ON THE EXTENSIONS OF GAUSS-MARKOV THEOREM TO SUBSETS OF THE PARAMETER SPACE UNDER COMPLEX MULTIVARIATE LINEAR MODELS*

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Summary

In this paper we continue the study of the extension of the Gauss-Markov theorem to certain general kinds of multiresponse models. In particular we obtain necessary and sufficient conditions, for the general incomplete multiresponse (GIM) model and the multiple design multiresponse (MDM) model, such that unique best linear unbiased estimates (BLUE's) exist for all elements in a subset of the set of all estimable linear functions of the location parameters. Also, the theory is illustrated by a couple of nontrivial examples.

1. Introduction and preliminaries

Assume that a set of n experimental units is divided into u disjoint subsets S_i, $(i=1, 2, \cdots, u)$, of n_i units. Further assume that the subset $\{V_{l_{i1}}, \cdots, V_{l_{ip_i}}\}$ of the p response $\{V_1, \cdots, V_p\}$ is measured on the units of S_i. Let Σ denote the unknown dispersion matrix of the p responses and let $Y_i(n_i \times p_i)$, $(i=1, \cdots, u)$, be the matrix of observations on the units of S_i. For Y_i, the standard multiresponse (SM) model is applicable, i.e.

$$\text{(1.1a)} \qquad \mathrm{E}(Y_i) = A_i(\boldsymbol{\xi}_{l_{i1}}, \cdots, \boldsymbol{\xi}_{l_{ip_i}}) ;$$

$$\text{(1.1b)} \qquad \mathrm{Var}(Y_i) = I_{n_i} \otimes \Sigma_i , \qquad i=1, 2, \cdots, u ;$$

where $A_i(n_i \times m)$ is the design-model matrix for S_i, $(\boldsymbol{\xi}_{l_{i1}}, \cdots, \boldsymbol{\xi}_{l_{ip_i}})$ is the set of p_i columns of the $(m \times p)$ matrix of unknown parameters $\xi = (\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_p)$ corresponding to the p_i responses measured on S_i and Σ_i is the principal submatrix of Σ corresponding to the p_i responses measured on S_i. Then the model given by (1.1a, b) is called the GIM model. Now,

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let $U_r = \{S_i | V_r$ is measured on the units of S_i, $(i=1, \cdots, u)\}$. If there is a permutation of $(1, \cdots, p)$, say $(r_1, \cdots, r_p)$, such that $U_{r_1} \supseteq \cdots \supseteq U_{r_p}$ then the GIM model reduces to an important special case, viz. the hierarchical multiresponse (HM) model. Without loss of generality, this can be written as

$$\text{(1.2a)} \qquad \mathrm{E}(Y_i) = A_i(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_i)$$

$$\text{(1.2b)} \qquad \mathrm{Var}(Y_i) = I_{n_i} \otimes \Sigma_i \; ; \qquad i=1, \cdots, p \; .$$

Actually, in Roy and Srivastava [2] where the HM model was first introduced, it appeared in a more general form than the above in that the $\boldsymbol{\xi}_i$ were allowed to be of different dimensions for different i. We shall however work with the less general form (1.2a, b) in this paper. Finally, the MDM model is given by

$$\text{(1.3a)} \qquad \mathrm{E}(Y) = (A_1\boldsymbol{\xi}_1, \cdots, A_p\boldsymbol{\xi}_p) \; ,$$

$$\text{(1.3b)} \qquad \mathrm{Var}(Y) = I_n \otimes \Sigma \; ,$$

where the symbols have the same meaning as before. Thus here we are allowed to have different design matrices A_r corresponding to the different responses V_r. However, the model is responsewise complete since on each unit, every response is measured.

In Srivastava [3], (here-after referred to as paper I), the problem of linear estimation of all parameters in the set $\Theta = \left\{\theta | \theta \text{ is of the form } \sum_{r=1}^{p} \boldsymbol{c}_r' \boldsymbol{\xi}_r, \text{ and } \theta \text{ is estimable}\right\}$ where $\boldsymbol{c}_r'$ are any row vectors of appropriate size, is considered. Also, necessary and sufficient conditions are derived (under each of the above models) such that there exists a (unique) best linear unbiased estimator (BLUE) for every $\theta \in \Theta$. A design satisfying these necessary and sufficient conditions is called an orthogonal multiresponse design (OMD) with respect to Θ.

Similarly, if $\Theta^* \subset \Theta$, then a design is said to be an OMD with respect to Θ^*, if there exists a BLUE for every $\theta \in \Theta^*$. In Section 2 of this paper we give necessary and sufficient conditions on the GIM and HM models such that a design is an OMD with respect to Θ^*, where Θ^* is a given subset of Θ. Section 3 is concerned similarly with the MDM model. Finally, for each model, the theory is illustrated by some important special cases.

2. Orthogonality under the GIM and the HM models

Consider first the standard uniresponse model

$$\text{(2.1a)} \qquad \mathrm{E}(\boldsymbol{y}) = [A_1 | A_2]\begin{bmatrix} \boldsymbol{\xi}_1 \\ \boldsymbol{\xi}_2 \end{bmatrix} = A_1\boldsymbol{\xi}_1 + A_2\boldsymbol{\xi}_2 \; ,$$

(2.1b) $$\mathrm{Var}\,(\boldsymbol{y})=\sigma^2 I_n\ ,$$

where $\boldsymbol{\xi}_1(\overline{m-k}\times 1)$, $\boldsymbol{\xi}_2(k\times 1)$ and σ^2 are unknown. Throughout this section, for any matrix B, the matrix B^- will designate the Moore-Penrose inverse of B, and B^* will denote a conditional inverse of B (i.e., B^* is such that $BB^*B=B$). Let W_0 and W_{01} denote respectively the column spaces of $A=[A_1|A_2]$ and $(I-A_1A_1^-)A_2$. Then it is easy to see that $W_{01}\subseteq W_0$.

LEMMA 2.1. *If $\boldsymbol{g}'\boldsymbol{y}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\xi}_2$ then $\boldsymbol{g}\in W_{01}$.*

PROOF. Let

(2.2) $$A^*=\left[\frac{A_1^- - A_1^-A_2(A_2'(I-A_1A_1^-)A_2)^-A_2'(I-A_1A_1^-)}{(A_2'(I-A_1A_1^-)A_2)^-A_2'(I-A_1A_1^-)}\right]\ .$$

Then it is easily checked that a conditional inverse of $(A'A)$ is $(A'A)^*=A^*(A^*)'$ and further that $(AA^*)'=(A^*)'A'=AA^*$, $A^*AA^*=A^*$, $AA^*A=A$. Thus

$$\boldsymbol{g}'=[\boldsymbol{O}'|\boldsymbol{c}'](A'A)^*A'=[\boldsymbol{O}'|\boldsymbol{c}']A^*=\boldsymbol{c}'(A_2'(I-A_1A_1^-)A_2)^-A_2'(I-A_1A_1^-)\ .$$

This completes the proof.

LEMMA 2.2. *Given $\boldsymbol{g}\in W_{01}$, there exists a vector $\boldsymbol{c}'$ in the row space of A_2 such that $\boldsymbol{g}'\boldsymbol{y}$ is the BLUE of $\boldsymbol{c}'\boldsymbol{\xi}_2$.*

PROOF. Let $\boldsymbol{g}=(I-A_1A_1^-)A_2\boldsymbol{l}$, then $\mathrm{E}\,(\boldsymbol{g}'\boldsymbol{y})=\boldsymbol{l}'A_2'(I-A_1A_1^-)A_2\boldsymbol{\xi}_2=\boldsymbol{c}'\boldsymbol{\xi}_2$ (say). Using the same argument as in the proof of the last lemma, the BLUE of $\boldsymbol{c}'\boldsymbol{\xi}_2$ is

(2.3) $$\boldsymbol{c}'\hat{\boldsymbol{\xi}}_2=\boldsymbol{l}'A_2'(I-A_1A_1^-)A_2[A_2'(I-A_1A_1^-)A_2]^-A_2'(I-A_1A_1^-)\boldsymbol{y}\ .$$

Noting that $(I-A_1A_1^-)$ is idempotent and that $X'=X'X(X'X)^-X'$ for any matrix X, Eq. (2.3) reduces to $\boldsymbol{c}'\hat{\boldsymbol{\xi}}_2=\boldsymbol{l}'A_2'(I-A_1A_1^-)\boldsymbol{y}=\boldsymbol{g}'\boldsymbol{y}$. This completes the proof.

We consider next the hierarchical model defined by (1.2a, b). Let $B_r'(m\times m_r)=[A_r'|A_{r+1}'|\cdots|A_p']$, $(r=1,\cdots,p)$, where $m_r=n_r+n_{r+1}+\cdots+n_p$. Letting $\boldsymbol{y}_r(m_r\times 1)$, $(r=1,\cdots,p)$, denote the vector of observations on response V_r, we can write $\mathrm{E}\,(\boldsymbol{y}_r)=B_r\boldsymbol{\xi}_r$. Given $\theta=\left(\sum_{r=1}^{p}\boldsymbol{c}_r'\boldsymbol{\xi}_r\right)\in\Theta$, let $\boldsymbol{g}_r'\boldsymbol{y}_r$, $(r=1,\cdots,p)$, be the BLUE of $\boldsymbol{c}_r'\boldsymbol{\xi}_r$ when the data on all responses except the rth (i.e., $\boldsymbol{y}_r$) are ignored. Then $\hat{\theta}=\sum_{r=1}^{p}\boldsymbol{g}_r'\boldsymbol{y}_r$ is called the 'piecewise estimator' of θ.

Let W_r, $(r=1,\cdots,p)$, be the column space of B_r, $\overline{W}_r$ (of dimensionality μ_r, say) be the space orthogonal to W_r and the columns of the $(m_r \times \mu_r)$ matrix K_r be an orthogonal basis of $\bar{W}_r$. Let O_{ij} denote a $(i \times j)$ matrix, each element of which is zero. From Theorems 2.4 and 2.5 of paper I, we have that under the HM model a necessary and sufficient condition that there exists a BLUE of $\theta \in \Theta$ is that

$$(2.3a) \qquad K_s'\left[\frac{O_{(m_s-m_r),1}}{g_r}\right]=O_{\mu_s,1}\ ,$$

$$(2.3b) \qquad [O_{\mu_r,(m_s-m_r)} \mid K_r']g_s=O_{\mu_r,1}\ , \qquad (r>s;\ r,s=1,\cdots,p)\ ,$$

where $\hat{\theta}=\sum_{r=1}^{p} g_r' y_r$ is the piecewise estimator of θ. Further if the BLUE of θ exists then it is given by the piecewise estimator $\hat{\theta}$.

For $r=1,\cdots,p$, let $\xi_r'=[\xi_{r1}' \mid \xi_{r2}']$ where ξ_{r1} and ξ_{r2} have respectively $(m-k_r)$ and k_r elements and where interest is supposed to lie in the parameter ξ_{r2}. Thus

$$(2.5) \quad E(y_r)=B_r\xi_r=[B_{r1} \mid B_{r2}]\left[\frac{\xi_{r1}}{\xi_{r2}}\right]=B_{r1}\xi_{r1}+B_{r2}\xi_{r2}\ , \qquad (r=1,\cdots,p)\ ,$$

where the partition of B_r is induced by that in ξ_r. Let W_{r1}, $(r=1,\cdots,p)$, denote the column space of the $(m_r \times k_r)$ matrix

$$(2.6) \qquad \Delta_r=(I-B_{r1}B_{r1}^{-})B_{r2}\ .$$

Now, we are assuming that interest lies in the estimation of the elements θ in the set

$$(2.7) \qquad \Theta^*=\left\{\theta \mid \theta \in \Theta \text{ and } \theta \text{ is of the form } \sum_{r=1}^{p} c_{r2}'\xi_{r2}\right\}\ .$$

If a HM design is an OMD with respect to Θ^* then clearly the BLUE of θ must exist for every $\theta \in \Theta^*$.

THEOREM 2.1. *A necessary and sufficient condition that a HM design is an OMD with respect to Θ^* is that*

$$(2.8a) \qquad K_s'\left[\frac{O_{(m_s-m_r),k_r}}{\Delta_r}\right]=O_{\mu_s,k_r}\ ,$$

$$(2.8b) \qquad [O_{\mu_r,(m_s-m_r)} \mid K_r']\Delta_s=O_{\mu_r,k_s}\ , \qquad (r>s;\ r,s=1,\cdots,p)\ .$$

PROOF OF SUFFICIENCY. Let $\theta \in \Theta^*$ and let $\hat{\theta}=\sum_{r=1}^{p} g_r' y_r$ denote the piecewise estimator of θ. By Lemma 2.1, $g_r \in W_{r1}$ for $r=1,\cdots,p$. Let $g_r=\Delta_r l_r$, $(r=1,\cdots,p)$. Now for $r>s$, we have from (2.8a) that

$$(2.9) \qquad K_s'\left[\frac{O_{(m_s-m_r),1}}{g_r}\right]=K_s'\left[\frac{O_{(m_s-m_r),k_r}}{\Delta_r}\right]l_r=O\ .$$

Similarly, from (2.8b)

$$[O_{\mu_r,(m_s-m_r)}|K_r']g_s=[O_{\mu_r,(m_s-m_r)}|K_r']\Delta_s l_s=\boldsymbol{O} . \tag{2.10}$$

Thus the BLUE of θ exists and is given by $\hat{\theta}$.

PROOF OF NECESSITY. Let $e_{ir}(k_r\times 1)$, $(i=1,\cdots,k_r;\ r=1,\cdots,p)$, denote a vector with unity in the ith position and zeros elsewhere. Now for fixed values of r and i, $(\Delta_r e_{ir})\in W_{r1}$, and by Lemma 2.2 there exists an element $\theta\in\Theta^*$ such that $(\Delta_r e_{ir})'y_r$ is the piecewise estimator of θ. Hence by assumption, $(\Delta_r e_{ir})'y_r$ is the BLUE of θ and from (2.4a, b) we have

$$K_s'\left[\frac{O_{(m_s-m_r),1}}{\Delta_r e_{ir}}\right]=\boldsymbol{O}_{\mu_s,1} , \tag{2.11a}$$

$$[O_{\mu_r,(m_s-m_r)}|K_r']\Delta_s e_{is}=\boldsymbol{O}_{\mu_r,1} , \qquad s=1,\cdots,r-1 . \tag{2.11b}$$

Letting r run from 1 to p and i run from 1 to k_r, this clearly implies that Eqs. (2.8a, b) hold.

Suppose $r>s$, $(r,s=1,\cdots,p)$. We must introduce some further notation at this point since we are allowing for the possibility that $\xi_{r2}(k_r\times 1)$, and $\xi_{s2}(k_s\times 1)$ are not of the same dimension, and since B_r consists of the last m_r rows of B_s. Let $(B_{sr}^*)'=[A_s'|\cdots|A_{r-1}']$. Thus

$$B_s'=[A_s'|\cdots|A_p']=[(B_{sr}^*)'|B_r'] . \tag{2.12}$$

Furthermore, we define the partitions $B_{sr}^*=[B_{sr1}^{*i}|B_{sr2}^{*i}]$ and $B_r=[B_{r1}^i|B_{r2}^i]$ to be such that the number of columns in B_{srj}^{*i} and in B_{rj}^i equals the number of elements in ξ_{ij}, $(j=1,2;\ i=r,s)$. Recall from (2.5) that ξ_{ij} has $(m-k_i)$ or k_i columns accordingly as $j=1$ or 2. Also note that $B_{rj}=B_{rj}^r$, $(r=1,\cdots,p;\ j=1,2)$, and

$$B_s=[B_{s1}^i|B_{s2}^i]=\left[\begin{array}{c|c} B_{sr1}^{*i} & B_{sr2}^{*i} \\ \hline B_{r1}^i & B_{r2}^i\end{array}\right], \qquad (r>s;\ r,s=1,\cdots,p;\ i=r,s) . \tag{2.13}$$

Consider Eq. (2.8b). The matrix $\Delta_s(m_s\times k_s)$, $(s=1,\cdots,p)$, can be written in the form

$$\Delta_s=\left(\left[\begin{array}{c|c} I_{(m_s-m_r)} & 0 \\ \hline 0 & I_{m_r}\end{array}\right]-\left[\frac{B_{sr1}^{*s}}{B_{r1}^s}\right]B_{s1}^-\right)B_{s2} , \qquad (r>s) . \tag{2.14}$$

Thus Eq. (2.8b) is equivalent to

$$K_r'([0|I_{m_r}]-B_{r1}^sB_{s1}^-)B_{s2}=0 . \tag{2.15}$$

Now $K_r'B_r=0$ by the definition of K_r. Thus since $B_r=[B_{r1}^s|B_{r2}^s]$ we have $K_r'B_{r1}^s=0$ and Eq. (2.15) is equivalent to

$$[0 \mid K_r'] B_{s2} = 0 \; . \tag{2.16}$$

But (2.12) and (2.13) indicate that this equation is always satisfied since the last m_r rows of B_{s2} (i.e., rows of B_{r2}^s) consist of part of the matrix B_r. Thus it is found that in Theorem 2.1 the condition (2.8b) may be removed without invalidating the result. We have established

COROLLARY 2.1. *In the HM model a necessary and sufficient condition that a design be an OMD with respect to Θ^* is that Eq.* (2.8a) *be satisfied.*

For $r > s$, $(r, s = 1, \cdots, p)$, let $K_s' = [(K_{s1}^r)' \mid (K_{s2}^r)']$, where $(K_{s1}^r)'$ and $(K_{s2}^r)'$ have respectively $(m_s - m_r)$ and m_r columns. Write B_s in the form

$$B_s = \left[\frac{B_{sr1}^{*r}}{B_{r1}} \middle| \frac{B_{sr2}^{*r}}{B_{r2}} \right] . \tag{2.17}$$

Since $K_s' B_s = 0$ we have

$$(K_{s1}^r)' B_{sr2}^{*r} = -(K_{s2}^r)' B_{r2} \; . \tag{2.18}$$

The left-hand side of (2.8a) equals

$$(K_{s2}^r)'(I_{m_r} - B_{r1} B_{r1}^-) B_{r2} = (K_{s2}^r)' B_{r2} - (K_{s2}^r)' B_{r1} B_{r1}^- B_{r2} \; . \tag{2.19}$$

Thus from (2.18), Eq. (2.8a) is equivalent to

$$(K_{s1}^r)' B_{sr2}^{*r} + (K_{s2}^r)' B_{r1} B_{r1}^- B_{r2} = 0 \; , \tag{2.20a}$$

or

$$K_s' \left[\frac{B_{sr2}^{*r}}{B_{r1} B_{r1}^- B_{r2}} \right] = 0 \; . \tag{2.20b}$$

In other words, (2.8a) is equivalent to the requirement that the rows of $[(B_{sr2}^{*r})' \mid (B_{r1} B_{r1}^- B_{r2})']$ belong to W_s, the column space of B_s, or

$$R(B_s) = R\left[B_s \middle| \frac{B_{sr2}^{*r}}{B_{r1} B_{r1}^- B_{r2}} \right] . \tag{2.21}$$

THEOREM 2.2. *Under the HM model a necessary and sufficient condition that a design be an OMD with respect to Θ^* is that*

$$R(B_1) = R(B_1 \mid \Delta_{1\cdot 2} \mid \Delta_{1\cdot 3} \mid \cdots \mid \Delta_{1\cdot p}) \; , \tag{2.22}$$

where $\Delta_{1\cdot r}' = [(B_{1r2}^{*r})' \mid (B_{r1} B_{r1}^- B_{r2})']$, $(r = 2, 3, \cdots, p)$.

PROOF OF NECESSITY. Hold $s = 1$ and let r take on each of the values $2, 3, \cdots, p$ in Eq. (2.21).

PROOF OF SUFFICIENCY. Let $r > s$, $(r, s = 1, \cdots, p)$, and recall that

$B_1'=[A_1'|\cdots|A_{s-1}'|B_s']$. Also note that B_{sr2}^{*i}, $(i=r, s)$, consists of the last (m_s-m_r) rows of B_{1r2}^{*i}. Eq. (2.22) implies that there exists a matrix L_r (say) such that $(B_1L_r)'=[(B_{1r2}^{*r})'|(B_{r1}B_{r1}^{-}B_{r2})']$. Hence $(B_sL_r)'=[(B_{sr2}^{*r})'|(B_{r1}B_{r1}^{-}B_{r2})']$. Thus Eq. (2.21) is satisfied for $r>s$, $(r, s=1,\cdots, p)$. This completes the proof.

The special case when $\Theta^*=\Theta$ was given in Theorem 5.4 (Paper I). This is easily checked since $\Theta^*=\Theta$ implies that (2.22) reduces to $R(B_1)=\sum_{r=1}^{p} R(A_r)$.

Consider now the GIM model defined in Eqs. (1.1a, b). Hold r fixed and select $s=1,\cdots, p$ such that $s\neq r$. Letting $\boldsymbol{y}_r$ denote the vector of observations on response V_r we can write

$$\text{(2.23)} \qquad \mathrm{E}\,(\boldsymbol{y}_r)=B_r\boldsymbol{\xi}_r=[B_{r1}|B_{r2}]\left[\frac{\boldsymbol{\xi}_{r1}}{\boldsymbol{\xi}_{r2}}\right], \qquad (r=1,\cdots, p)\,.$$

Without loss of generality we can assume that $B_r'=[(B_{rs0}^*)'|B_{rs0}']$, where B_{rs0} consists of the rows of B_r which correspond to units on which both responses V_r and V_s are measured and B_{rs0}^* consists of the remaining rows of B_r.

For notation let

$$\text{(2.24)} \qquad B_r=\left[\frac{B_{rs0}^*}{B_{rs0}}\right]=\left[\begin{array}{c|c} B_{rs1}^{*i} & B_{rs2}^{*i} \\ \hline B_{rs1}^{i} & B_{rs2}^{i}\end{array}\right]=[B_{r1}|B_{r2}]\,,$$
$$(r\neq s;\ r, s=1,\cdots, p;\ i=r, s)\,,$$

where the number of columns in B_{rsj}^i, B_{rsj}^{*i}, or B_{ij} is the same as the number of elements in $\boldsymbol{\xi}_{ij}$ for all permissible values of the suffixes. Also, note that

$$\text{(2.25)} \qquad \begin{aligned} B_{r1}&=\left[\frac{B_{rs1}^{*r}}{B_{rs1}^{r}}\right], & B_{r2}&=\left[\frac{B_{rs2}^{*r}}{B_{rs2}^{r}}\right], \\ B_{rs0}^*&=[B_{rs1}^{*i}|B_{rs2}^{*i}]\,, & B_{rs0}&=[B_{rs1}^{i}|B_{rs2}^{i}]\,. \end{aligned}$$

Starting with Theorem 3.2 (Paper I), and proceeding in a manner similar to that used to prove Theorems 2.1 and 2.2 above, we can also establish Theorem 2.3, the generalization of Theorem 5.3 (Paper I). We are omitting the proof here because of its length.

THEOREM 2.3. *Under the GIM model, a necessary and sufficient condition that a design be an OMD with respect to* $\Theta^*=\left\{\theta=\sum_{r=1}^{p} \boldsymbol{c}_r'\boldsymbol{\xi}_{r2}\,|\,\theta\in\Theta\right\}$ *is that*

$$\text{(2.26)} \qquad R(B_r)=R(B_r|E_{rs})\,, \qquad (r\neq s;\ r, s=1,\cdots, p)\,,$$

where

$$E'_{rs}=[(B^{*s}_{rs2})' \mid (B^{s}_{rs1}B^{-}_{s1}B_{s2})'] \,. \tag{2.27}$$

The verification in special cases of the conditions (2.22) and (2.26) of Theorems 2.2 or 2.3, is often made easier by using the identity

$$XX^{-}=X(X'X)^{*}X' \tag{2.28}$$

where X is an arbitrary matrix and $(X'X)^*$ denotes any conditional inverse of $(X'X)$.

We now illustrate the theory of this section by

Example 1. Let $p=2$ and $u=3$, where response V_r, $(r=1, 2)$, is measured on the sets S_r and S_3. Let

$$A_r(n_r\times 2)=[\boldsymbol{x}_r \mid \rho\boldsymbol{x}_r] \,, \qquad \boldsymbol{\xi}'_r=(\alpha_r, \beta_r) \,, \qquad (r=1, 2) \,, \tag{2.29a}$$

$$A_3(n_3\times 2)=\left[\begin{array}{c|c} \boldsymbol{x}_3 & \rho\boldsymbol{x}_3 \\ \hline \boldsymbol{0} & \boldsymbol{x}_4 \end{array}\right] , \tag{2.29b}$$

where $\boldsymbol{x}_i\neq 0$, $(i=1, 2, 3, 4)$, and $\rho\neq 0$. Now $R(B'_1)=R(A'_1 \mid A'_3)=2$, $R(B_{120})=R(A_3)=2$ and $R(B^{*}_{120})=R(A_1)=1$. Thus since $R(B_1)\neq R(B_{120})+R(B^{*}_{120})$, we have by Theorem 5.3 (Paper I) that the design is not orthogonal with respect to Θ. Define $\Theta^*=\{\theta \mid \theta=c_1\beta_1+c_2\beta_2$, where c_1 and c_2 are real numbers$\}$. From (2.24) and (2.25), we have $B^{*2}_{122}=\rho\boldsymbol{x}_1$, $B^{2}_{121}=[\boldsymbol{x}'_3 \mid \boldsymbol{0}']$, $B'_{21}=[\boldsymbol{x}'_2 \mid \boldsymbol{x}'_3 \mid \boldsymbol{0}']$, and $B'_{22}=[\rho\boldsymbol{x}'_2 \mid \rho\boldsymbol{x}'_3 \mid \boldsymbol{x}'_4]$. Thus $B^{-}_{21}=(\boldsymbol{x}'_2\boldsymbol{x}_2+\boldsymbol{x}'_3\boldsymbol{x}_3)^{-1}[\boldsymbol{x}'_2 \mid \boldsymbol{x}'_3 \mid \boldsymbol{0}']$, and $(B^{2}_{121}B^{-}_{21}B_{22})'=[\rho\boldsymbol{x}'_3 \mid \boldsymbol{0}']$. From (2.27), we have $E'_{12}=[\rho\boldsymbol{x}'_1 \mid \rho\boldsymbol{x}'_3 \mid \boldsymbol{0}']$, and hence

$$R(B_1 \mid E_{12})=R\left[\begin{array}{c|c|c} \boldsymbol{x}_1 & \rho\boldsymbol{x}_1 & \rho\boldsymbol{x}_1 \\ \hline \boldsymbol{x}_3 & \rho\boldsymbol{x}_3 & \rho\boldsymbol{x}_3 \\ \hline \boldsymbol{0} & \boldsymbol{x}_4 & \boldsymbol{0} \end{array}\right]=2 \,. \tag{2.30}$$

Similarly $R(B_2 \mid E_{21})=2$. Thus from Theorem 2.3, the design is orthogonal with respect to Θ^*. Similarly, it can be checked that the hierarchical design obtained by ignoring one of the sets S_1 or S_2, is also orthogonal with respect to Θ^*.

3. The MDM model

Consider the MDM model defined by (1.3a, b). Let W_r, $(r=1,\cdots, p)$, denote the column space of A_r and let $\overline{W}_r$ denote the space orthogonal to W_r. It is shown in paper I that a necessary and sufficient condition that there exists a BLUE of $\theta\in\Theta$ is that

$$\boldsymbol{g}'_r\boldsymbol{\theta}_{sj}=0 \,, \qquad (r\neq s;\ r, s=1,\cdots, p;\ j=1,\cdots, \mu_s) \,, \tag{3.1}$$

where $\mu_s = n - R(A_s)$, the columns of $K_s = [\theta_{s1} | \cdots | \theta_{s\mu_s}]$ form an orthogonal basis of $\bar{W}_s$, and $\hat{\theta} = \sum_{r=1}^{p} g_r' y_r$ is the piecewise estimator of θ.

Let $\xi_r' = [\xi_{r1}' | \xi_{r2}']$, $(r=1, \cdots, p)$, where ξ_{r1} and ξ_{r2} have respectively $(m-k_r)$ and k_r elements, and define

$$\Theta^* = \left\{ \theta \,|\, \theta \in \Theta \text{ and } \theta \text{ is of the form } \sum_{r=1}^{p} c_r' \xi_{r2} \right\} . \tag{3.2}$$

Also let W_{r1} denote the column space of $\Delta_r = (I_n - A_{r1} A_{r1}^-) A_{r2}$ where the partition $A_r = [A_{r1} | A_{r2}]$ is induced by the partition of $\xi_r' = [\xi_{r1}' | \xi_{r2}']$.

THEOREM 3.1. *Under the MDM model a necessary and sufficient condition that the total design be an OMD with respect to Θ^* is that $W^* \subseteq W$ where $W^* = \bigcup_{r=1}^{p} W_{r1}$ and $W = \bigcap_{r=1}^{p} W_r$.*

PROOF OF SUFFICIENCY. Let $\theta \in \Theta^*$ and let $\hat{\theta} = \sum_{r=1}^{p} g_r' y_r$. By Lemma 2.1, $g_r \in W_{r1}$, $(r=1, \cdots, p)$. Thus $g_r \in W^*$, and by assumption $g_r \in W_s$ for all $s=1, \cdots, p$. Hence $g_r' K_s = 0$, $(r \neq s;\ r, s=1, \cdots, p)$.

PROOF OF NECESSITY. Let $g_0 \in W^*$, i.e. $g_0 \in W_{r1}$ for some value of $r=1, 2, \cdots, p$. Suppose $g_0 \in W_{r1}$, where r is now considered fixed. By Lemma 2.2 there exists an element $\hat{\theta}_0 \in \Theta^*$ such that $\hat{\theta}_0 = g_0' y_r$ is the piecewise estimator of θ_0. Now using conditions (3.1) with fixed r and $s=1, \cdots, p$ we find that $g_0' K_s = 0$. Thus $g_0 \in W_s$ for $s=1, \cdots, p$, and we can conclude that $W^* \subseteq W = \bigcap_{s=1}^{p} W_s$.

COROLLARY 3.1. *Under the MDM model a necessary and sufficient condition that a design be an OMD with respect to Θ^* is that*

$$R(A_r | \Delta_s) = R(A_r) , \qquad (r, s=1, \cdots, p) . \tag{3.3}$$

PROOF OF SUFFICIENCY. If $R(A_r | \Delta_s) = R(A_r)$ for $r, s=1, \cdots, p$, then clearly the columns of Δ_s belong to W_r. Hence W_{s1} is contained in W_r for $r, s=1, \cdots, p$, $\left(\text{that is, } \bigcup_{s=1}^{p} W_{s1} \subseteq \bigcap_{r=1}^{p} W_r\right)$ and we conclude from Theorem 3.1 that the design is an OMD with respect to Θ^*.

PROOF OF NECESSITY. Reversing the above argument we have by Theorem 3.1 that if a design is an OMD with respect to Θ^* then $\bigcup_{s=1}^{p} W_{s1} \subseteq \bigcap_{r=1}^{p} W_r$. Thus $W_{s1} \subseteq W_r$ and hence $R(A_r | \Delta_s) = R(A_r)$ for $r, s=1, \cdots, p$.

We now illustrate the above by

Example 2. Consider the following system

(3.4)

		Column Blocks (1)	(2)	(3)	(4)
Row Blocks	I	A	B	C	D
	II	B	C	D	A
	III	C	D	A	B
	IV	D	A	B	C

Suppose that there are two responses and four treatments (A, B, C, D) under study and that the four row blocks (I–IV) above constitute a good block system for response V_1, while the four column blocks ((1)–(4)) constitute a good block system for response V_2. Also suppose that the row blocks are homogeneous with respect to V_1, and the column blocks are homogeneous with respect to V_2. Assume that the rows of the observation matrix $Y=(\boldsymbol{y}_1, \boldsymbol{y}_2)$ are arranged such that the four observations on treatment A occur first, the four observations on treatment B occur next, etc. Let

$$\boldsymbol{\xi}'_{s1}=(\mu_s, \beta_{s1}, \beta_{s2}, \beta_{s3}, \beta_{s4}) \quad \text{and} \quad \boldsymbol{\xi}'_{s2}=(\tau_{sa}, \tau_{sb}, \tau_{sc}, \tau_{sd}) , \qquad (s=1, 2)$$

where μ_s denotes the overall mean for the sth response, β_{sj} denotes the effect of block j on the sth response and $\tau_{sa}, \tau_{sb}, \tau_{sc}, \tau_{sd}$ denote, respectively, the effects of treatments A, B, C, D on the sth response. We have

$$\text{(3.5)} \qquad \mathrm{E}\,(\boldsymbol{y}_s)=[A_{s1}\,|\,A_{s2}]\left[\frac{\boldsymbol{\xi}_{s1}}{\boldsymbol{\xi}_{s2}}\right]=A_{a1}\boldsymbol{\xi}_{s1}+A_{s2}\boldsymbol{\xi}_{s2} , \qquad (s=1, 2) ,$$

where $A_{s1}=[\boldsymbol{J}_{16,1}\,|\,X_s]$ (say), and $A_{s2}=(I_4\otimes \boldsymbol{J}_{4,1})$. Here $J_{a,b}$ and I_a denote respectively the $(a\times b)$ matrix each element of which is unity, and the $(a\times a)$ identity matrix. Also, $C\otimes D$ denotes the Kronecker product of the matrices C and D, i.e. $C\otimes D=((c_{ij}D))$, where c_{ij} are the elements of C.

Let $\Theta^*=\left\{\theta\,|\,\theta \text{ is a treatment contrast of the form } \sum_{s=1}^{2} \boldsymbol{c}'_s\boldsymbol{\xi}_{s2}\right\}$. Now it is easily checked by writing out the matrices in full that $R(A_1)=7$ while $R(A_1\,|\,A_2)\geqq 8$. Thus by Theorem 5.1 (Paper I) the design is not orthogonal with respect to Θ. However, for $s=1, 2$, we have

$$\text{(3.6)} \quad A'_{s1}A_{s1}=\left[\begin{array}{c|c} 16 & 4\boldsymbol{J}_{1,4} \\ \hline 4\boldsymbol{J}_{4,1} & 4I_4 \end{array}\right] , \qquad (A'_{s1}A_{s1})^*=\left[\begin{array}{c|c} 0 & 0' \\ \hline 0 & (4)^{-1}I_4 \end{array}\right] , \qquad (s=1, 2) .$$

Thus $A_{s1}A^-_{s1}=A_{s1}(A'_{s1}A_{s1})^*A'_{s1}=(4)^{-1}(J_{4,4}\otimes I_4)$, and $\Delta_s=[(I_4\otimes \boldsymbol{J}_{4,1})-(4)^{-1}(J_{4,4}\otimes$

$\boldsymbol{J}_{4,1})]$, $(s=1, 2)$. Now $A_s=[\boldsymbol{J}_{16,1}|X_s|(I_4\otimes \boldsymbol{J}_{4,1})]$, $(s=1, 2)$, and clearly Eq. (3.3) of Corollary 3.1 is satisfied. Hence the design is orthogonal with respect to Θ^*.

As the above example indicates, orthogonality under the MDM model often exists with respect to important subsets Θ^*. In a separate communication we would attempt to characterize such situations using the concept of the relationship algebra (see, for example [1]) of a design.

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