ASYMPTOTIC EXPANSIONS OF THE DISTRIBUTIONS OF BARTLETT'S TEST AND SPHERICITY TEST UNDER THE LOCAL ALTERNATIVES

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1. Introduction and summary

The purpose of this paper is to give the asymptotic expansions of the distributions of Bartlett's test criterion (M test) due to Bartlett [2] for testing homogeneity of variances and of sphericity test criterion (see Anderson [1], p. 262) under the local alternative sequences. In the previous paper [12], the M test was compared with Lehmann's test (L test) due to Lehmann [5] by giving the asymptotic expansions of the distributions of the M test and L test under the fixed alternative. However under the null hypothesis, the variances of the two limiting distributions of the M and L tests vanish. Hence we can not compute the power when the alternative hypothesis is near to the null hypothesis. Similarly the asymptotic expansion of the distribution of the sphericity test under the fixed alternative given by Sugiura [10] can not evaluate the power under the local alternative. So Nagao [6] has treated the asymptotic expansions of the M test and sphericity test under the local alternatives. However the author [6] could not obtain a good approximation for power under the local alternative since the order of these formulas is only up to $n^{-1/2}$.

In this paper, along the previous paper [6], the asymptotic expansions for the local alternatives of the M test and sphericity test are given up to the terms of order n^{-1}.

2. Expression of the characteristic function of M

Let $X_{i1}, X_{i2}, \cdots, X_{iN_i}$ be a random sample from a normal distribution with mean μ_i and variance σ_i^2 $(i=1, 2, \cdots, k)$. For testing the hypothesis H: $\sigma_1^2=\sigma_2^2=\cdots=\sigma_k^2$ against all alternatives K: $\sigma_i^2\neq\sigma_j^2$ for some i and j $(i\neq j)$ with unspecified μ_i, the M test criterion due to Bartlett [2] is given by

$$M=n\log\left(\sum_{\alpha=1}^{k} S_\alpha/n\right)-\sum_{\alpha=1}^{k} n_\alpha\log(S_\alpha/n_\alpha)\ , \tag{2.1}$$

where $S_\alpha=\sum_{\beta=1}^{N_\alpha}(X_{\alpha\beta}-\bar{X}_\alpha)^2$ with $\bar{X}_\alpha=N_\alpha^{-1}\sum_{\beta=1}^{N_\alpha}X_{\alpha\beta}$ and $n_\alpha=N_\alpha-1$ with $n=\sum_{\alpha=1}^{k}n_\alpha$. The M test rejects the hypothesis H, when the observed value of M is larger than a preassigned constant. The M test is equivarent to the modified likelihood ratio test known to be unbiased (Pitman [8]).

Let $c=1+o(1)$, then the statistic cM is expressed as follows:

$$cM=m\log\left(\sum_{\alpha=1}^{k}S_\alpha/m\right)-\sum_{\alpha=1}^{k}m_\alpha\log(S_\alpha/m_\alpha)\,, \tag{2.2}$$

where $m_\alpha=cn_\alpha$ and $\sum_{\alpha=1}^{k}m_\alpha=m$.

We put $Y_\alpha=\sqrt{m_\alpha/2}\{\log(S_\alpha/m_\alpha)-\log\sigma_\alpha^2\}$, which has asymptotically normal distribution with mean zero and variance 1. Since we may assume $\sigma_1^2=1$ under the alternatives without loss of generality, we can express the statistic cM in (2.2) in terms of $Y_1, Y_2,\cdots, Y_k$ under $\sigma_\alpha^2=1+m^{-1/2}\theta_\alpha$ and fixed $\rho_\alpha=m_\alpha/m$ $(\alpha=1, 2,\cdots, k)$ as follows:

$$cM=q_0(Y)+m^{-1/2}q_1(Y)+m^{-1}q_2(Y)+O_p(m^{-3/2})\,, \tag{2.3}$$

where

$$q_0(Y)=\sum_{\alpha=1}^{k}\left(Y_\alpha+\sqrt{\frac{\rho_\alpha}{2}}\theta_\alpha\right)^2-\left\{\sum_{\alpha=1}^{k}\left(\sqrt{\rho_\alpha}\,Y_\alpha+\frac{1}{\sqrt{2}}\rho_\alpha\theta_\alpha\right)\right\}^2, \tag{2.4}$$

$$\begin{aligned}q_1(Y)=&-\frac{1}{3}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^3+\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^2+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}Y_\alpha^3/\sqrt{\rho_\alpha}\\&-\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\sum_{\alpha=1}^{k}(Y_\alpha^2+\sqrt{2\rho_\alpha}\,\theta_\alpha Y_\alpha)\\&+\frac{1}{3}\left\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\right\}^3,\end{aligned}$$

$$\begin{aligned}q_2(Y)=&\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^4+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^3/\sqrt{\rho_\alpha}+\frac{1}{6}\sum_{\alpha=1}^{k}Y_\alpha^4/\rho_\alpha\\&-\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\left(\sum_{\alpha=1}^{k}\theta_\alpha Y_\alpha^2+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}Y_\alpha^3/\sqrt{\rho_\alpha}\right)\\&+\left\{\sum_{\alpha=1}^{k}(\sqrt{2\rho_\alpha}\,Y_\alpha+\rho_\alpha\theta_\alpha)\right\}^2\left(\sqrt{2}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Y_\alpha+\sum_{\alpha=1}^{k}Y_\alpha^2\right)\\&-\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha Y_\alpha+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}Y_\alpha^2\right)^2-\left\{\sum_{\alpha=1}^{k}\left(\sqrt{\rho_\alpha}\,Y_\alpha+\frac{1}{\sqrt{2}}\rho_\alpha\theta_\alpha\right)\right\}^4.\end{aligned}$$

Hence the characteristic function of cM is expressed as

$$C_M(t)=\mathrm{E}\left[\exp(itq_0(Y))\left\{1+m^{-1/2}(it)q_1(Y)+m^{-1}\right.\right. \tag{2.5}$$

$$\cdot\left[(it)q_2(Y)+\frac{(it)^2}{2}q_1(Y)^2\right]\Big\}\Big]+O(m^{-3/2})\,.$$

In order to calculate (2.5), we require the distribution of $z_\alpha=\sqrt{m_\alpha/2}\cdot\log(S_\alpha/m_\alpha)$, which is given by the previous paper [6] as

$$c_{m_\alpha}\sigma_\alpha^{-m_\alpha-2\Delta_\alpha}\exp\left[\sqrt{\frac{m_\alpha}{2}}z_\alpha+\Delta_\alpha\sqrt{\frac{2}{m_\alpha}}z_\alpha-\frac{m_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{m_\alpha}}z_\alpha\right], \quad -\infty<z_\alpha<\infty\,, \tag{2.6}$$

where $c_{m_\alpha}=(m_\alpha/2)^{(m_\alpha-1)/2+\Delta_\alpha}\{\Gamma[m_\alpha/2+\Delta_\alpha]\}^{-1}$ and $\Delta_\alpha=(n_\alpha-m_\alpha)/2$. Expressing $q_0(Y)$, $q_1(Y)$ and $q_2(Y)$ in terms of z_α under $\sigma_\alpha^2=1+m^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$, we have

$$\begin{aligned}q_0'(z)=&\sum_{\alpha=1}^k z_\alpha^2-\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^2+\frac{m^{-1/2}}{\sqrt{2}}\Big\{\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha-\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big\}\\&+m^{-1}\Big\{\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^3\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha-\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^3 z_\alpha\\&+\frac{1}{8}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^4-\frac{1}{8}\Big(\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big)^2\Big\}+O(m^{-3/2})\,,\end{aligned}\tag{2.7}$$

$$\begin{aligned}q_1'(z)=&\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k z_\alpha^3/\sqrt{\rho_\alpha}+\frac{2\sqrt{2}}{3}\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^3-\sqrt{2}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k z_\alpha^2\\&-\frac{1}{2}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\\&+m^{-1/2}\Big\{\frac{1}{2}\sum_{\alpha=1}^k\theta_\alpha^2 z_\alpha^2-\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha\\&-\frac{1}{2}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k z_\alpha^2+\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^2\\&-\frac{1}{4}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^4+\frac{1}{4}\Big(\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big)^2\Big\}+O(m^{-1})\,,\end{aligned}$$

$$\begin{aligned}q_2'(z)=&\frac{1}{6}\sum_{\alpha=1}^k z_\alpha^4/\rho_\alpha-\frac{1}{2}\Big(\sum_{\alpha=1}^k z_\alpha^2\Big)^2-\frac{2}{3}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k z_\alpha^3/\sqrt{\rho_\alpha}\\&+2\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^2\sum_{\alpha=1}^k z_\alpha^2-\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^4-\frac{1}{2}\sum_{\alpha=1}^k\theta_\alpha^2 z_\alpha^2\\&+\frac{1}{2}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k z_\alpha^2+\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha\\&-\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^2-\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^3\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\\&+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^3 z_\alpha+\frac{1}{8}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^4-\frac{1}{8}\Big(\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big)^2+O(m^{-1/2})\,.\end{aligned}$$

Hence the expected value $\mathrm{E}[\exp(itq_0(Y))]$ is given by

$$\text{(2.8)}\qquad \mathrm{E}[\exp(itq_0(Y))]=\left\{\prod_{\alpha=1}^{k} c_{m_\alpha}\sigma_\alpha^{-m_\alpha-2\varDelta_\alpha}\right\}\int \exp[(it)q_0'(z)]$$
$$\cdot\exp\left[\sum_{\alpha=1}^{k}\sqrt{\frac{m_\alpha}{2}}z_\alpha+\sum_{\alpha=1}^{k}\varDelta_\alpha\sqrt{\frac{2}{m_\alpha}}z_\alpha\right.$$
$$\left.-\sum_{\alpha=1}^{k}\frac{m_\alpha}{2\sigma_\alpha^2}\exp\sqrt{\frac{2}{m_\alpha}}z_\alpha\right]dz_1\cdots dz_k\,.$$

Expanding the last term of the second exponential part in the above integrand, we can rewrite (2.8) as $\mathrm{E}[\exp(itq_0(Y))]=E_1(t)E_2^{(1)}(t)$. The above two terms $E_1(t)$ and $E_2^{(1)}(t)$ are given by

$$\text{(2.9)}\qquad E_1(t)=\left\{\prod_{\alpha=1}^{k} c_{m_\alpha}\sigma_\alpha^{-m_\alpha-2\varDelta_\alpha}\exp\left[-\frac{m_\alpha}{2\sigma_\alpha^2}\right]\right\}\exp\left[\frac{1}{4}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\right](2\pi)^{k/2}$$
$$\cdot|\varSigma|^{1/2}\exp\left[\frac{it}{2(1-2it)}\sum_{\alpha=1}^{k}\rho_\alpha(\theta_\alpha-\tilde{\theta})^2\right]$$

and

$$\text{(2.10)}\qquad E_2^{(1)}(t)=\mathrm{E}\left\{1+m^{-1/2}\gamma_0(z)+m^{-1}\left(\gamma_1(z)+\frac{1}{2}\gamma_0(z)^2\right)\right\}+O(m^{-3/2})\,,$$

where $\tilde{\theta}=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha$, $\varDelta=\sum_{\alpha=1}^{k}\varDelta_\alpha$ and

$$\text{(2.11)}\qquad \gamma_0(z)=\frac{it}{\sqrt{2}}\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha-\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\right)-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}z_\alpha^3/\sqrt{\rho_\alpha}$$
$$+\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha z_\alpha^2-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha+\sqrt{2}\,\varDelta\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\,,$$

$$\text{(2.12)}\qquad \gamma_1(z)=it\left\{-\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^3 z_\alpha+\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^3\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\right.$$
$$\left.+\frac{1}{8}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^4-\frac{1}{8}\left(\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\right)^2\right\}-\frac{1}{12}\sum_{\alpha=1}^{k}z_\alpha^4/\rho_\alpha$$
$$+\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^{k}\theta_\alpha z_\alpha^3/\sqrt{\rho_\alpha}-\frac{1}{2}\sum_{\alpha=1}^{k}\theta_\alpha^2 z_\alpha^2+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^3 z_\alpha\,.$$

The symbol E in (2.10) denotes the expectation of $z=(z_1,z_2,\cdots,z_k)'$ with respect to the k-variate normal distribution having mean $\eta=(\eta_1,\eta_2,\cdots,\eta_k)'$ and covariance matrix $\varSigma=(\sigma_{\alpha\beta})$ with $\eta_\alpha=\sqrt{\rho_\alpha/2}\,(\theta_\alpha-2it\tilde{\theta})(1-2it)^{-1}$ and $\sigma_{\alpha\beta}=(\delta_{\alpha\beta}-2it\sqrt{\rho_\alpha\rho_\beta})(1-2it)^{-1}$. Applying stirling's formula to $E_1(t)$, we have the following asymptotic formula:

$$\text{(2.13)}\qquad E_1(t)=(1-2it)^{-f/2}\exp\left[\frac{it}{2(1-2it)}\nu_2\right]$$

$$\cdot\left[1+m^{-1/2}\left(\frac{1}{3}\nu_3+\tilde{\theta}\nu_2+\frac{1}{3}\tilde{\theta}^3-\Delta\tilde{\theta}\right)+m^{-1}\left\{-\frac{3}{8}\nu_4+\frac{1}{18}\nu_3^2\right.\right.$$
$$+\frac{1}{3}\tilde{\theta}\nu_2\nu_3+\frac{1}{9}\tilde{\theta}^3\nu_3-\frac{1}{6}(2\Delta+9)\tilde{\theta}\nu_3+\frac{1}{2}\tilde{\theta}^2\nu_2^2+\frac{1}{3}\tilde{\theta}^4\nu_2$$
$$-\frac{1}{4}(4\Delta+9)\tilde{\theta}^2\nu_2+\frac{\Delta}{2}\nu_2+\frac{1}{18}\tilde{\theta}^6-\frac{1}{24}(8\Delta+9)\tilde{\theta}^4$$
$$\left.\left.+\frac{1}{2}(\Delta^2+\Delta)\tilde{\theta}-\Delta^2+k\Delta-\frac{1}{6}\tilde{\rho}\right\}+O(m^{-3/2})\right],$$

where $\nu_\alpha=\sum_{\beta=1}^{k}\rho_\beta(\theta_\beta-\tilde{\theta})^\alpha$, $\tilde{\rho}=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}$ and $f=k-1$. Also by Appendix and the previous paper ([6], pp. 164–167 and p. 174), we can compute the term $E_2^{(1)}(t)$ as follows:

$$E_2^{(1)}(t)=1+m^{-1/2}\left\{-\frac{1}{12}\nu_3(t)_3+\left(\frac{1}{4}\nu_3-\frac{1}{2}\zeta_1\right)(t)_2+\left(-\frac{1}{4}\nu_3+\frac{1}{2}\zeta_1\right)(t)_1\right.$$
$$\left.-\frac{1}{4}\nu_3-\tilde{\theta}\nu_2-\frac{1}{3}\tilde{\theta}^3+\Delta\tilde{\theta}\right\}+m^{-1}\sum_{\alpha=0}^{6}h_{2\alpha}^{(1)}(t)_\alpha+O(m^{-3/2}), \tag{2.14}$$

where $\zeta_\alpha=\sum_{\beta=1}^{k}(\theta_\beta-\tilde{\theta})^\alpha$, $(t)_\alpha=(1-2it)^{-\alpha}$ and

$$h_{12}^{(1)}=\frac{1}{288}\nu_3^2,\qquad h_{10}^{(1)}=\frac{1}{16}\nu_4-\frac{1}{48}\nu_3^2+\frac{1}{24}\zeta_1\nu_3-\frac{1}{16}\nu_2^2, \tag{2.15}$$

$$h_8^{(1)}=-\frac{13}{48}\nu_4+\frac{5}{96}\nu_3^2-\frac{1}{6}\zeta_1\nu_3+\frac{5}{16}\nu_2^2-\frac{1}{4}(k+2)\nu_2+\frac{1}{2}\zeta_2+\frac{1}{8}\zeta_1^2,$$

$$h_6^{(1)}=\frac{11}{24}\nu_4-\frac{1}{24}\nu_3^2+\frac{1}{12}\tilde{\theta}\nu_2\nu_3+\frac{1}{36}\tilde{\theta}^3\nu_3-\frac{\Delta}{12}\tilde{\theta}\nu_3+\frac{1}{4}\zeta_1\nu_3-\frac{5}{8}\nu_2^2$$
$$+\frac{3}{4}(k+2)\nu_2-\frac{5}{4}\zeta_2-\frac{1}{4}\zeta_1^2+\frac{5}{12}\tilde{\rho}-\frac{1}{4}k^2-\frac{1}{2}k+\frac{1}{3},$$

$$h_4^{(1)}=-\frac{3}{8}\nu_4-\frac{1}{32}\nu_3^2-\frac{1}{4}\tilde{\theta}\nu_2\nu_3-\frac{1}{12}\tilde{\theta}^3\nu_3+\frac{\Delta}{4}\tilde{\theta}\nu_3+\frac{5}{8}\nu_2^2+\frac{1}{2}\tilde{\theta}\zeta_1\nu_2$$
$$-\frac{1}{4}(3k+6+2\Delta)\nu_2+\zeta_2+\frac{1}{8}\zeta_1^2+\frac{1}{6}\tilde{\theta}^3\zeta_1-\frac{\Delta}{2}\tilde{\theta}\zeta_1-\frac{1}{4}\tilde{\rho}$$
$$+\frac{1}{4}k^2+\frac{1}{2}k-\frac{1}{2},$$

$$h_2^{(1)}=\frac{7}{48}\nu_4+\frac{1}{16}\nu_3^2+\frac{1}{4}\tilde{\theta}\nu_2\nu_3+\frac{1}{12}\tilde{\theta}^3\nu_3-\frac{\Delta}{4}\tilde{\theta}\nu_3-\frac{1}{8}\zeta_1\nu_3-\frac{5}{16}\nu_2^2$$
$$-\frac{1}{12}\tilde{\theta}\zeta_1\nu_2+\frac{1}{4}(k+4\Delta+2)\nu_2-\frac{1}{4}\zeta_2-\frac{1}{6}\tilde{\theta}^3\zeta_1+\frac{\Delta}{2}\tilde{\theta}\zeta_1-\Delta(k-1),$$

$$h_0^{(1)}=\frac{11}{48}\nu_4+\frac{1}{32}\nu_3^2+\frac{1}{4}\tilde{\theta}\nu_2\nu_3+\frac{1}{12}\tilde{\theta}^3\nu_3-\frac{1}{4}(\Delta-6)\tilde{\theta}\nu_3+\frac{1}{2}\tilde{\theta}^2\nu_2^2$$
$$+\frac{1}{3}\tilde{\theta}^4\nu_2+\frac{1}{16}\nu_2^2-\frac{1}{4}(4\Delta-9)\tilde{\theta}^2\nu_2-\Delta\nu_2+\frac{1}{18}\tilde{\theta}^6-\frac{1}{24}(8\Delta-9)\tilde{\theta}^4$$
$$+\frac{1}{2}(\Delta^2-\Delta)\tilde{\theta}^2+\Delta^2-\Delta+\frac{1}{6}\,.$$

Similarly the second term in (2.5) is expressed in terms of z_α ($\alpha=1,2,\cdots,k$) as follows: $\mathrm{E}\,[(it)q_1(Y)\exp(itq_0(Y))]=E_1(t)E_2^{(2)}(t)$ where

$$(2.16)\quad E_2^{(2)}(t)=(it)\,\mathrm{E}\Bigg[\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k z_\alpha^3/\sqrt{\rho_\alpha}-\sqrt{2}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k z_\alpha^2$$
$$+\frac{2\sqrt{2}}{3}\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^3-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2$$
$$\cdot\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha+m^{-1/2}\Big\{\frac{1}{2}\sum_{\alpha=1}^k\theta_\alpha^2 z_\alpha^2-\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha$$
$$-\frac{1}{2}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k z_\alpha^2+\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^2$$
$$-\frac{1}{4}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^4+\frac{1}{4}\Big(\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\Big)^2+\Big(\frac{it}{\sqrt{2}}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha$$
$$-\frac{it}{\sqrt{2}}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha-\frac{1}{3\sqrt{2}}\sum_{\alpha=1}^k z_\alpha^3/\sqrt{\rho_\alpha}+\frac{1}{2}\sum_{\alpha=1}^k\theta_\alpha z_\alpha^2$$
$$-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha+\sqrt{2}\,\Delta\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)\Big(\frac{\sqrt{2}}{3}\sum_{\alpha=1}^k z_\alpha^3/\sqrt{\rho_\alpha}$$
$$-\sqrt{2}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^k z_\alpha^2+\frac{2\sqrt{2}}{3}\Big(\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)^3$$
$$-\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^k\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^k\sqrt{\rho_\alpha}\,z_\alpha\Big)\Big\}\Bigg]+O(m^{-1})\,.$$

Applying Appendix to $E_2^{(2)}(t)$, we have

$$(2.17)\quad E_2^{(2)}(t)=\frac{1}{12}\nu_3(t)_3+\Big(-\frac{1}{12}\nu_3+\frac{1}{2}\zeta_1\Big)(t)_2+\Big(-\frac{1}{4}\nu_3-\frac{1}{2}\tilde{\theta}\nu_2-\frac{1}{2}\zeta_1\Big)(t)_1$$
$$+\frac{1}{4}\nu_3+\frac{1}{2}\tilde{\theta}\nu_2+m^{-1/2}\sum_{\alpha=0}^{6}h_{2\alpha}^{(2)}(t)_\alpha+O(m^{-1})\,,$$

where

$$(2.18)\quad h_{12}^{(2)}=-\frac{1}{144}\nu_3^2\,,\qquad h_{10}^{(2)}=-\frac{1}{8}\nu_4+\frac{1}{36}\nu_3^2-\frac{1}{12}\zeta_1\nu_3+\frac{1}{8}\nu_2^2\,,$$

$$h_8^{(2)}=\frac{3}{8}\nu_4-\frac{1}{48}\nu_3^2+\frac{1}{24}\tilde{\theta}\nu_2\nu_3+\frac{1}{4}\zeta_1\nu_3-\frac{3}{8}\nu_2^2+\frac{1}{2}(k+2)\nu_2-\zeta_2-\frac{1}{4}\zeta_1^2\,,$$

$$h_6^{(2)} = -\frac{1}{4}\nu_4 - \frac{1}{12}\nu_3^2 - \frac{1}{4}\tilde{\theta}\nu_2\nu_3 - \frac{1}{36}\tilde{\theta}^3\nu_3 + \frac{1}{12}(\Delta+3)\tilde{\theta}\nu_3 - \frac{1}{6}\zeta_1\nu_3$$
$$+\frac{1}{4}\nu_2^2 + \frac{1}{4}\tilde{\theta}\zeta_1\nu_2 - (k+2)\nu_2 + 2\zeta_2 + \frac{1}{2}\zeta_1^2 - \frac{5}{6}\tilde{\rho} + \frac{k^2}{2} + k - \frac{2}{3},$$

$$h_4^{(2)} = -\frac{1}{4}\nu_4 + \frac{7}{48}\nu_3^2 + \frac{1}{3}\tilde{\theta}\nu_2\nu_3 + \frac{1}{36}\tilde{\theta}^3\nu_3 - \frac{1}{12}(\Delta+9)\tilde{\theta}\nu_3 - \frac{1}{4}\zeta_1\nu_3$$
$$+\frac{1}{4}\nu_2^2 - \tilde{\theta}\zeta_1\nu_2 + \frac{1}{2}(k+2)\nu_2 - \zeta_2 - \frac{1}{4}\zeta_1^2 - \frac{1}{6}\tilde{\theta}^3\zeta_1$$
$$+\frac{1}{2}(\Delta+1)\tilde{\theta}\zeta_1 + \frac{5}{6}\tilde{\rho} - \frac{1}{2}k^2 - k + \frac{2}{3},$$

$$h_2^{(2)} = \frac{3}{8}\nu_4 + \frac{1}{4}\tilde{\theta}\nu_2\nu_3 - \frac{1}{4}(\Delta-3)\tilde{\theta}\nu_3 + \frac{1}{12}\tilde{\theta}^3\nu_3 + \frac{1}{4}\zeta_1\nu_3 + \frac{1}{2}\tilde{\theta}^2\nu_2^2$$
$$-\frac{3}{8}\nu_2^2 + \frac{1}{6}\tilde{\theta}^4\nu_2 - \frac{\Delta}{2}\tilde{\theta}^2\nu_2 + \frac{3}{4}\tilde{\theta}\zeta_1\nu_2 + \frac{1}{6}\tilde{\theta}^3\zeta_1 - \frac{1}{2}(\Delta+1)\tilde{\theta}\zeta_1,$$

$$h_0^{(2)} = -\frac{1}{8}\nu_4 - \frac{1}{16}\nu_3^2 - \frac{3}{8}\tilde{\theta}\nu_2\nu_3 - \frac{1}{12}\tilde{\theta}^3\nu_3 + \frac{1}{4}(\Delta-1)\tilde{\theta}\nu_3 - \frac{1}{2}\theta^2\nu_2^2$$
$$+\frac{1}{8}\nu_2^2 - \frac{1}{6}\tilde{\theta}^4\nu_2 + \frac{\Delta}{2}\tilde{\theta}^2\nu_2 .$$

By the same argument, the third term in (2.5) is given by

$$\text{(2.19)} \qquad \mathrm{E}\left[\left\{(it)q_2(Y) + \frac{(it)^2}{2}q_1(Y)^2\right\}\exp\left(itq_0(Y)\right)\right] = E_1(t)E_2^{(3)}(t),$$

where

$$\text{(2.20)} \quad E_2^{(3)}(t) = \mathrm{E}\left\{(it)q_2'(z) + \frac{(it)^2}{2}\left[\frac{\sqrt{2}}{3}\sum_{\alpha=1}^{k} z_\alpha^3/\sqrt{\rho_\alpha} + \frac{2\sqrt{2}}{3}\left(\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\right)^3\right.\right.$$
$$-\sqrt{2}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\sum_{\alpha=1}^{k}z_\alpha^2 - \frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha$$
$$\left.\left.+\frac{1}{\sqrt{2}}\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha^2\sum_{\alpha=1}^{k}\sqrt{\rho_\alpha}\,z_\alpha\right]^2\right\} + O(m^{-1/2}).$$

Thus for the value $E_2^{(3)}(t)$, we obtain

$$\text{(2.21)} \qquad E_2^{(3)}(t) = \sum_{\alpha=0}^{6} h_{2\alpha}^{(3)}(t)_\alpha + O(m^{-1/2}),$$

where

$$\text{(2.22)} \quad h_{12}^{(3)} = \frac{1}{288}\nu_3^2, \qquad h_{10}^{(3)} = \frac{1}{16}\nu_4 - \frac{1}{144}\nu_3^2 + \frac{1}{24}\zeta_1\nu_3 - \frac{1}{16}\nu_2^2,$$

$$h_8^{(3)}=-\frac{5}{48}\nu_4-\frac{5}{288}\nu_3^2-\frac{1}{24}\tilde{\theta}\nu_2\nu_3-\frac{1}{12}\zeta_1\nu_3+\frac{1}{16}\nu_2^2-\frac{1}{4}(k+2)\nu_2$$
$$+\frac{1}{2}\zeta_2+\frac{1}{8}\zeta_1^2\,,$$

$$h_6^{(3)}=-\frac{1}{12}\nu_4+\frac{1}{24}\nu_3^2+\frac{1}{12}\tilde{\theta}\nu_2\nu_3-\frac{1}{4}\tilde{\theta}\nu_3-\frac{1}{12}\zeta_1\nu_3+\frac{1}{8}\nu_2^2-\frac{1}{4}\tilde{\theta}\zeta_1\nu_2$$
$$+\frac{1}{4}(k+2)\nu_2-\frac{3}{8}\zeta_2-\frac{1}{4}\zeta_1^2+\frac{5}{12}\tilde{\rho}-\frac{1}{4}k^2-\frac{1}{2}k+\frac{1}{3}\,,$$

$$h_4^{(3)}=\frac{1}{8}\nu_4+\frac{1}{96}\nu_3^2+\frac{1}{12}\tilde{\theta}\nu_2\nu_3+\frac{1}{4}\tilde{\theta}\nu_3+\frac{1}{4}\zeta_1\nu_3+\frac{1}{8}\tilde{\theta}^2\nu_2^2-\frac{1}{8}\nu_2^2$$
$$+\frac{1}{2}\tilde{\theta}\zeta_1\nu_2+\frac{k}{4}\nu_2+\frac{1}{8}\zeta_1^2-\frac{1}{2}\tilde{\theta}\zeta_1-\frac{7}{12}\tilde{\rho}+\frac{1}{4}k^2+\frac{1}{2}k-\frac{1}{6}\,,$$

$$h_2^{(3)}=\frac{11}{48}\nu_4-\frac{1}{16}\nu_3^2-\frac{1}{4}\tilde{\theta}\nu_2\nu_3+\frac{3}{4}\tilde{\theta}\nu_3-\frac{1}{8}\zeta_1\nu_3-\frac{1}{4}\tilde{\theta}^2\nu_2^2-\frac{1}{16}\nu_2^2$$
$$+\frac{3}{4}\tilde{\theta}^2\nu_2-\frac{1}{4}\tilde{\theta}\zeta_1\nu_2-\frac{k}{4}\nu_2+\frac{1}{4}\zeta_2+\frac{1}{2}\tilde{\theta}\zeta_1+\frac{1}{6}\tilde{\rho}-\frac{1}{6}\,,$$

$$h_0^{(3)}=-\frac{11}{48}\nu_4+\frac{1}{32}\nu_3^2+\frac{1}{8}\tilde{\theta}\nu_2\nu_3-\frac{3}{4}\tilde{\theta}\nu_3+\frac{1}{8}\tilde{\theta}^2\nu_2^2+\frac{1}{16}\nu_2^2-\frac{3}{4}\tilde{\theta}^2\nu_2\,.$$

Hence the characteristic function (2.5) can be calculated as follows:

$$(2.23)\quad C_M(t)=(1-2it)^{-f/2}\exp\left[\frac{it}{2(1-2it)}\nu_2\right]\left[1+m^{-1/2}\left\{\frac{1}{6}\nu_3(t)_2+\left(-\frac{1}{2}\nu_3\right.\right.\right.$$
$$\left.\left.\left.-\frac{1}{2}\tilde{\theta}\nu_2\right)(t)_1+\frac{1}{3}\nu_3+\frac{1}{2}\tilde{\theta}\nu_2\right\}+m^{-1}\sum_{\alpha=0}^{4}h'_{2\alpha}(t)_\alpha\right]+O(m^{-3/2})\,,$$

where

$$(2.24)\quad h'_8=\frac{1}{72}\nu_3^2\,,\qquad h'_6=\frac{1}{8}\nu_4-\frac{1}{12}\nu_3^2-\frac{1}{12}\tilde{\theta}\nu_2\nu_3-\frac{1}{4}\nu_2^2\,,$$

$$h'_4=-\frac{1}{2}\nu_4+\frac{13}{72}\nu_3^2+\frac{1}{3}\tilde{\theta}\nu_2\nu_3-\frac{1}{2}\tilde{\theta}\nu_3+\frac{1}{8}\tilde{\theta}^2\nu_2^2+\frac{3}{4}\nu_2^2-\frac{1}{2}(\Delta+1)\nu_2\,,$$

$$h'_2=\frac{3}{4}\nu_4-\frac{1}{6}\nu_3^2-\frac{5}{12}\tilde{\theta}\nu_2\nu_3+\frac{3}{2}\tilde{\theta}\nu_3-\frac{1}{4}\tilde{\theta}^2\nu_2^2+\frac{3}{4}\tilde{\theta}^2\nu_2-\frac{3}{4}\nu_2^2$$
$$+\frac{1}{2}(2\Delta+1)\nu_2-\Delta(k-1)+\frac{1}{6}\tilde{\rho}-\frac{1}{6}\,,$$

$$h'_0=-\frac{3}{8}\nu_4+\frac{1}{18}\nu_3^2+\frac{1}{6}\tilde{\theta}\nu_2\nu_3-\tilde{\theta}\nu_3+\frac{1}{8}\tilde{\theta}^2\nu_2^2-\frac{3}{4}\tilde{\theta}^2\nu_2+\frac{1}{4}\nu_2^2$$
$$-\frac{\Delta}{2}\nu_2+(k-1)\Delta+\frac{1}{6}-\frac{1}{6}\tilde{\rho}\,.$$

3. The final result

Taking a correction factor c for the M test, because of Box [3], as

$$c=1-(\tilde{\rho}-1)/3(k-1)n\ , \tag{3.1}$$

we have $(k-1)\Delta=(\tilde{\rho}-1)/6$. Therefore the coefficients h_2' and h_0' in (2.23) are slightly modified.

THEOREM 3.1. *Under the sequence of alternatives* $\sigma_\alpha^2=1+m^{-1/2}\theta_\alpha$ $(\alpha=1,2,\cdots,k)$, *the distribution of the M test with the correction factor* (3.1) *can be expanded asymptotically for large m as*

$$\begin{aligned}\mathrm{P}\,(cM\leqq x)=P_f(\delta^2)&+m^{-1/2}\left\{\frac{\nu_3}{6}P_{f+4}(\delta^2)+\left(-\frac{1}{2}\nu_3-\frac{1}{2}\tilde{\theta}\nu_2\right)P_{f+2}(\delta^2)\right.\\&\left.+\left(\frac{1}{3}\nu_3+\frac{1}{2}\tilde{\theta}\nu_2\right)P_f(\delta^2)\right\}+m^{-1}\sum_{\alpha=0}^{4}h_{2\alpha}P_{f+2\alpha}(\delta^2)+O(m^{-3/2})\ ,\end{aligned} \tag{3.2}$$

where the symbol $P_f(\delta^2)$ *means the distribution function of the noncentral* χ^2 *variate with* $f=k-1$ *degrees of freedom and noncentrality parameter* $\delta^2=\nu_2/4$. *The coefficients* $h_{2\alpha}$ *are given as follows with*

$$\nu_\alpha=\sum_{\beta=1}^{k}\rho_\beta(\theta_\beta-\tilde{\theta})^\alpha\ ,\qquad \tilde{\theta}=\sum_{\alpha=1}^{k}\rho_\alpha\theta_\alpha\qquad\text{and}\qquad \tilde{\rho}=\sum_{\alpha=1}^{k}\rho_\alpha^{-1}\ .$$

$$\begin{aligned}&h_8=h_8'\ ,\qquad h_6=h_6'\ ,\qquad h_4=h_4'\ ,\\&h_2=\frac{3}{4}\nu_4-\frac{1}{6}\nu_3^2-\frac{5}{12}\tilde{\theta}\nu_2\nu_3+\frac{3}{2}\tilde{\theta}\nu_3-\frac{1}{4}\tilde{\theta}^2\nu_2^2+\frac{3}{4}\tilde{\theta}^2\nu_2-\frac{3}{4}\nu_2^2\\&\qquad+\frac{1}{2}(2\Delta+1)\nu_2\ ,\\&h_0=-\frac{3}{8}\nu_4+\frac{1}{18}\nu_3^2+\frac{1}{6}\tilde{\theta}\nu_2\nu_3-\tilde{\theta}\nu_3+\frac{1}{8}\tilde{\theta}^2\nu_2^2-\frac{3}{4}\tilde{\theta}^2\nu_2+\frac{1}{4}\nu_2^2-\frac{\Delta}{2}\nu_2\ .\end{aligned} \tag{3.3}$$

It may be interesting to note that each sum of the coefficients of order $m^{-1/2}$ and m^{-1}, respectively, is zero.

4. Sphericity test

As in Sugiura and Nagao [11], the modified likelihood ratio criterion for testing the sphericity hypothesis, based on a random sample of size $N=n+1$ from a p-variate normal population, is given by

$$\lambda^*=|S|^{n/2}\left(\frac{1}{p}\operatorname{tr}S\right)^{-np/2}\ , \tag{4.1}$$

where S has a Wishart distribution $W(\Sigma, n)$. Since the criterion λ^* remains invariant by the transformation $S\to cH'SH$ for any orthogonal $p\times p$ matrix H and positive scalar c, we may assume $\Sigma=\Lambda=\mathrm{diag}(1, \lambda_2, \cdots, \lambda_p)$ without loss of generality. We shall consider the sequence of alternatives $\lambda_\alpha=1+m^{-1/2}\theta_\alpha$, where $m=\rho n$ with

$$\rho=1-(2p^2+p+2)/6pn\ , \tag{4.2}$$

and derive the asymptotic expansion of $-2\rho\log\lambda^*$ directly from one of cM. The relationship between Bartlett's test and the sphericity test λ^* was used by Gleser [4] to prove the unbiasedness of the latter. Thus the characteristic function of $-2\rho\log\lambda^*$ is given by

$$\begin{aligned} C_{\lambda^*}(t)=c_{p,n}\int |S|^{-mit}\left(\frac{1}{p}\operatorname{tr} S\right)^{mpit}|S|^{(n-p-1)/2}|\Lambda|^{-n/2} \\ \cdot \operatorname{etr}\left[-\frac{1}{2}\Lambda^{-1}S\right]dS\ , \end{aligned} \tag{4.3}$$

where the coefficient $c_{p,n}$ is given by (see Anderson [1], p. 154)

$$c_{p,n}^{-1}=2^{np/2}\pi^{p(p-1)/4}\prod_{\alpha=1}^{p}\Gamma\left[\frac{1}{2}(n-\alpha+1)\right]\ . \tag{4.4}$$

Transform the variable S to D and R by $S=D^{1/2}RD^{1/2}$ such that the matrix D is diagonal and composed of the diagonal elements of S. Then $|\partial S/\partial(D, R)|=|D|^{(p-1)/2}$, so we have

$$\begin{aligned} C_{\lambda^*}(t)=\left\{\prod_{\alpha=1}^{p}\frac{\Gamma[(m/2)(1-2it)+(1-\alpha)/2+\Delta]}{\Gamma[m/2+(1-\alpha)/2+\Delta]}\right\} \\ \cdot\frac{\Gamma^p[m/2+\Delta]}{\Gamma^p[(m/2)(1-2it)+\Delta]}C_M(t)\ , \end{aligned} \tag{4.5}$$

where $\Delta=(n-m)/2$ and $C_M(t)$ is the characteristic function with the correction factor $c=\rho$ in (4.2). Thus the first factor of (4.5) is expanded as

$$\begin{aligned} &\left\{\prod_{\alpha=1}^{p}\frac{\Gamma[(m/2)(1-2it)+(1-\alpha)/2+\Delta]}{\Gamma[m/2+(1-\alpha)/2+\Delta]}\right\}\frac{\Gamma^p[m/2+\Delta]}{\Gamma^p[(m/2)(1-2it)+\Delta]} \\ &=(1-2it)^{-p(p-1)/4}\Big[1+m^{-1}\{(1-2it)^{-1}-1\} \\ &\quad\cdot\left\{\frac{1}{24}p(p-1)(2p+5)-\frac{\Delta}{2}p(p-1)\right\}+O(m^{-2})\Big]\ . \end{aligned} \tag{4.6}$$

On the other hand, replacing k, ρ_α, m_α, m, θ_α and Δ with p, p^{-1}, m, pm, $\sqrt{p}\,\theta_\alpha$ and $p\Delta$ in (2.23), respectively, the characteristic function (4.5) is given by

$$\text{(4.7)}\quad C_{\lambda^*}(t)=(1-2it)^{-f/2}\exp\left[\frac{2it}{(1-2it)}\delta^2\right]\left[1+m^{-1/2}\left\{\frac{1}{6}(t_3-3p^{-1}t_1t_2+2p^{-2}t_1^3)(t)_2+\frac{1}{2}(-t_3+2p^{-1}t_1t_2-p^{-2}t_1^3)(t)_1+\frac{1}{6}(2t_3-3p^{-1}t_1t_2+p^{-2}t_1^3)\right\}+m^{-1}\sum_{\alpha=0}^{4}h_{2\alpha}(t)_\alpha\right]+O(m^{-3/2}),$$

where $\delta^2=(t_2-p^{-1}t_1^2)/4$, $\theta=\mathrm{diag}(0,\theta_2,\cdots,\theta_p)$, $t_j=\mathrm{tr}\,\theta^j$ and $f=p(p+1)/2-1$. The coefficients $h_{2\alpha}$ are given by

$$\text{(4.8)}\quad h_8=\frac{1}{72}(t_3-3p^{-1}t_1t_2+2p^{-2}t_1^3)^2,$$

$$h_6=\frac{1}{24}(3t_4-2t_3^2)-\frac{p^{-1}}{12}(6t_1t_3-5t_1t_2t_3+3t_2^2)+\frac{p^{-2}}{4}(5t_1^2t_2-2t_1^2t_2^2-t_1^3t_3)+\frac{p^{-3}}{24}(14t_1^4t_2-15t_1^4)-\frac{p^{-4}}{6}t_1^6,$$

$$h_4=\frac{1}{72}(13t_3^2-36t_4-36\Delta t_2)-\frac{p^{-1}}{4}(3t_1t_2t_3-6t_1t_3-2\Delta t_1^2-3t_2^2+2t_2)+\frac{p^{-2}}{36}(27t_1^2t_2^2+14t_1^3t_3-108t_1^2t_2+18t_1^2)+\frac{p^{-3}}{4}(5t_1^4-3t_1^4t_2)+\frac{13}{72}p^{-4}t_1^6,$$

$$h_2=\frac{1}{12}(-2t_3^2+9t_4+12\Delta t_2)+\frac{p^{-1}}{12}(7t_1t_2t_3-18t_1t_3-9t_2^2-12\Delta t_1^2+6t_2)-\frac{p^{-2}}{4}(2t_1^2t_2^2+t_1^3t_3-9t_1^2t_2+2t_1^2)+\frac{p^{-3}}{12}(5t_1^4t_2-9t_1^4)-\frac{p^{-4}}{12}t_1^6,$$

$$h_0=\frac{1}{72}(-27t_4-36\Delta t_2+4t_3^2)+\frac{p^{-1}}{12}(-2t_1t_2t_3+6t_1t_3+6\Delta t_1^2+3t_2^2)+\frac{p^{-2}}{72}(9t_1^2t_2^2+4t_1^3t_3-36t_1^2t_2)+\frac{p^{-3}}{24}(3t_1^4-2t_1^4t_2)+\frac{p^{-4}}{72}t_1^6.$$

Inverting this characteristic function, we have the following theorem:

THEOREM 4.1. *Under the sequence of alternatives $\Lambda=I+m^{-1/2}\theta$, the distribution of the LR criterion given by* (4.1) *can be expanded asymptotically for large $m=\rho n$ as follows:*

$$\text{(4.9)}\quad \mathrm{P}(-2\rho\log\lambda^*\leqq x)=P_f(\delta^2)+m^{-1/2}\left\{\frac{1}{6}(2t_3-3p^{-1}t_1t_2+p^{-2}t_1^3)P_f(\delta^2)+\frac{1}{2}(-t_3+2p^{-1}t_1t_2-p^{-2}t_1^3)P_{f+2}(\delta^2)+\frac{1}{6}(t_3-3p^{-1}t_1t_2+2p^{-2}t_1^3)P_{f+4}(\delta^2)\right\}+m^{-1}\sum_{\alpha=0}^{4}h_{2\alpha}P_{f+2\alpha}(\delta^2)+O(m^{-3/2}),$$

where the correction factor ρ *is given by* (4.2) *and* $t_j=\mathrm{tr}\,\theta^j$. *The symbol* $P_f(\delta^2)$ *means the distribution function of the noncentral* χ^2 *variate with* $f=p(p+1)/2-1$ *degrees of freedom and noncentrality parameter* $\delta^2=(t_2-p^{-1}t_1^2)/4$ *and the coefficients* $h_{2\alpha}$ *are given by* (4.8).

Under the fixed alternative, Nagao [7] gave the asymptotic expansion of the distribution of the LR criterion for testing a covariance matrix Σ has all off-diagonal elements zero and variances in multiplicities, the special case of which is sphericity test.

5. Numerical examples

We shall give some numerical values of the asymptotic power of Bartlett's test and sphericity test when the alternatives are near to the null hypothesis in the following special cases.

Example 5.1. When $k=2$ and $n_1=4$, $n_2=20$, the exact values of the power of Bartlett's test for some alternatives have been given by Ramachandran [9] in his Table 744a. Using the 5% point of cM 3.801 in our previous paper [12], and specifying the alternatives K: $\sigma_2^2=d\sigma_1^2$, we have the following approximate powers of the cM test:

P_K $(cM\geqq 3.801)$

d	10/7	10/8	10/13	10/14	10/15	10/16
First term	0.0843	0.0623	0.0607	0.0658	0.0711	0.0765
Second term	−0.0188	−0.0037	0.0029	0.0056	0.0088	0.0126
Third term	0.0039	−0.0006	−0.0006	−0.0003	0.0004	0.0015
Approx. power	0.0694	0.0580	0.0630	0.0711	0.0803	0.0906
Exact power	0.068	0.057	0.062	0.070	0.080	0.091

Example 5.2. When $p=2$ and $n=50$, the approximate 5% point of $-2\rho\log\lambda^*$ is given by Nagao [6] as 5.9915. Let us specify the alternatives K as $\lambda_1=1$, $\lambda_2=1+d$, then we have the following approximate powers of the sphericity test.

P_K $(-2\rho\log\lambda^*\geqq 5.9915)$

d	0.3	−0.3
First term	0.1419	0.1419
Second term	−0.0298	0.0298
Third term	0.0056	0.0056
Approx. power	0.1177	0.1773

6. Appendix

Let a random vector u be distributed according to a multivariate normal distribution with mean zero and covariance matrix $\Sigma=(\sigma_{\alpha\beta})$. First of all we shall state the moments concerning a random vector $u=(u_1, u_2, \cdots, u_k)'$.

$$\text{(6.1)}\quad \mathrm{E}\,(u_\alpha u_\beta u_\gamma u_\delta)=\sigma_{\alpha\beta}\sigma_{\gamma\delta}+\sigma_{\alpha\gamma}\sigma_{\beta\delta}+\sigma_{\beta\gamma}\sigma_{\alpha\delta}\ ,$$

$$\text{(6.2)}\quad \begin{aligned}\mathrm{E}\,(u_\alpha u_\beta u_\gamma u_\delta u_\epsilon u_\zeta)=&\sigma_{\alpha\beta}(\sigma_{\gamma\delta}\sigma_{\epsilon\zeta}+\sigma_{\gamma\epsilon}\sigma_{\delta\zeta}+\sigma_{\gamma\zeta}\sigma_{\delta\epsilon})+\sigma_{\alpha\gamma}(\sigma_{\beta\delta}\sigma_{\epsilon\zeta}+\sigma_{\beta\epsilon}\sigma_{\delta\zeta}+\sigma_{\beta\zeta}\sigma_{\delta\epsilon})\\&+\sigma_{\alpha\delta}(\sigma_{\beta\gamma}\sigma_{\epsilon\zeta}+\sigma_{\beta\epsilon}\sigma_{\gamma\zeta}+\sigma_{\beta\zeta}\sigma_{\gamma\epsilon})+\sigma_{\alpha\epsilon}(\sigma_{\beta\gamma}\sigma_{\delta\zeta}+\sigma_{\beta\delta}\sigma_{\gamma\zeta}+\sigma_{\beta\zeta}\sigma_{\gamma\delta})\\&+\sigma_{\alpha\zeta}(\sigma_{\beta\gamma}\sigma_{\delta\epsilon}+\sigma_{\beta\delta}\sigma_{\gamma\epsilon}+\sigma_{\beta\epsilon}\sigma_{\gamma\delta})\ .\end{aligned}$$

By the above moments, we can compute the expectations of $z=(z_1, z_2, \cdots, z_k)'$ with respect to a k-variate normal distribution having mean $\eta=(\eta_1, \eta_2, \cdots, \eta_k)'$ and covariance matrix $\Sigma=(\sigma_{\alpha\beta})$ with $\eta_\alpha=\sqrt{\rho_\alpha/2}\,(\theta_\alpha-2it\tilde{\theta})\cdot(1-2it)^{-1}$ and $\sigma_{\alpha\beta}=(\delta_{\alpha\beta}-2it\sqrt{\rho_\alpha\rho_\beta})(1-2it)^{-1}$.

$$\text{(6.3)}\quad \mathrm{E}(\sum\sqrt{\rho_\alpha}\,z_\alpha)^2=\frac{1}{2}\tilde{\theta}^2+1\ ,$$

$$\text{(6.4)}\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha)(\sum\sqrt{\rho_\alpha}\,z_\alpha)\}=\frac{1}{2}(\tilde{\theta}\nu_3+2\tilde{\theta}^2\nu_2)(t)_1+\frac{1}{2}(\nu_2+\tilde{\theta}^2)(\tilde{\theta}^2+2)\ ,$$

$$\text{(6.5)}\quad \mathrm{E}(\sum z_\alpha^2)=\frac{1}{2}\nu_2(t)_2+(k-1)(t)_1+\frac{1}{2}\tilde{\theta}^2+1\ ,$$

$$\text{(6.6)}\quad \begin{aligned}\mathrm{E}(\sum\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha)^2=&\frac{1}{2}(\nu_3+2\tilde{\theta}\nu_2)^2(t)_2+(\tilde{\theta}\nu_2\nu_3+2\tilde{\theta}^2\nu_2^2+\tilde{\theta}^3\nu_3\\&+2\tilde{\theta}^4\nu_2+\nu_4+4\tilde{\theta}\nu_3+4\tilde{\theta}^2\nu_2-\nu_2^2)(t)_1\\&+\frac{1}{2}\tilde{\theta}^2\nu_2^2+\tilde{\theta}^4\nu_2+\nu_2^2+2\tilde{\theta}^2\nu_2+\frac{1}{2}\tilde{\theta}^6+\tilde{\theta}^4\ ,\end{aligned}$$

$$\text{(6.7)}\quad \begin{aligned}&\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)(\sum\theta_\alpha z_\alpha^2)\}\\&=\frac{1}{2\sqrt{2}}(\tilde{\theta}\nu_3+\tilde{\theta}^2\nu_2)(t)_2+\frac{1}{\sqrt{2}}\{\tilde{\theta}^2\nu_2+2\nu_2+\tilde{\theta}\zeta_1+(k-1)\tilde{\theta}^2\}(t)_1\\&\quad+\frac{1}{2\sqrt{2}}(\tilde{\theta}^4+6\tilde{\theta}^2)\ ,\end{aligned}$$

$$\text{(6.8)}\quad \mathrm{E}(\sum\sqrt{\rho_\alpha}\,z_\alpha)^4=\frac{1}{4}\tilde{\theta}^4+3\tilde{\theta}^2+3\ ,$$

$$\text{(6.9)}\quad \begin{aligned}&\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^3(\sum\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha)\}\\&=\frac{1}{4}(\tilde{\theta}\nu_3+2\tilde{\theta}^2\nu_2)(\tilde{\theta}^2+6)(t)_1+\frac{1}{4}(\nu_2+\tilde{\theta}^2)(\tilde{\theta}^4+12\tilde{\theta}^2+12)\ ,\end{aligned}$$

$$\text{(6.10)}\quad \begin{aligned}&\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^2(\sum z_\alpha^2)\}\\&=\frac{1}{4}(\tilde{\theta}^2+2)\nu_2(t)_2+\frac{1}{2}(k-1)(\tilde{\theta}^2+2)(t)_1+\frac{1}{4}(\tilde{\theta}^4+12\tilde{\theta}^2+12)\ ,\end{aligned}$$

$$(6.11)\quad \mathrm{E}(\sum z_\alpha^2)^2=\frac{1}{4}\nu_2^2(t)_4+(k+1)\nu_2(t)_3+\frac{1}{2}(\tilde{\theta}^2\nu_2+2\nu_2+2k^2-2)(t)_2$$
$$+(k-1)(\tilde{\theta}^2+2)(t)_1+\frac{1}{4}(\tilde{\theta}^4+12\tilde{\theta}^2+12)\ ,$$

$$(6.12)\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)(\sum z_\alpha^3/\sqrt{\rho_\alpha})\}$$
$$=\frac{1}{4}\tilde{\theta}\nu_3(t)_3+\frac{3}{4}(\tilde{\theta}^2\nu_2+2\nu_2+2\tilde{\theta}\zeta_1)(t)_2+\frac{3}{2}(k-1)(\tilde{\theta}^2+2)(t)_1$$
$$+\frac{1}{4}(\tilde{\theta}^4+12\tilde{\theta}^2+12)\ ,$$

$$(6.13)\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)(\sum\sqrt{\rho_\alpha}\,\theta_\alpha^2 z_\alpha)(\sum z_\alpha^2)\}$$
$$=\frac{1}{4}(\tilde{\theta}\nu_2\nu_3+2\tilde{\theta}^2\nu_2^2)(t)_3+\frac{1}{4}\{2(k+1)\tilde{\theta}\nu_3+\tilde{\theta}^2\nu_2^2+2\nu_2^2+\tilde{\theta}^4\nu_2$$
$$+2(2k+3)\tilde{\theta}^2\nu_2\}(t)_2+\frac{1}{4}\{\tilde{\theta}^3\nu_3+6\tilde{\theta}\nu_3+2\tilde{\theta}^4\nu_2+2(k+5)\tilde{\theta}^2\nu_2$$
$$+4(k-1)\nu_2+2(k-1)\tilde{\theta}^4+4(k-1)\tilde{\theta}^2\}(t)_1$$
$$+\frac{1}{4}(\nu_2+\tilde{\theta}^2)(\tilde{\theta}^4+12\tilde{\theta}^2+12)\ ,$$

$$(6.14)\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^3(\sum\theta_\alpha z_\alpha^2)\}$$
$$=\frac{1}{4\sqrt{2}}(\tilde{\theta}\nu_3+\tilde{\theta}^2\nu_2)(\tilde{\theta}^2+6)(t)_2+\frac{1}{2\sqrt{2}}\{\tilde{\theta}^4\nu_2+12\tilde{\theta}^2\nu_2+12\nu_2$$
$$+\tilde{\theta}^3\zeta_1+6\tilde{\theta}\zeta_1+(k-1)\tilde{\theta}^4+6(k-1)\tilde{\theta}^2\}(t)_1$$
$$+\frac{1}{4\sqrt{2}}(\tilde{\theta}^6+20\tilde{\theta}^4+60\tilde{\theta}^2)\ ,$$

$$(6.15)\quad \mathrm{E}(\sum\sqrt{\rho_\alpha}\,z_\alpha)^6=\frac{1}{8}(\tilde{\theta}^6+30\tilde{\theta}^4+180\tilde{\theta}^2+120)\ ,$$

$$(6.16)\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^4(\sum z_\alpha^2)\}$$
$$=\frac{1}{8}(\tilde{\theta}^4\nu_2+12\tilde{\theta}^2\nu_2+12\nu_2)(t)_2+\frac{1}{4}(k-1)(\tilde{\theta}^4+12\tilde{\theta}^2+12)(t)_1$$
$$+\frac{1}{8}(\tilde{\theta}^6+30\tilde{\theta}^4+180\tilde{\theta}^2+120)\ ,$$

$$(6.17)\quad \mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^3(\sum z_\alpha^3/\sqrt{\rho_\alpha})\}$$
$$=\frac{1}{8}(\tilde{\theta}^3\nu_3+6\tilde{\theta}\nu_3)(t)_3+\frac{1}{8}(3\tilde{\theta}^4\nu_2+36\tilde{\theta}^2\nu_2+36\nu_2+6\tilde{\theta}^3\zeta_1+36\tilde{\theta}\zeta_1)(t)_2$$
$$+\frac{1}{4}(k-1)(3\tilde{\theta}^4+36\tilde{\theta}^2+36)(t)_1+\frac{1}{8}(\tilde{\theta}^6+30\tilde{\theta}^4+180\tilde{\theta}^2+120)\ ,$$

(6.18) $$\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)^2(\sum z_\alpha^2)^2\}$$
$$=\frac{1}{8}(\tilde\theta^2\nu_2^2+2\nu_2^2)(t)_4+\frac{1}{2}(k+1)(\tilde\theta^2\nu_2+2\nu_2)(t)_3$$
$$+\frac{1}{4}\{\tilde\theta^4\nu_2+12\tilde\theta^2\nu_2+12\nu_2+2(k^2-1)\tilde\theta^2+4(k^2-1)\}(t)_2$$
$$+\frac{1}{2}(k-1)(\tilde\theta^4+12\tilde\theta^2+12)(t)_1+\frac{1}{8}(\tilde\theta^6+30\tilde\theta^4+180\tilde\theta^2+120)\,,$$

(6.19) $$\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)(\sum z_\alpha^2)(\sum\theta_\alpha z_\alpha^2)\}$$
$$=\frac{1}{4\sqrt{2}}(\tilde\theta\nu_2\nu_3+\tilde\theta^2\nu_2^2)(t)_4+\frac{1}{2\sqrt{2}}\{(k+3)\tilde\theta\nu_3+\tilde\theta^2\nu_2^2+2\nu_2^2$$
$$+2(k+1)\tilde\theta^2\nu_2+\tilde\theta\zeta_1\nu_2\}(t)_3+\frac{1}{4\sqrt{2}}\{\tilde\theta^3\nu_3+6\tilde\theta\nu_3+2\tilde\theta^4\nu_2$$
$$+4(k+4)\tilde\theta^2\nu_2+8(k+1)\nu_2+4(k+1)\tilde\theta\zeta_1+4(k^2-1)\tilde\theta^2\}(t)_2$$
$$+\frac{1}{2\sqrt{2}}\{\tilde\theta^4\nu_2+12\tilde\theta^2\nu_2+12\nu_2+\tilde\theta^3\zeta_1+6\tilde\theta\zeta_1+2(k-1)\tilde\theta^4$$
$$+12(k-1)\tilde\theta^2\}(t)_1+\frac{1}{4\sqrt{2}}(\tilde\theta^6+20\tilde\theta^4+60\tilde\theta^2)\,,$$

(6.20) $$\mathrm{E}\{(\sum\sqrt{\rho_\alpha}\,z_\alpha)(\sum z_\alpha^2)(\sum z_\alpha^3/\sqrt{\rho_\alpha})\}$$
$$=\frac{1}{8}\tilde\theta\nu_2\nu_3(t)_5+\frac{1}{8}\{2(k+5)\tilde\theta\nu_3+3\tilde\theta^2\nu_2^2+6\nu_2^2+3\tilde\theta\zeta_1\nu_2\}(t)_4$$
$$+\frac{1}{8}\{\tilde\theta^3\nu_3+6\tilde\theta\nu_3+12(k+1)\tilde\theta^2\nu_2+24(k+1)\nu_2+12(k+3)\tilde\theta\zeta_1\}(t)_3$$
$$+\frac{1}{4}\{2\tilde\theta^4\nu_2+24\tilde\theta^2\nu_2+24\nu_2+3\tilde\theta^3\zeta_1+18\tilde\theta\zeta_1+6(k^2-1)\tilde\theta^2$$
$$+12(k^2-1)\}(t)_2+(k-1)(\tilde\theta^4+12\tilde\theta^2+12)(t)_1$$
$$+\frac{1}{8}(\tilde\theta^6+30\tilde\theta^4+180\tilde\theta^2+120)\,.$$

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