AN ASYMPTOTIC EXPANSION FOR THE DISTRIBUTION OF THE DETERMINANT OF A MULTIVARIATE QUADRATIC FORM IN A NORMAL SAMPLE*

TAKESI HAYAKAWA

(Received Oct. 16, 1972; revised Oct. 19, 1973)

1. Introduction and summary

The exact representations of the probability density function (p.d.f.) of the multivariate quadratic form of the central case in normal sample were obtained by Hayakawa [4], Khatri [13], and Shah [14] by the use of the zonal polynomials and Laguerre polynomials of matrix argument. The exact representations of the p.d.f.'s of the latent roots of the multivariate quadratic form in non-central case were also treated in Hayakawa [5], [6] in terms of the new polynomial $P_\kappa(T, A)$. Recently Hayakawa [8] obtained the asymptotic expansion of the distribution of the determinant of the multivariate quadratic form in central case. In this paper, we will extend the result of [8] to the non-central case. A certain condition on the limiting behaviour of the latent roots of the quadratic form is assumed together with the boundedness of the non-centrality parameters. We also give the asymptotic expansion of the p.d.f. of the trace of the multivariate quadratic form. Using these results, we have the asymptotic expansion of Cornish-Fisher type for these statistics. All the results could be extended similarly to complex normal sample case.

2. Some useful results

Let T and U be $m\times n$ ($m\leqq n$) matrices, and let A be an $n\times n$ diagonal matrix, i.e., $A=\operatorname{diag}(a_1, a_2, \cdots, a_n)$, $a_1\geqq a_2\geqq\cdots\geqq a_n>0$. Then the polynomial $P_\kappa(T, A)$ is defined by

$$(1)\qquad \operatorname{etr}(-TT')P_\kappa(T, A) =(-1)^k\pi^{-mn/2}\int_U \operatorname{etr}(-2iTU')\operatorname{etr}(-UU')C_\kappa(UAU')dU\ ,$$

* This work was begun in Department of Mathematics in Indiana University, U.S.A., and was supported by NSF Grant No. GP27715, and was completed at the Institute of Statistical Mathematics.

where $dU=\prod_{\alpha=1}^{m}\prod_{\beta=1}^{n} du_{\alpha\beta}$ and the integral is done on whole mn dimensional Euclidian space, and $C_\kappa(UAU')$ is a zonal polynomial of UAU' corresponding to a partition κ of k, James [12].

We can see the following relations.

$$(2)\qquad P_\kappa(0, A)=(-1)^k(m/2)_\kappa C_\kappa(A)^{1)}$$

$$(3)\qquad P_\kappa(T, I_n)=H_\kappa(T)=(-1)^k L_\kappa^{n/2-p}(TT'),\qquad p=(m+1)/2,$$

where $H_\kappa(T)$ and $L_\kappa^{n/2-p}(TT')$ are the generalized Hermite polynomial and the generalized Laguerre polynomial, respectively, Hayakawa [5] and Constantine [1].

The exact expressions for $P_\kappa(T, A)$'s up to $k=4$ are listed in Hayakawa [6].

LEMMA 1. *Let $P_\kappa(T, A)$ be a polynomial corresponding to a partition $\kappa=\{k_1, k_2, \cdots, k_m\}$ of k into not more than m parts for an $m\times n$ matrix T and an $n\times n$ matrix A, and $a_1(\kappa)=\sum_{\alpha=1}^{m} k_\alpha(k_\alpha-\alpha)$, and put $\tilde{T}=T(I+xA)^{-1/2}$, $\tilde{A}=(I+xA)^{-1/2}A(I+xA)^{-1/2}$ for $\|xA\|<1$, where $\|A\|$ implies the maximum value of the absolute values of the latent roots of A. Then the following equalities hold.*

$$(4)\qquad \sum_{k=r}^{\infty}(x^k/(k-r)!)\sum_\kappa P_\kappa(T, A)=d(x, T, A)x^r\sum_\tau P_\tau(\tilde{T}, \tilde{A}),\qquad r=0, 1, 2, \cdots,$$

where $d(x, T, A)=\det(I+xA)^{-m/2}\operatorname{etr}\{T(I-(I+xA)^{-1})T'\}$, and τ is a partition of r.

$$(5)\qquad \sum_{k=0}^{\infty}(x^k/k!)\sum_\kappa a_1(\kappa)P_\kappa(T, A)=d(x, T, A)x^2[P_{(2)}(\tilde{T}, \tilde{A})-(1/2)P_{(1^2)}(\tilde{T}, \tilde{A})].$$

PROOF. To prove (4) and (5), we need two formulas which were obtained by Sugiura and Fujikoshi [15].

$$(6)\qquad \sum_{k=r}^{\infty}(x^k/(k-r)!)\sum_\kappa C_\kappa(UAU')=x^r\operatorname{etr}(xUAU')\sum_\tau C_\tau(UAU'),$$

$$(7)\qquad \sum_{k=0}^{\infty}(x^k/k!)\sum_\kappa a_1(\kappa)C_\kappa(UAU')=x^2\operatorname{etr}(xUAU')[C_{(2)}(UAU')-(1/2)C_{(1^2)}(UAU')].$$

Using the definition of $P_\kappa(T, A)$ and (6), we have

$$\sum_{k=r}^{\infty}(x^k/(k-r)!)\sum_\kappa P_\kappa(T, A)$$

1) The right hand side of (36) in [5] should be multiplied by $(n/2)_\kappa$.

$$
\begin{aligned}
&= \operatorname{etr}(TT')(-x)^r \pi^{-mn/2} \int_U \operatorname{etr}(-2iTU') \operatorname{etr}(-UU') \\
&\quad \cdot \operatorname{etr}(-xUAU') \sum_\tau C_\tau(UAU') dU \\
&= x^r \operatorname{etr}(TT') \det(I+xA)^{-m/2}(-1)^r \pi^{-mn/2} \\
&\quad \cdot \int_U \operatorname{etr}(-UU') \operatorname{etr}(-2iT(I+xA)^{-1/2}U') \\
&\quad \cdot \sum_\tau C_\tau\{U(I+xA)^{-1/2}A(I+xA)^{-1/2}U'\} dU \\
&= x^r \operatorname{etr}(TT') \operatorname{etr}(-T(I+xA)^{-1}T') \\
&\quad \cdot \det(I+xA)^{-m/2} \sum_\tau P_\tau(\tilde{T}, \tilde{A}) .
\end{aligned}
$$

We have (5) by similar way as (4) by the use of (7).

Note. More details for the weighted sums of $P_\kappa(T, A)$ may be found in Hayakawa [9].

COROLLARY 1. *If we set* $A=I_n$ *and* $x=-y$, *we have same results as Fujikoshi* ([3], *Lemma* 8) *with* $S=TT'$ *by* (3). *If we set* $T=0$ *and* $x=-1$, *we also have same results as Fujikoshi* ([3], *Lemma* 3) *with* $b=m/2$ *by* (2).

3. Asymptotic expansions of the distributions for two functions of XAX'

3.1. *The asymptotic expansion of the distribution of* $\det XAX'$

Let X be an $m\times n$ matrix with probability density function (p.d.f.)

$$
(8) \qquad \pi^{-mn/2}(\det 2\Sigma)^{-n/2} \operatorname{etr}(-\Sigma^{-1}XX'/2) ,
$$

and A a diagonal matrix such that $A=\operatorname{diag}(a_1, a_2, \cdots, a_n)$, $a_1 \geqq a_2 \geqq \cdots \geqq a_n > 0$, and M an $m\times n$ arbitrary matrix with rank m. Then the following theorem holds.

PROPOSITION 1 (Hayakawa [6], Theorem 3). Let X, A and M be defined above, then the joint p.d.f. of the latent roots $\Lambda=\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$ of $Z=n^{-1}\Sigma^{-1/2}(X-M)A(X-M)'\Sigma^{-1/2}$ is expressed as follows.

$$
(9) \qquad \sum_{k=0}^{\infty}\sum_\kappa R_\kappa f_\kappa(\Lambda) , \qquad \text{for the minimum value of } \{a_k\}=a_n>q>0 ,
$$

where

$$
\begin{aligned}
f_\kappa(\Lambda) &= (n/(2q))^{mn/2}\pi^{m^2/2}(\Gamma_m(m/2)\Gamma_m(n/2;\kappa)C_\kappa(I_m))^{-1} \\
&\quad \cdot \operatorname{etr}\{-(n/2q)\Lambda\}(\det\Lambda)^{(n-m-1)/2}C_\kappa((n/2q)\Lambda)\prod_{i<j}(\lambda_i-\lambda_j) ,
\end{aligned}
$$

and

(10) $$k!R_\kappa=(\det B)^{-m/2}\operatorname{etr}(-\Omega)(-1)^k \cdot P_\kappa((i/\sqrt{2})\Sigma^{-1/2}M(B-I)^{-1/2}, I-B^{-1}) ,$$

where $B=A/q$, $\Omega=M'\Sigma^{-1}M/2$ and $i=\sqrt{-1}$. Furthermore, $R_\kappa>0$ for all partition κ of k, and

(11) $$\sum_{k=0}^{\infty}\sum_{\kappa} R_\kappa=1 .$$

We have the similar set of results of Lemma 1 as follows.

LEMMA 2. *Let R_κ's be defined as (10), and*

$$t_j=\operatorname{tr}(B-I)^j , \qquad s_j=\operatorname{tr} B(B-I)^{j-1}\Omega , \qquad j=1, 2, 3 ,$$
$$\textit{and} \qquad \omega=\operatorname{tr}(B\Omega)^2 ,$$

then the following equalities hold.

(12) $$F_1(T, A)\equiv 2\sum_{k=1}^{\infty}\sum_{\kappa} kR_\kappa=mt_1+s_1 .$$

(13) $$F_2(T, A)\equiv 2\sum_{k=2}^{\infty}\sum_{\kappa} k(k-1)R_\kappa =m[(m/2)t_1^2+t_2]+mt_1s_1+(1/2)s_1^2+2s_2 .$$

(14) $$F_3(T, A)\equiv 4\sum_{k=3}^{\infty}\sum_{\kappa} k(k-1)(k-2)R_\kappa =(m^3/2)t_1^3+3m^2t_1t_2+mt_3+(3/2)(m^2t_1^2+2mt_2)s_1 +(3/2)mt_1(s_1^2+4s_2)+12s_3+6s_1s_2+(1/2)s_1^3 .$$

(15) $$F_4(T, A)\equiv 4\sum_{k=0}^{\infty}\sum_{\kappa} a_1(\kappa)R_\kappa =mt_1^2+m(m+1)t_2+2(m+1)s_2+2t_1s_1+\omega .$$

Using the p.d.f. (9), the characteristic function of $\lambda=\sqrt{n/(2m)}\log\{\det(Z/q)\}$ is expressed as

(16) $$\varphi(t)=\varphi_0(t)\varphi_{T,A}(t) ,$$

where

(17) $$\varphi_0(t)=\left(\frac{2}{n}\right)^{itm\sqrt{n/(2m)}}\frac{\Gamma_m(n/2+it\sqrt{n/(2m)})}{\Gamma_m(n/2)} ,$$

(18) $$\varphi_{T,A}(t)=\sum_{k=0}^{\infty}\sum_{\kappa} R_\kappa\frac{(n/2+it\sqrt{n/2})_\kappa}{(n/2)_\kappa} .$$

The asymptotic expansion of $\varphi_0(t)$ is given by (12) in Hayakawa [7], and

$$(19)\qquad \varphi_{T,A}(t)=\sum_{k=0}^{\infty}\sum_{\kappa} R_{\kappa}\{1+(2/\sqrt{2mn})(it)k+(2(it)^2/(2mn))k(k-1)$$
$$-(1/(2mn\sqrt{2mn}))\{4m(it)a_1(\kappa)-(4/3)(it)^3k(k-1)(k-2)\}$$
$$+O(1/n^2)\}\ .$$

We can rewrite (19) by using Lemma 2,

$$(20)\qquad \varphi_{T,A}(t)=1+(F_1(T,A)/\sqrt{2mn})(it)+(F_2(T,A)/(2mn))(it)^2$$
$$-(1/(2mn\sqrt{2mn}))\{mF_4(T,A)(it)-(1/3)F_3(T,A)(it)^3\}$$
$$+O(1/n^2)\ .$$

By combining (12) of [7] and (20), we have the asymptotic expansion of $\varphi(t)$ as follows.

$$(21)\qquad \varphi(t)=\exp(-t^2/2)\Big[1-(1/\sqrt{2mn})\sum_{\alpha=1}^{2} l_{1\alpha}(it)^{2\alpha-1}+(1/2mn)\sum_{\alpha=1}^{3} l_{2\alpha}(it)^{2\alpha}$$
$$-(1/2mn\sqrt{2mn})\sum_{\alpha=1}^{5} l_{3\alpha}(it)^{2\alpha-1}+O(1/n^2)\Big]\ ,$$

where

$$l_{11}=mp-F_1\ ,\qquad l_{12}=1/3\ ,$$
$$l_{21}=F_2-mpF_1+mp(mp+2)/2\ ,$$
$$l_{22}=(mp+1-F_1)/3\ ,\qquad l_{23}=1/18\ ,$$
$$(22)\qquad l_{31}=m^2(2m^2+3m-1)/6+mF_4\ ,$$
$$l_{32}=mp(mp+2)(mp+4)/6-mp(mp+2)F_1/2+mpF_2-F_3/3\ ,$$
$$l_{33}=(5m^2p^2+20mp+12)/30-(mp+1)F_1/3+F_2/3\ ,$$
$$l_{34}=(mp+2-F_1)/18\ ,\qquad l_{35}=1/162\ ,$$

and $F_\alpha=F_\alpha(T,A)$, $\alpha=1,2,3,4$ are given in Lemma 2. Inverting (21), we have a following theorem.

THEOREM 1. *Let X be an $m\times n$ matrix with p.d.f.* (9), *M an $m\times n$ arbitrary matrix with rank m, A an $n\times n$ diagonal matrix,* $\operatorname{diag}(a_1, a_2,\cdots,a_n)$, $a_1\geqq a_2\geqq\cdots\geqq a_n>0$. *Let* $nZ=\Sigma^{-1/2}(X-M)A(X-M)'\Sigma^{-1/2}$. *Then the asymptotic expansion of the distribution of* $\lambda=\sqrt{n/(2m)}\log\{\det(Z/q)\}$ *is given under the conditions* $\operatorname{tr}(A-qI_n)=q\operatorname{tr}(B-I)=O(1)$ *and* $\operatorname{tr}\Sigma^{-1}MM'=\operatorname{tr}\Omega=O(1)$ *as follows.*

$$(23)\quad \mathrm{P}(\lambda\leqq x)=\Phi(x)+(1/\sqrt{2mn})\sum_{\alpha=1}^{2} l_{1\alpha}\Phi^{(2\alpha-1)}(x)+(1/(2mn))\sum_{\alpha=1}^{3} l_{2\alpha}\Phi^{(2\alpha)}(x)$$
$$+(1/(2mn\sqrt{2mn}))\sum_{\alpha=1}^{5} l_{3\alpha}\Phi^{(2\alpha-1)}(x)+O(1/n^2)\ ,$$

where $l_{i\alpha}$'s are given by (22) and $\Phi^{(\alpha)}(x)$ stands for the αth derivative of the distribution function $\Phi(x)$ of a standard normal random variable.

Note. The term of order $1/n^2$ of this expansion has been obtained in Hayakawa [10].

COROLLARY 2. *If we set $M=0$ in (23), then we have the asymptotic expansion of the distribution of the determinant of the central quadratic form which is same result as (7) in Hayakawa [7].*

3.2. *The asymptotic expansion of the p.d.f. of* $\mathrm{tr}\,\Sigma^{-1}(X-M)A(X-M)'$

The exact p.d.f. of $mnT=\mathrm{tr}\,\Sigma^{-1}(X-M)A(X-M)'$ for $0<q<a_n$ is given by

$$(24)\qquad (mn/2q)^{mn/2}(1/\Gamma(mn/2))\exp\{-(mn/2q)T\}\,T^{mn/2-1}\cdot\sum_{k=0}^{\infty}((mn/2q)T)^k/(mn/2)_k\sum_{\kappa}R_\kappa\,,$$

where R_κ's are given as (10), and the power series converges absolutely for all $T>0$, ([2], (62)).

THEOREM 2. *Let T be distributed with p.d.f. (24), then the asymptotic expansion of the p.d.f. of $x=\sqrt{mn/2}\log(T/q)$ is given by*

$$(25)\quad f(x)=\phi(x)\{1+B_1/\sqrt{2mn}+B_2/(2mn)+B_3/(2mn\sqrt{2mn})+O(1/n^2)\}\,,$$

where

$$(26)\qquad \begin{aligned}\phi(x)&=\exp(-x^2/2)/\sqrt{2\pi}\,,\\ B_1&=-\{x^3/3-xF_1\}\,,\\ B_2&=x^6/18-x^4(1+2F_1)/6+x^2(F_1+F_2)-(1/3+F_2)\\ B_3&=-[x^9/162-x^7(1+F_1)/18+x^5(2+15F_1+10F_2)/30\\ &\quad -x^3(1+6F_1+21F_2+3F_3)/9+x(F_1+12F_2+3F_3)/3]\,,\end{aligned}$$

where F_α's are same as ones of Lemma 2.

PROOF. Since $T=q\exp(\sqrt{2/mn}\,x)$, the p.d.f. of x is expressed as

$$(27)\qquad \begin{aligned}&(mn/2)^{(mn-1)/2}(1/\Gamma(mn/2))\exp(\sqrt{mn/2}\,x)\\ &\cdot\exp\{-(mn/2)\exp(\sqrt{2/mn}\,x)\}\\ &\cdot\sum_{k=0}^{\infty}((mn/2)^k/(mn/2)_k)\exp\{k\sqrt{2/mn}\,x\}\sum_{\kappa}R_\kappa\,,\end{aligned}$$

which converges absolutely for all x. We can check easily that the first term and the second term of (27) can be expanded in the following forms, respectively.

(28) $$\exp(-x^2/2)/\sqrt{2\pi}\,[1-x^3/(3\sqrt{2mn})+(1/(2mn))\{x^6/18-x^4/6-1/3\} \\ +(1/2mn\sqrt{2mn})\{x^9/162-x^7/18+x^5/15-x^3/9\}+(1/n^2)]$$

(29) $$1+F_1/\sqrt{2mn}+(1/2mn)\{x^2(F_2+F_1)-F_2\}+(1/2mn\sqrt{2mn}) \\ \cdot\{x^3(F_3+6F_2+2F_1)/3-x(F_3+4F_2)\}+O(1/n^2)\ .$$

Combining (28) and (29), we have (25).

Remark. Let $X=[\boldsymbol{x}_1', \boldsymbol{x}_2', \cdots, \boldsymbol{x}_m']'$ and $M=[\boldsymbol{\mu}_1', \boldsymbol{\mu}_2', \cdots, \boldsymbol{\mu}_m']'$, and put $\boldsymbol{x}=[\boldsymbol{x}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_m]$ and $\boldsymbol{\mu}=[\boldsymbol{\mu}_1, \boldsymbol{\mu}_2, \cdots, \boldsymbol{\mu}_m]$, then

$$\mathrm{tr}\,\Sigma^{-1}(X-M)A(X-M)'=(\boldsymbol{x}-\boldsymbol{\mu})(\Sigma^{-1}\otimes A)(\boldsymbol{x}-\boldsymbol{\mu})'\ .$$

This implies that the asymptotic expansion of the distribution of $\sqrt{mn/2}\cdot\log(T/q)$ can be obtained from Theorem 2 by replacing m with 1, n with mn, and noticing

$$F_4=4\sum_{k=0}^{\infty}\sum_{\kappa}a_1(\kappa)R_\kappa=4\sum_{k=0}^{\infty}\sum_{\kappa}k(k-1)R_\kappa=2F_2\ .$$

Hill and Davis [11] obtained the general formula of the asymptotic expansion of Cornish-Fisher type. Using (23) of [11], we have the following corollary.

COROLLARY 3. *Let u be the pth percentile of the standard normal random variable, and x the pth one of $\lambda=\sqrt{n/2m}\log\{\det(z/q)\}$, then*

(30) $$x=u-G_1/\sqrt{2mn}+G_2/(2mn)-G_3/(2mn\sqrt{2mn})+O(1/n^2)\ ,$$

where

$$G_1=mp-1/3-mt_1-s_1+u^2/3\ ,$$

(31) $$G_2=\{mp-4/9+mt_2+2s_2\}u+u^3/9\ ,$$

$$G_3=G_{31}+G_{32}u^2+G_{33}u^4\ ,$$

$$G_{31}=(4/9)mt_3+m^2t_1^2+(m(3m^2+3m+2)/3)t_2 \\ +(2/3)(3m^2+3m+2)s_3+2mt_1s_1+m\omega+4s_3 \\ +m^2(2m^2+3m-1)/6-2mp/3-116/405\ ,$$

$$G_{32}=2mp/3-152/405-(4/9)mt_3-(2/3)mt_2-(4/3)s_2-4s_3\ ,$$

$$G_{33}=4/135\ .$$

Let y be the pth percentile of $\sqrt{mn/2}\,\mathrm{tr}\{\Sigma^{-1}(X-M)A(X-M)'/q\}$, then

(32) $$y=u-\bar{G}_1/\sqrt{2mn}+\bar{G}_2/(2mn)-\bar{G}_3/(2mn\sqrt{2mn})+O(1/n^2)\ ,$$

where

$$
\begin{aligned}
&\bar{G}_1=2/3-mt_1-s_1+u^2/3\;, \\
(33)\qquad &\bar{G}_2=\{mt_2+2s_2+5/9\}u+u^3/9\;, \\
&\bar{G}_3=\bar{G}_{31}+\bar{G}_{32}u^2+\bar{G}_{33}u^4\;, \\
&\quad \bar{G}_{31}=-116/405+(4/9)mt_3+mt_1^2+(m(3m+5)/3)t_2 \\
&\qquad\qquad +(2/3)(3m+5)s_2+2t_1s_1+w+4s_3\;, \\
&\quad \bar{G}_{32}=118/405-(4/9)mt_3-(2/3)mt_2-(4/3)s_2-4s_3\;, \\
&\quad \bar{G}_{33}=4/135\;.
\end{aligned}
$$

4. Corresponding results for complex Gaussian variables

In this section we shall state the results for complex Gaussian distribution.

Let $T=T^R+iT^I=(t^R_{\alpha\beta})+i(t^I_{\alpha\beta})$ and $U=U^R+iU^I=(U^R_{\alpha\beta})+i(U^I_{\alpha\beta})$ be $m\times n$ $(m<n)$ complex arbitrary matrices whose ranks are m, respectively, and A an $n\times n$ positive definite Hermitian matrix. We define $\tilde{P}_\kappa(T, A)$ as

$$
\begin{aligned}
(34)\qquad &\operatorname{etr}(-T\bar{T}')\tilde{P}_\kappa(T, A) \\
&=(-1)^k\pi^{-mn}\int_U \operatorname{etr}(-U\bar{U}')\operatorname{etr}(-i(T\bar{U}'+U\bar{T}'))\tilde{C}_\kappa(UA\bar{U}')dU\;,
\end{aligned}
$$

where $dU=\prod_{\alpha=1}\prod_{\beta=1} du^R_{\alpha\beta}du^I_{\alpha\beta}$, and the integral is done on whole $2mn$ dimensional Euclidian space. The explicit expressions for $\tilde{P}_\kappa(T, A)$'s up to $k=3$ may be found in [5]. We have completely similar set of results as Lemma 1.

LEMMA 3. *Let $\tilde{P}_\kappa(T, A)$ defined by (34) be a polynomial corresponding to partition $\kappa=\{k_1,\cdots,k_m\}$ of k into not more than m parts for T and A defined above. Put $\hat{T}=T(I+xA)^{-1/2}$ and $\hat{A}=(I+xA)^{-1/2}A(I+xA)^{-1/2}$ for $\|xA\|<1$. Then the following equalities hold.*

$$
(35)\qquad \sum_{k=r}^{\infty}\sum_{\kappa}(x^k/(k-r)!)\tilde{P}_\kappa(T, A)=\tilde{d}(x, T, A)x^r\sum_{\tau}\tilde{P}_\tau(\hat{T}, \hat{A})\;,
$$

$$
\begin{aligned}
(36)\qquad &\sum_{k=0}^{\infty}\sum_{\kappa}(x^k/k!)(\tilde{a}_1(\kappa)+k)\tilde{P}_\kappa(T, A) \\
&=\tilde{d}(x, T, A)x^2[\tilde{P}_{(2)}(\hat{T}, \hat{A})-\tilde{P}_{(1^2)}(\hat{T}, \hat{A})]\;,
\end{aligned}
$$

where $\tilde{a}_1(\kappa)=\sum_{\alpha=1}^{m}k_\alpha(k_\alpha-2\alpha)$ *and* $\tilde{d}(x, T, A)=\det(I+xA)^{-m}\operatorname{etr}\{T(I-(I+xA)^{-1})\bar{T}'\}$.

PROOF. We can show (35) and (36) completely same way as Lemma 1 by using (2.10) and (2.11) in [8].

Note. We have similar set of the more weighted formulas for $\tilde{P}_\kappa(T, A)$ in [10].

To have the similar theorem as Theorem 1 for complex case, we use the following proposition.

PROPOSITION 2 (Theorem 8, [6]). Let X be an $m\times n$ complex matrix with p.d.f. $\pi^{-mn}\,\mathrm{etr}\,(-\Sigma^{-1}M\bar{M}')$, $A=\mathrm{diag}\,(a_1,\cdots,a_n)$, $a_1\geqq a_2\geqq\cdots\geqq a_n>0$, and M an $m\times n$ arbitrary complex matrix, then the joint p.d.f. of the latent roots $\Lambda=\mathrm{diag}\,(\lambda_1,\cdots,\lambda_m)$ of $nZ=\Sigma^{-1/2}(X-M)A(\overline{X-M})'\Sigma^{-1/2}$ is expressed as follows,

$$\sum_{k=0}^{\infty}\sum_{\kappa}\tilde{R}_\kappa\tilde{f}_\kappa(\Lambda)\ , \qquad \text{for } 0<q<a_n\ , \tag{37}$$

where

$$\tilde{f}_\kappa(\Lambda)=(n/q)^{mn}(\pi^{m^2}/(\tilde{\Gamma}_m(n\,;\kappa)\tilde{\Gamma}_m(m)\tilde{C}_\kappa(I_m)))\,\mathrm{etr}\,(-(n/q)\Lambda)(\det\Lambda)^{n-m} \cdot \tilde{C}_\kappa((n/q)\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)^2\ , \tag{38}$$

and

$$k!\tilde{R}_\kappa=(\det\tilde{B})^{-m}\,\mathrm{etr}\,(-\tilde{\Omega})(-1)^k\tilde{P}_\kappa(iT^*,\, I-B^{-1})\ , \tag{39}$$

where $T^*=[T^R, T^I]_{m\times 2n}$ and $T=T^R+iT^I=\Sigma^{-1/2}M(\tilde{B}-I)^{-1/2}$, and $\tilde{B}=A/q$ and $\tilde{\Omega}=\bar{M}'\Sigma^{-1}M$. $\tilde{R}_\kappa>0$, for all partition κ of k with probability one,

$$\sum_{k=0}^{\infty}\sum_{\kappa}\tilde{R}_\kappa=1\ . \tag{40}$$

Combining Lemma 3 and this proposition, we have the following theorem by completely same way as Theorem 2.

LEMMA 4. *Let R_κ's be defined as* (39), *and* $\tilde{t}_j=\mathrm{tr}\,(\tilde{B}-I)^j$, $\tilde{s}_j=\mathrm{tr}\,\tilde{B}(\tilde{B}-I)^{j-1}\tilde{\Omega}$, $j=1, 2, 3$, *and* $\tilde{\omega}=\mathrm{tr}\,(\tilde{B}\tilde{\Omega})^2$, *then the following equalities hold,*

$$\tilde{F}_1\equiv\sum_{k=1}^{\infty}\sum_{\kappa}k\tilde{R}_\kappa=m\tilde{t}_1+\tilde{s}_1 \tag{41}$$

$$\begin{aligned}\tilde{F}_2&\equiv\sum_{k=2}^{\infty}\sum_{\kappa}k(k-1)\tilde{R}_\kappa\\&=m^2+\tilde{t}_1^2+m\tilde{t}_2+2\tilde{t}_1\tilde{s}_1+2\tilde{s}_2+\tilde{s}_1^2\end{aligned} \tag{42}$$

$$\begin{aligned}\tilde{F}_3&\equiv\sum_{k=3}^{\infty}\sum_{\kappa}k(k-1)(k-2)\tilde{R}_\kappa\\&=m^3\tilde{t}_1^3+3m^2\tilde{t}_1\tilde{t}_2+2m\tilde{t}_3+3\tilde{s}_1(m^2\tilde{t}_1^2+m\tilde{t}_2)+3m\tilde{t}_1(\tilde{s}_1^2+2s_2)+6\tilde{s}_3\\&\quad+6\tilde{s}_1\tilde{s}_2+\tilde{s}_1^3\end{aligned} \tag{43}$$

$$(44)\qquad \tilde{F}_4 \equiv \sum_{k=0}^{\infty} \sum_{\kappa} (\tilde{a}_1(\kappa)+k)\tilde{R}_\kappa = m\tilde{t}_1^2 + m^2\tilde{t}_2 + 2m\tilde{s}_2 + 2\tilde{t}_1\tilde{s}_1 + \tilde{\omega} .$$

THEOREM 3. *Let X, M, A and nZ be defined as in Proposition 2. Then the asymptotic expansion of the distribution function of $\hat{\lambda}=\sqrt{n/m}\cdot\log(\det(Z/q))$ is given as*

$$(45)\qquad \Phi(x)+(1/\sqrt{mn})\sum_{\alpha=1}^{2}\tilde{l}_{1\alpha}\Phi^{(2\alpha-1)}(x)+(1/mn)\sum_{\alpha=1}^{3}\tilde{l}_{2\alpha}\Phi^{(2\alpha)}(x) + (1/mn\sqrt{mn})\sum_{\alpha=1}^{5}\tilde{l}_{3\alpha}\Phi^{(2\alpha-1)}(x)+O(1/n^2) ,$$

and all the coefficients are obtained by the following transformations of parameters in (22),

$$(46)\qquad \begin{aligned} &p\to m \\ &l_{j\alpha}\to 2^j\tilde{l}_{j\alpha} , \qquad j=1, 2, 3, \text{ except } \tilde{l}_{31} , \\ &\tilde{l}_{31}=(1/12)\{m^2(2m^2-1)+6m\tilde{F}_4\} , \\ &(F_1, F_2)\to(2\tilde{F}_1, 2\tilde{F}_2) , \qquad (F_3, F_4)\to(4\tilde{F}_3, 4\tilde{F}_4) , \end{aligned}$$

respectively.

Note. The term of order $1/n^2$ has been calculated in [10].

Since the p.d.f. of $mn\tilde{T}=\operatorname{tr}\Sigma^{-1}(X-M)A(\overline{X-M})'$ is given by

$$(47)\qquad f(\tilde{T})=(mn/q)^{mn}(1/\Gamma(mn))\exp(-(mn/q)\tilde{T})\tilde{T}^{mn-1} \cdot\sum_{k=0}^{\infty}(((mn/q)\tilde{T})^k/(mn)_k)\sum_{\kappa}\tilde{R}_\kappa , \qquad 0<q<a_n,\ ([6],\ (85)) ,$$

we have the following theorem.

THEOREM 4. *Let $\tilde{T}$ be distributed with p.d.f.* (47), *then the asymptotic expansion of the p.d.f. of $x=\sqrt{mn}\log(\tilde{T}/q)$ is given by*

$$(48)\qquad \tilde{f}(x)=\phi(x)\left\{1+\sum_{k=1}^{3}\tilde{B}_k/(\sqrt{mn})^k+O(1/n^2)\right\} ,$$

and all the coefficients are obtained by changing the parameters in (26) *as following rule:*

$$(49)\qquad \begin{aligned} &B_j\to 2^j\tilde{B}_j , \qquad j=1, 2, 3 \\ &(F_1, F_2)\to(2\tilde{F}_1, 2\tilde{F}_2) , \qquad (F_3, F_4)\to(4\tilde{F}_3, 4\tilde{F}_4) . \end{aligned}$$

We have the asymptotic expansion of the pth percentile of $\hat{\lambda}=\sqrt{n/m}\cdot\log\{\det(Z/q)\}$ and $\sqrt{mn}\operatorname{tr}\Sigma^{-1}(X-M)A(\overline{X-M})'$, similar as the real case.

COROLLARY 4. *Let u be the pth percentile of the standard normal random variable, and x the pth one of $\hat{\lambda}$, then*

$$x=u-\tilde{B}_1/\sqrt{mn}+\tilde{B}_2/mn-\tilde{B}_3/mn\sqrt{mn}+O(1/n^2)\ , \tag{50}$$

where

$$\begin{aligned}
&\tilde{B}_1=(1/6)(3m^2-1)-m\tilde{t}_1-\tilde{s}_1+u^2/6\\
&\tilde{B}_2=\{m^2/4-1/9+(1/2)\{m\tilde{t}_1^2+2\tilde{s}_2\}\}u+u^3/36\\
&\tilde{B}_3=\tilde{B}_{31}+\tilde{B}_{32}u^2+\tilde{B}_{33}u^4\ ,\\
&\qquad\tilde{B}_{31}=(1/6)[2m\tilde{t}_3+3m^2\tilde{t}_1^2+(3m^3+m)\tilde{t}_2+6m\tilde{t}_1\tilde{s}_1+3m\tilde{\omega}+6\tilde{s}_3\\
&\qquad\qquad+(6m^2+2)\tilde{s}_2+m^4-m^2+29/135]\\
&\qquad\tilde{B}_{32}=(1/6)[m^2/2-38/135-2m\tilde{t}_3-m\tilde{t}_2-2\tilde{s}_2-6\tilde{s}_3]\\
&\qquad\tilde{B}_{33}=1/270\ .
\end{aligned}\tag{51}$$

Let y be the pth percentile of $\sqrt{mn}\,\mathrm{tr}\,\Sigma^{-1}(X-M)A(\overline{X-M})'$, then

$$y=u-\hat{\tilde{B}}_1/\sqrt{mn}+\hat{\tilde{B}}_2/mn-\hat{\tilde{B}}_3/mn\sqrt{mn}+O(1/n^2)\ , \tag{52}$$

where

$$\begin{aligned}
&\hat{\tilde{B}}_1=1/3-m\tilde{t}_1-\tilde{s}_1+u^2/6\\
&\hat{\tilde{B}}_2=\{5/36+(1/2)m\tilde{t}_2+\tilde{s}_2\}u+u^3/36\\
&\hat{\tilde{B}}_3=\hat{\tilde{B}}_{31}+\hat{\tilde{B}}_{32}u^2+\hat{\tilde{B}}_{33}u^4\ ,\\
&\qquad\hat{\tilde{B}}_{31}=(1/6)[2m\tilde{t}_3+3m^2\tilde{t}_1^2+(3m^3+m)\tilde{t}_2+6m\tilde{t}_1\tilde{s}_1+3m\tilde{\omega}+6\tilde{s}_3\\
&\qquad\qquad+(6m^2+2)\tilde{s}_2+29/135]\\
&\qquad\hat{\tilde{B}}_{32}=(1/6)[59/270-2m\tilde{t}_3-m\tilde{t}_2-2\tilde{s}_2-6\tilde{s}_3]\\
&\qquad\hat{\tilde{B}}_{33}=1/270\ .
\end{aligned}\tag{53}$$

Acknowledgements

The author heartly thanks the referee who has given him some useful advices to revise the original version.

THE INSTITUTE OF STATISTICAL MATHEMATICS

REFERENCES

[1] Constantine, A. G. (1963). Some non-central distribution problems in multivariate analysis, *Ann. Math. Statist.*, **34**, 1270-1285.

[2] Fujikoshi, Y. (1968). Asymptotic expansion of the distribution of the generalized variance in the non-central case, *J. Sci. Hiroshima Univ.*, Ser. A-I, **32**, 293-299.

[3] Fujikoshi, Y. (1970). Asymptotic expansions of the distributions of test statistics in multivariate analysis, *J. Sci. Hiroshima Univ.*, Ser. A-I, **34**, 73-144.

[4] Hayakawa, T. (1966). On the distribution of a quadratic form in multivariate normal sample, *Ann. Inst. Statist. Math.*, **18**, 191-201.

[5] Hayakawa, T. (1969). On the distribution of the latent roots of a positive definite random symmetric matrix I, *Ann. Inst. Statist. Math.*, **21**, 1-21.

[6] Hayakawa, T. (1972). On the distribution of the multivariate quadratic form in multivariate normal samples, *Ann. Inst. Math. Statist.*, **24**, 205-230.

[7] Hayakawa, T. (1972). Note on the asymptotic distributions of the functions of an multivariate quadratic form in normal sample, *Ann. Inst. Statist. Math.*, **24**, 509-515.

[8] Hayakawa, T. (1972). The asymptotic distributions of the statistics based on the complex Gaussian distribution, *Ann. Inst. Statist. Math.*, **24**, 231-244.

[9] Hayakawa, T. (1972). On the asymptotic distribution of the determinant of the multivariate quadratic form in normal sample, *Research Memorandum*, No. 47, Inst of Statist. Math., Tokyo.

[10] Hayakawa, T. (1972). Further contribution to the asymptotic distribution of the determinant of the multivariate quadratic form in normal sample, *Research Memorandum*, No. 48, Inst. Statist. Math., Tokyo.

[11] Hill, G. W. and Davis, A. W. (1968). Generalized asymptotic expansions of Cornish-Fisher type, *Ann. Math. Statist.*, **39**, 1264-1273.

[12] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[13] Khatri, C. G. (1966). On certain distribution problems based on positive definite quadratic functions in normal vectors, *Ann. Math. Statist.*, **37**, 468-479.

[14] Shah, B. H. (1970). Distribution theory of positive definite quadratic form with matrix argument, *Ann. Math. Statist.*, **41**, 692-697.

[15] Sugiura, N. and Fujikoshi, Y. (1969). Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence, *Ann. Math. Statist.*, **40**, 942-952.