-p-1)/2}{}_0F_1(n_3/2; n_3 itS)dS$$

by using the recurrence relation for the hypergeometric function of matrix argument due to Constantine [3]. By considering the expansion of ${}_0F_1$ up to order n^{-3} which is obtained by the same technique as in ${}_1F_1$ in (2.1), we can write the expression (4.1) as follows:

$$(4.2)\qquad \begin{aligned}&(1-2it)^{-pq/2}L_{(1-2it)I}[1-n_3^{-1}(2it)^2 \operatorname{tr} S^2+(6n_3^2)^{-1}(2it)^2\{6(\operatorname{tr} S)^2\\ &+6\operatorname{tr} S^2+32it\operatorname{tr} S^3+3(2it)^2(\operatorname{tr} S^2)^2\}-(6n_3^3)^{-1}(2it)^2\{6(\operatorname{tr} S)^2\\ &+18\operatorname{tr} S^2+96it(\operatorname{tr} S)\operatorname{tr} S^2+96it\operatorname{tr} S^3+6(2it)^2(\operatorname{tr} S)^2\operatorname{tr} S^2\\ &+6(2it)^2(\operatorname{tr} S^2)^2+60(2it)^2\operatorname{tr} S^4+16(2it)^3(\operatorname{tr} S^2)\operatorname{tr} S^3\\ &+(2it)^4(\operatorname{tr} S^2)^3\}+O(n^{-4})]\,.\end{aligned}$$

Applying the formulas for $L_R[\{\ \}]$ given in Section 5 to the above expression, we have the following final result:

$$(4.3)\qquad \begin{aligned}\mathrm{P}(V<x)=&\,\mathrm{P}(\chi_f^2<x)\\ &-f\gamma(4n_3)^{-1}\{\mathrm{P}(\chi_f^2<x)-2\,\mathrm{P}(\chi_{f+2}^2<x)+\mathrm{P}(\chi_{f+4}^2<x)\}\\ &+f(96n_3^2)^{-1}\sum_{\alpha=0}^{4}(-1)^\alpha h_\alpha\,\mathrm{P}(\chi_{f+2\alpha}^2<x)\\ &-f(384n_3^3)^{-1}\sum_{\alpha=0}^{6}(-1)^\alpha g_\alpha\,\mathrm{P}(\chi_{f+2\alpha}^2<x)+O(n^{-4})\,,\end{aligned}$$

where $f=pq$, the coefficients h_α $(\alpha=0,1,\cdots,4)$ and g_α $(\alpha=0,1,\cdots,6)$ are given in (3.8) and (3.11), respectively. This shows that the asymptotic expansion of the null distribution of V is obtained from the same formula (3.10) for T by replacing n_2 by $-n_3$. Hence the percentile expansion of V up to order n^{-3} is obtained from (3.12) by making the same transformation, i.e. $n_2\to -n_3$.

5. Some formulas

For each partition κ of the integer k we put $a_i(\kappa)$ as follows:

$$(5.1)\quad a_1(\kappa)=\sum_{\alpha=1}^{p} k_\alpha(k_\alpha-\alpha)\,,\qquad a_2(\kappa)=\sum_{\alpha=1}^{p} k_\alpha(4k_\alpha^2-6k_\alpha\alpha+3\alpha^2)\,,$$

$$a_3(\kappa)=\sum_{\alpha=1}^{p} k_\alpha(2k_\alpha^3-4k_\alpha^2\alpha+3k_\alpha\alpha^2-\alpha^3)\,,$$

$$a_4(\kappa)=\sum_{\alpha=1}^{p} k_\alpha(16k_\alpha^4-40k_\alpha^3\alpha+40k_\alpha^2\alpha^2-20k_\alpha\alpha^3+5\alpha^4)\,.$$

Let Z be any symmetric matrix and put $z_j=\mathrm{tr}\, Z^j$. Then the following identities hold:

$$(5.2)\quad \sum_{k=l}^{\infty}\sum_{\kappa} C_\kappa(Z)/(k-l)!=z_1^l e^{z_1}\,,$$

$$(5.3)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)/k!=z_2 e^{z_1}\,,$$

$$(5.4)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)^2/k!=\{z_1^2+z_2+4z_3+z_2^2\}e^{z_1}\,,$$

$$(5.5)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_2(\kappa)/k!=\{z_1+3z_1^2+3z_2+4z_3\}e^{z_1}\,,$$

$$(5.6)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_3(\kappa)/k!=\{z_1^2+4z_2+4z_1z_2+4z_3+2z_4\}e^{z_1}\,,$$

$$(5.7)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)a_2(\kappa)/k!=\{3z_1^2+11z_2+25z_1z_2+24z_3+3z_1^2z_2+3z_2^2$$
$$+24z_4+4z_2z_3\}e^{z_1}\,,$$

$$(5.8)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)^3/k!=\{z_1^2+3z_2+16z_1z_2+16z_3+3z_1^2z_2+3z_2^2+32z_4$$
$$+12z_2z_3+z_2^3\}e^{z_1}\,,$$

$$(5.9)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_4(\kappa)/k!=\{z_1+25z_1^2+35z_2+(40/3)z_1^3+60z_1z_2+(380/3)z_3$$
$$+20z_2^2+40z_1z_3+60z_4+16z_5\}e^{z_1}\,,$$

$$(5.10)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_2(\kappa)^2/k!=\{z_1+40z_1^2+57z_2+54z_1^3+186z_1z_2+324z_3+9z_1^4$$
$$+18z_1^2z_2+81z_2^2+224z_1z_3+288z_4+24z_1^2z_3+24z_2z_3+144z_5+16z_3^2\}e^{z_1}\,,$$

$$(5.11)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)a_3(\kappa)/k!=\{4z_1^2+6z_2+4z_1^3+20z_1z_2+44z_3+z_1^2z_2$$
$$+16z_2^2+24z_1z_3+36z_4+4z_1z_2^2+4z_2z_3+16z_5+2z_2z_4\}e^{z_1}\,,$$

$$(5.12)\quad \sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)^2a_2(\kappa)/k!=\{11z_1^2+17z_2+25z_1^3+109z_1z_2+216z_3$$
$$+3z_1^4+12z_1^2z_2+121z_2^2+220z_1z_3+312z_4+16z_1^2z_3+64z_2z_3$$
$$+49z_1z_2^2+240z_5+3z_1^2z_2^2+48z_2z_4+3z_2^3+16z_3^2+4z_2^2z_3\}e^{z_1}\,,$$

(5.13) $$\sum_{k=0}^{\infty}\sum_{\kappa} C_\kappa(Z)a_1(\kappa)^4/k! = \{3z_1^2+5z_2+16z_1^3+68z_1z_2+124z_3+3z_1^4 +10z_1^2z_2+111z_2^2+216z_1z_3+312z_4+24z_1^2z_3+88z_2z_3+64z_1z_2^2 +400z_5+6z_1^2z_2^2+6z_2^3+128z_2z_4+48z_3^2+24z_2^2z_3+z_2^4\}e^{z_1}\,.$$

The formulas (5.2)-(5.5) have been proved by Sugiura and Fujikoshi [22]. The formulas (5.6)-(5.13) may be derived by using the relation $|I-(1/n)Z|^{-n}={}_1F_0(n; (1/n)Z)$ due to Constantine [3] and the following differential relations for zonal polynomials due to the author [6]:

(5.14) $$a_1(\kappa)C_\kappa(Z)=\mathrm{tr}\,(\Lambda\partial)^2C_\kappa(\Lambda)|_{\Sigma=\Lambda}\,,$$

$$\{3a_1(\kappa)^2-a_2(\kappa)+k\}C_\kappa(Z)=[3\{\mathrm{tr}\,(\Lambda\partial)^2\}^2+8\,\mathrm{tr}\,(\Lambda\partial)^3]C_\kappa(\Lambda)|_{\Sigma=\Lambda}\,,$$

where ∂ denotes the matrix of differential operators having $\{(1+\delta_{rs})/2\}\cdot\partial/\partial\sigma_{rs}$ as its (r, s) element for a symmetric matrix $\Sigma=(\sigma_{rs})$ with Kronecker's delta δ_{rs} and $\Lambda=\mathrm{dia}\,(\lambda_1, \lambda_2, \cdots, \lambda_p)$ is a diagonal matrix with p characteristic roots of Z as its non-zero elements. In Sections 2 and 4, we sometimes need the following type of summations, for example

$$\sum_{k=1}^{\infty}\sum_{\kappa} C_\kappa(Z)a_i(\kappa)/(k-1)!\,, \qquad \sum_{k=2}^{\infty}\sum_{\kappa} C_\kappa(Z)a_i(\kappa)/(k-2)!\,, \qquad \text{etc.}$$

The formulas for this type of summations are obtained by differentiating the formulas (5.2)-(5.13) replaced Z by xZ with respect to x.

Put $s_j=\mathrm{tr}\,(ZS)^j$ and let us consider the polynomials in $s_1, s_2, \cdots$ such as $s_1^{\nu_1}s_2^{\nu_2}\cdots$ with $\nu_1+2\nu_2+3\nu_3+\cdots=k$. The author [6] has obtained the formulas for $L_R[s_1^{\nu_1}s_2^{\nu_2}\cdots]$ up to 4th degree. In the following we list the formulas up to 6th degree. Let $v^{(\alpha)}(Z, S)$ $(\alpha=1, 2, \cdots, 6)$ be the following vectors with the elements such as $s_1^{\nu_1}s_2^{\nu_2}\cdots$:

(5.15) $$v^{(1)}(Z, S)'=(s_1)\,, \qquad v^{(2)}(Z, S)'=(s_1^2, s_2)\,,$$

$$v^{(3)}(Z, S)'=(s_1^3, s_1s_2, s_3)\,, \qquad v^{(4)}(Z, S)'=(s_1^4, s_1^2s_2, s_2^2, s_1s_3, s_4)\,,$$

$$v^{(5)}(Z, S)'=(s_1^5, s_1^3s_2, s_1s_2^2, s_1^2s_3, s_2s_3, s_1s_4, s_5)\,,$$

$$v^{(6)}(Z, S)'=(s_1^6, s_1^4s_2, s_1^2s_2^2, s_2^3, s_1^3s_3, s_1s_2s_3, s_3^2, s_1^2s_4, s_2s_4, s_1s_5, s_6)\,.$$

Then the following identities hold:

(5.16) $$L_R[v^{(1)}(Z, S)]=2^{-1}qv^{(1)}(Z, R^{-1})\,,$$

(5.17) $$L_R[v^{(2)}(Z, S)]=2^{-2}q\{qI_2+A_1^{(2)}\}v^{(2)}(Z, R^{-1})\,,$$

(5.18) $$L_R[v^{(3)}(Z, S)]=2^{-3}q\{q^2I_3+qA_1^{(3)}+A_2^{(3)}\}v^{(3)}(Z, R^{-1})\,,$$

(5.19) $$L_R[v^{(4)}(Z, S)]=2^{-4}q\{q^3I_5+q^2A_1^{(4)}+qA_2^{(4)}+A_3^{(4)}\}v^{(4)}(Z, R^{-1})\,,$$

(5.20) $$L_R[v^{(5)}(Z, S)]=2^{-5}q\{q^4I_7+q^3A_1^{(5)}+q^2A_2^{(5)}+qA_3^{(5)}+A_4^{(5)}\}v^{(5)}(Z, R^{-1})\,,$$

$$L_R[v^{(6)}(Z, S)]=2^{-6}q\{q^5 I_{11}+q^4 A_1^{(6)}+q^3 A_2^{(6)}+q^2 A_3^{(6)}+qA_4^{(6)}+A_5^{(6)}\} \cdot v^{(6)}(Z, R^{-1}) , \tag{5.21}$$

where the matrices $A_\beta^{(\alpha)}$ are defined as follows:

$$A_1^{(2)}=\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad A_1^{(3)}=\begin{bmatrix} 0 & 6 & 0 \\ 1 & 1 & 4 \\ 0 & 3 & 3 \end{bmatrix}, \quad A_2^{(3)}=\begin{bmatrix} 0 & 0 & 8 \\ 0 & 4 & 4 \\ 1 & 3 & 4 \end{bmatrix}, \tag{5.22}$$

$$A_1^{(4)}=\begin{bmatrix} 0 & 12 & 0 & 0 & 0 \\ 1 & 1 & 2 & 8 & 0 \\ 0 & 2 & 2 & 0 & 8 \\ 0 & 3 & 0 & 3 & 6 \\ 0 & 0 & 2 & 4 & 6 \end{bmatrix}, \quad A_2^{(4)}=\begin{bmatrix} 0 & 0 & 12 & 32 & 0 \\ 0 & 10 & 2 & 8 & 24 \\ 1 & 2 & 5 & 16 & 20 \\ 1 & 3 & 6 & 16 & 18 \\ 0 & 6 & 5 & 12 & 21 \end{bmatrix},$$

$$A_3^{(4)}=\begin{bmatrix} 0 & 0 & 0 & 0 & 48 \\ 0 & 0 & 8 & 16 & 24 \\ 0 & 8 & 4 & 16 & 20 \\ 0 & 6 & 6 & 12 & 24 \\ 1 & 6 & 5 & 16 & 20 \end{bmatrix}, \quad A_1^{(5)}=\begin{bmatrix} 0 & 20 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6 & 12 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 8 & 8 & 0 \\ 0 & 3 & 0 & 3 & 2 & 12 & 0 \\ 0 & 0 & 3 & 1 & 4 & 0 & 12 \\ 0 & 0 & 2 & 4 & 0 & 6 & 8 \\ 0 & 0 & 0 & 0 & 5 & 5 & 10 \end{bmatrix},$$

$$A_2^{(5)}=\begin{bmatrix} 0 & 0 & 60 & 80 & 0 & 0 & 0 \\ 0 & 18 & 6 & 12 & 32 & 72 & 0 \\ 1 & 2 & 13 & 24 & 16 & 20 & 64 \\ 1 & 3 & 18 & 28 & 6 & 36 & 48 \\ 0 & 4 & 6 & 3 & 31 & 36 & 60 \\ 0 & 6 & 5 & 12 & 24 & 45 & 48 \\ 0 & 0 & 10 & 10 & 25 & 30 & 65 \end{bmatrix},$$

$$A_3^{(5)}=\begin{bmatrix} 0 & 0 & 0 & 0 & 160 & 240 & 0 \\ 0 & 0 & 48 & 56 & 32 & 72 & 192 \\ 0 & 16 & 12 & 24 & 72 & 116 & 160 \\ 0 & 14 & 18 & 24 & 80 & 120 & 144 \\ 1 & 4 & 27 & 40 & 64 & 96 & 168 \\ 1 & 6 & 29 & 40 & 64 & 92 & 168 \\ 0 & 10 & 25 & 30 & 70 & 105 & 160 \end{bmatrix},$$

$$A_4^{(5)}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 384 \\ 0 & 0 & 0 & 0 & 96 & 96 & 192 \\ 0 & 0 & 32 & 32 & 64 & 96 & 160 \\ 0 & 0 & 24 & 24 & 72 & 72 & 192 \\ 0 & 12 & 24 & 36 & 60 & 108 & 144 \\ 0 & 8 & 24 & 24 & 72 & 96 & 160 \\ 1 & 10 & 25 & 40 & 60 & 100 & 148 \end{bmatrix},$$

$$A_1^{(6)}=\begin{bmatrix}
0 & 30 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 12 & 0 & 16 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 2 & 2 & 0 & 16 & 0 & 8 & 0 & 0 & 0\\
0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 24 & 0 & 0\\
0 & 3 & 0 & 0 & 3 & 6 & 0 & 18 & 0 & 0 & 0\\
0 & 0 & 3 & 0 & 1 & 4 & 4 & 0 & 6 & 12 & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 18\\
0 & 0 & 2 & 0 & 4 & 0 & 0 & 6 & 2 & 16 & 0\\
0 & 0 & 0 & 2 & 0 & 4 & 0 & 1 & 7 & 0 & 16\\
0 & 0 & 0 & 0 & 0 & 5 & 0 & 5 & 0 & 10 & 10\\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 6 & 6 & 15
\end{bmatrix},$$

$$A_2^{(6)}=\begin{bmatrix}
0 & 0 & 180 & 0 & 160 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 28 & 12 & 12 & 16 & 128 & 0 & 144 & 0 & 0 & 0\\
1 & 2 & 25 & 4 & 32 & 32 & 32 & 20 & 64 & 128 & 0\\
0 & 3 & 6 & 15 & 0 & 48 & 0 & 24 & 84 & 0 & 160\\
1 & 3 & 36 & 0 & 40 & 18 & 8 & 54 & 36 & 144 & 0\\
0 & 4 & 6 & 6 & 3 & 59 & 16 & 42 & 24 & 60 & 120\\
0 & 0 & 9 & 0 & 2 & 24 & 35 & 0 & 54 & 72 & 144\\
0 & 6 & 5 & 4 & 12 & 56 & 0 & 69 & 12 & 96 & 80\\
0 & 0 & 8 & 7 & 4 & 16 & 24 & 6 & 75 & 64 & 136\\
0 & 0 & 10 & 0 & 10 & 25 & 20 & 30 & 40 & 105 & 100\\
0 & 0 & 0 & 5 & 0 & 30 & 24 & 15 & 51 & 60 & 155
\end{bmatrix},$$

$$A_3^{(6)}=\begin{bmatrix}
0 & 0 & 0 & 120 & 0 & 960 & 0 & 720 & 0 & 0 & 0\\
0 & 0 & 156 & 12 & 128 & 128 & 128 & 144 & 336 & 768 & 0\\
0 & 26 & 24 & 26 & 32 & 272 & 64 & 260 & 136 & 320 & 640\\
1 & 3 & 39 & 25 & 48 & 96 & 128 & 60 & 312 & 384 & 704\\
0 & 24 & 36 & 36 & 36 & 336 & 24 & 288 & 108 & 432 & 480\\
1 & 4 & 51 & 12 & 56 & 124 & 124 & 114 & 258 & 456 & 600\\
0 & 6 & 18 & 24 & 6 & 186 & 114 & 108 & 252 & 360 & 726\\
1 & 6 & 65 & 10 & 64 & 152 & 96 & 164 & 234 & 528 & 480\\
0 & 7 & 17 & 26 & 12 & 172 & 112 & 117 & 265 & 336 & 736\\
0 & 10 & 25 & 20 & 30 & 190 & 100 & 165 & 210 & 400 & 650\\
0 & 0 & 30 & 22 & 20 & 150 & 121 & 90 & 276 & 390 & 701
\end{bmatrix},$$

$$A_4^{(6)}=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 640 & 0 & 1440 & 2304 & 0\\
0 & 0 & 0 & 96 & 0 & 704 & 128 & 432 & 336 & 768 & 1920\\
0 & 0 & 128 & 24 & 96 & 256 & 288 & 240 & 664 & 1088 & 1600\\
0 & 24 & 36 & 44 & 48 & 432 & 288 & 348 & 540 & 960 & 1664\\
0 & 0 & 108 & 36 & 80 & 312 & 320 & 216 & 720 & 1152 & 1440\\
0 & 22 & 48 & 54 & 52 & 436 & 256 & 348 & 576 & 912 & 1680\\
1 & 6 & 81 & 54 & 80 & 384 & 268 & 288 & 630 & 1008 & 1584\\
0 & 18 & 60 & 58 & 48 & 464 & 256 & 336 & 568 & 896 & 1680\\
1 & 7 & 83 & 45 & 80 & 384 & 280 & 284 & 604 & 1008 & 1608\\
1 & 10 & 85 & 50 & 80 & 380 & 280 & 280 & 630 & 988 & 1600\\
0 & 15 & 75 & 52 & 60 & 420 & 264 & 315 & 603 & 960 & 1620
\end{bmatrix},$$

$$A_5^{(6)}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3840 \\ 0 & 0 & 0 & 0 & 0 & 0 & 384 & 0 & 768 & 768 & 1920 \\ 0 & 0 & 0 & 64 & 0 & 384 & 256 & 192 & 576 & 768 & 1600 \\ 0 & 0 & 96 & 32 & 64 & 384 & 224 & 288 & 480 & 960 & 1312 \\ 0 & 0 & 0 & 48 & 0 & 288 & 288 & 144 & 576 & 576 & 1920 \\ 0 & 0 & 72 & 48 & 48 & 336 & 240 & 216 & 576 & 864 & 1440 \\ 0 & 18 & 72 & 42 & 72 & 360 & 216 & 324 & 504 & 864 & 1368 \\ 0 & 0 & 48 & 48 & 32 & 288 & 288 & 144 & 624 & 768 & 1600 \\ 0 & 16 & 72 & 40 & 64 & 384 & 224 & 312 & 488 & 896 & 1344 \\ 0 & 10 & 60 & 50 & 40 & 360 & 240 & 240 & 560 & 800 & 1480 \\ 1 & 15 & 75 & 41 & 80 & 360 & 228 & 300 & 504 & 888 & 1348 \end{bmatrix}.$$

The formulas for $k=7$ and 8 have been also obtained by using the tables of zonal polynomials due to Kitchen [13] but because of its complexity it is not given here.

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