ON THE EXACT NON-NULL DISTRIBUTION OF LIKELIHOOD RATIO CRITERIA FOR COVARIANCE MATRICES[1]

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1. Introduction

Let $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_N$ be independent normal random p-vectors with mean vector $\boldsymbol{\mu}$ and positive definite covariance matrix Σ, i.e., $\boldsymbol{x}_i \sim N(\boldsymbol{\mu}, \Sigma)$, $\Sigma > 0$. Let

$$(1.1) \qquad \bar{\boldsymbol{x}} = N^{-1} \sum_{i=1}^{N} \boldsymbol{x}_i \,, \quad \text{and} \quad S = \sum_{i=1}^{N} (\boldsymbol{x}_i - \bar{\boldsymbol{x}})(\boldsymbol{x}_i - \bar{\boldsymbol{x}})' \,, \qquad n = N-1 \,.$$

Then $\bar{\boldsymbol{x}}$ and S are independently distributed. $N^{1/2}(\bar{\boldsymbol{x}} - \boldsymbol{\mu}) \sim N(\boldsymbol{0}, \Sigma)$ and S has a Wishart distribution with mean $n\Sigma$, i.e., $S \sim W(\Sigma, p, n)$. In Sections 3 and 4 we derive the exact non-null moments and distribution of the likelihood ratio criterion (LRC) for testing:

(a) H_1: $\Sigma = \sigma^2 I$, $\boldsymbol{\mu} = \boldsymbol{\mu}_0$, σ^2 unknown, against the alternative $A_1 \neq H_1$; $\boldsymbol{\mu}_0$ known.

(b) H_2: $\Sigma = \sigma^2[(1-\rho)I + \rho \boldsymbol{e}\boldsymbol{e}']$, σ and ρ unknown, against the alternative $A_2 \neq H_2$; $\boldsymbol{e}' = (1, 1, \cdots, 1)$, ($\mu$ unknown).

These distributions are in the form of Meijer's G-functions [3] and H-functions [2]. The non-null moments derived here have been used to obtain chi-square type asymptotic expansions of the non-null distribution of the likelihood ratio criteria for alternatives close to the hypothesis. These results will be published elsewhere.

2. Preliminaries

Let $C_\kappa(\Delta)$ denote the zonal polynomial, a symmetric function in the roots of the symmetric matrix Δ (see James [4]) of degree k corresponding to the partition $\kappa = (k_1, k_2, \cdots, k_p)$, $k_1 \geqq k_2 \geqq \cdots \geqq k_p \geqq 0$, $\sum_{i=1}^{p} k_i = k$ and

$$(2.1) \qquad (a)_\kappa = \prod_{i=1}^{p} (a - (i-1)/2)_{k_i} = \Gamma_p(a, \kappa)/\Gamma_p(a) \,,$$

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$$\Gamma_p(a)=\pi^{p(p-1)/4}\prod_{i=1}^{p}\Gamma(a-(i-1)/2)\ ,$$

$$(a)_k=a(a+1)\cdots(a+k-1)\ .$$

We also make use of the Gauss and Legendre multiplication theorem,

$$\prod_{i=1}^{p}\Gamma(a+p^{-1}(i-1))=(2\pi)^{(p-1)/2}p^{-pa/2}\Gamma(pa)\ , \tag{2.2}$$

and the definition of Meijer's G-function [3] given by

$$G_{p,q}^{m,n}\left[x\left|\begin{matrix}a_1,\cdots,a_p\\ b_1,\cdots,b_q\end{matrix}\right.\right] \tag{2.3}$$

$$=(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}\frac{\prod_{j=1}^{m}\Gamma(b_j-s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j-s)}x^s ds\ ,$$

where $0\leqq m\leqq q$, $0\leqq n\leqq p$ and the poles of $\Gamma(b_j-s)$ do not coincide with the poles of $\Gamma(1-a_k+s)$ for any j and k, $j=1,2,\cdots,m$, $k=1,2,\cdots,n$.

In the appendix, some relationship between G- and H-functions are derived.

Finally, we recall the following two results from [5] and [6] respectively.

LEMMA 1. *Let* $S\sim W(\Sigma,p,n)$, $n=N-1$. *Let*

$$\lambda^*=|S|/(p^{-1}\operatorname{tr}S)^p\ .$$

Then

$$\mathrm{E}\,(\lambda^{*h})=\frac{p^{ph}\Gamma_p((2h+n)/2)\,|q\Sigma^{-1}|^{n/2}}{\Gamma_p(n/2)}$$

$$\times\sum_{k=0}^{k}\sum_{\kappa}\frac{((2h+n)/2)_\kappa C_\kappa(I-q\Sigma^{-1})}{k!}\ \frac{\Gamma(np/2+k)}{\Gamma_p(p(2h+n)/2+k)}\ ,$$

where $0<q<\infty$ *and may be taken to be* $2\lambda_1\lambda_p/(\lambda_1+\lambda_p)$ *for the rapid convergence of the series*; $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_p$ *are the ordered roots of* Σ.

LEMMA 2. *The problem* (b) *of testing* H: $\Sigma=\sigma^2[(1-\rho)I+\rho ee']$ *is equivalent to testing that* $\Sigma=\mathrm{diag}\,(a,b,\cdots,b)$.

This follows from the well known fact (see e.g., [6]) that any orthogonal matrix with first row $e'/\sqrt{p}$ diagonalizes Σ and the elements of this orthogonal matrix does not depend upon Σ.

3. Exact distribution of the LRC for problem (a)

The LRC for testing the hypothesis H_1: $\Sigma=\sigma^2 I$, $\mu=\mu_0$, against the alternative $A_1 \neq H_1$ is based on the statistic

$$\lambda = \{p^p|S|/[\mathrm{tr}\, S + N(\bar{\boldsymbol{x}}-\boldsymbol{\mu}_0)'(\bar{\boldsymbol{x}}-\boldsymbol{\mu}_0)^p]\}^{N/2} \tag{3.1}$$

where N is the size of the sample taken from a p-variate $N(\boldsymbol{\mu}, \Sigma)$. We may note that the test based on λ given in (3.1) is unbiased and can be established easily.

In this section we shall obtain the exact distribution of $\lambda^* = \lambda^{2/N}$. Since $\bar{\boldsymbol{x}}$ and S are independently distributed, we shall consider the joint distribution of S and $\boldsymbol{y} = N^{1/2}(\bar{\boldsymbol{x}}-\boldsymbol{\mu}_0)$. From this we can easily obtain the joint distribution of

$$V = S + \boldsymbol{y}\boldsymbol{y}' \quad \text{and} \quad u = \boldsymbol{y}'V^{-1}\boldsymbol{y} , \tag{3.2}$$

which is given by

$$C|V|^{(N-p-1)/2} \exp\left(-\frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}V\right) \sum_{j=0}^{\infty} \left(\frac{N}{4}\boldsymbol{\nu}'\Sigma^{-1}V\Sigma^{-1}\boldsymbol{\nu}\right)^j \times (1-u)^{(N-p)/2-1} u^{p/2+j-1} \Big/ j!\,\Gamma\left(\frac{1}{2}p+j\right) \tag{3.3}$$

where $\boldsymbol{\nu} = \boldsymbol{\mu} - \boldsymbol{\mu}_0$, and C is a constant given by

$$C^{-1} = 2^{pN/2}\Gamma_p\left(\frac{1}{2}(N-1)\right)|\Sigma|^{N/2} \exp\left(+\frac{N}{2}\boldsymbol{\nu}'\Sigma^{-1}\boldsymbol{\nu}\right) . \tag{3.4}$$

With this notation, λ in (3.1) can be written as

$$\lambda = (p^p|V|/(\mathrm{tr}\, V)^p)^{N/2}(1-u)^{N/2} . \tag{3.5}$$

Hence, (from the monotone convergence theorem, the interchange of integral and summation is justified),

$$\mathrm{E}\,(\lambda^h) = C\sum_{j=0}^{\infty} A_j \Gamma\left(\frac{N(1+h)-p}{2}\right)\Big/\Gamma\left(\frac{N(1+h)}{2}+j\right) \tag{3.6}$$

where A_j is given by

$$A_j = \int_{V>0} \frac{1}{j!}\left(\frac{N}{4}\boldsymbol{\nu}'\Sigma^{-1}V\Sigma^{-1}\boldsymbol{\nu}\right)^j (p^p|V|/(\mathrm{tr}\, V)^p)^{Nh/2}|V|^{(N-p-1)/2} \times \exp\left(-\frac{1}{2}\,\mathrm{tr}\,\Sigma^{-1}V\right) dV . \tag{3.7}$$

To obtain A_j, we define the generating function of A_j by

$$g(\theta) = \sum_{j=0}^{\infty} A_j\theta^j = \int_{V>0} (p^p|V|/(\mathrm{tr}\, V)^p)^{Nh/2}|V|^{(N-p-1)/2} \times \exp\left(-\frac{1}{2}\,\mathrm{tr}\,\{R(\theta)\}^{-1}V\right) dV \tag{3.8}$$

where θ is such that $(\Sigma^{-1}-(N/2)\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\Sigma^{-1}\theta)\equiv\{R(\theta)\}^{-1}$ is positive definite. This can be rewritten as

$$g(\theta)=2^{pN/2}\Gamma_p(N/2)\,|R(\theta)|^{N/2}\,\mathrm{E}\,(p^p|V|/(\operatorname{tr}V)^p)^{Nh/2} \tag{3.9}$$

where V is distributed as Wishart whose density function is given by

$$\left\{2^{pN/2}\Gamma_p\left(\frac{1}{2}N\right)|R(\theta)|^{N/2}\right\}^{-1}|V|^{(N-p-1)/2}\exp\left(-\frac{1}{2}\operatorname{tr}\{R(\theta)\}^{-1}V\right). \tag{3.10}$$

Using the result of Lemma 1 in Section 2, we can write

$$\begin{aligned}g(\theta)=&\,2^{pN/2}p^{Nph/2}q^{pN/2}\Gamma_p\left(\frac{N(1+h)}{2}\right)\\&\times\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(N(1+h)/2)_\kappa C_\kappa[I-q(\Sigma^{-1}-\theta(1/2)\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\Sigma^{-1}N)]\Gamma(Np/2+k)}{k!\,\Gamma(pN(1+h)/2+k)}\end{aligned} \tag{3.11}$$

for $0<q<\infty$. To obtain the coefficient of θ^j from (3.11), we note that for $0<q<2\lambda_p$,

$$\begin{aligned}g(\theta)=&\,\frac{2^{pN/2}p^{Nph/2}q^{pN/2}\Gamma_p(N(1+h)/2)}{\Gamma(Nhp/2)}\\&\times\int_0^1|I-x[I-q(\Sigma^{-1}-\theta(1/2)\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\Sigma^{-1}N)]|^{-N(1+h)/2}x^{pN/2-1}\\&\times(1-x)^{Nph/2-1}dx \qquad\qquad \text{for } \operatorname{Re}(h)>0\,.\end{aligned} \tag{3.11$'$}$$

We note that

$$\begin{aligned}&|I-x(I-q\Sigma^{-1})-(1/2)xq\theta\Sigma^{-1}\boldsymbol{\nu}\boldsymbol{\nu}'\Sigma^{-1}N|^{-N(1+h)/2}\\&\quad=|I-x(I-q\Sigma^{-1})|^{-N(1+h)/2}\\&\quad\quad\times\left[1-xq\theta\frac{N}{2}\boldsymbol{\nu}'\Sigma^{-1}(I-x(I-q\Sigma^{-1}))^{-1}\Sigma^{-1}\boldsymbol{\nu}\right]^{-N(1+h)/2}.\end{aligned}$$

Hence, the coefficient A_j of θ^j from $g(\theta)$ is given by

$$\begin{aligned}A_j=&\,\frac{(2q)^{pN/2}p^{Nph/2}\Gamma_p(N(1+h)/2)\Gamma(N(1+h)/2+j)}{\Gamma(Nhp/2)j!\,\Gamma(N(1+h)/2)}\\&\times\int_0^1|I-x(I-q\Sigma^{-1})|^{-N(1+h)/2}x^{pN/2+j-1}(1-x)^{pNh/2-1}\\&\times[\boldsymbol{\nu}'\Sigma^{-1}(I-x(I-q\Sigma^{-1}))^{-1}\Sigma^{-1}\boldsymbol{\nu}Nq/2]^j dx\,.\end{aligned} \tag{3.12}$$

For $0<q\leqq 2\lambda_p$, and $0<x<1$, we can write

$$|I-x(I-q\Sigma^{-1})|^{-N(1+h)/2}=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(N(1+h)/2)_\kappa}{k!}C_\kappa(I-q\Sigma^{-1})x^k \tag{3.13}$$

and

$$\left[\frac{Nq}{2}\boldsymbol{\nu}'\Sigma^{-1}(I-x(I-q\Sigma^{-1}))^{-1}\Sigma^{-1}\boldsymbol{\nu}\right]^j=\sum_{\delta=0}^{\infty}a_\delta(j)x^\delta\,, \tag{3.14}$$

$$\begin{aligned} A_j=&\frac{(2q)^{pN/2}p^{Nph/2}\Gamma_p(N(1+h)/2)\Gamma(N(1+h)/2+j)}{j!\Gamma(N(1+h)/2)}\\ &\times\sum_{k=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{\kappa}a_\delta(j)\frac{(N(1+h)/2)_\kappa}{k!}C_\kappa(I-q\Sigma^{-1})\\ &\times\frac{\Gamma(pN/2+j+\delta+k)}{\Gamma(Np(1+h)/2+j+\delta+k)}\end{aligned} \tag{3.15}$$

and, we get

$$\begin{aligned} \mathrm{E}\,(\lambda^h)=&\frac{|q\Sigma^{-1}|^{N/2}\Gamma_p((N(1+h)-1)/2)p^{Nph/2}}{\Gamma_p((N-1)/2)}\\ &\times\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}a_\delta(j)\frac{(N(1+h)/2)_\kappa}{k!j!}C_\kappa(I-q\Sigma^{-1})\\ &\times\frac{\Gamma(pN/2+j+\delta+k)}{\Gamma(Np(1+h)/2+j+\delta+k)}\,,\qquad 0<q\leqq 2\lambda_p\end{aligned} \tag{3.16}$$

where $a_\delta(j)$'s are defined by (3.14). Hence

$$\begin{aligned} \mathrm{E}\,(\lambda^{*h})=&\frac{|q\Sigma^{-1}|^{N/2}\Gamma_p((N+2h-1)/2)}{p^{-ph}\Gamma_p((N-1)/2)}\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}a_\delta(j)\\ &\times\left[\frac{(N/2+h)_\kappa}{k!j!}C_\kappa(I-q\Sigma^{-1})\frac{\Gamma(pN/2+j+\delta+k)}{\Gamma(Np/2+ph+j+\delta+k)}\right].\end{aligned} \tag{3.17}$$

Using the results of (2.2) and (2.3), we get the density function of λ^*

$$\begin{aligned} p(\lambda^*)=&\frac{\pi^{\{(p-1)(p+2)\}/4}|q\Sigma^{-1}|^{N/2}}{p^{(Np-1)/2}2^{-(p-1)/2}\Gamma_p((N-1)/2)}(\lambda^*)^{(N-p-2)/2}\\ &\times\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}\left[\frac{a_\delta(j)C_\kappa(I-q\Sigma^{-1})}{k!j!}\frac{\Gamma(pN/2+j+\delta+k)}{p^{j+\delta+k}}G_{p+1,p+1}^{p+1,0}\right.\\ &\left.\times\left(\lambda^*\left|\begin{matrix}p/2,\ p/2+p^{-1}(j+k+\delta)+p^{-1}(i-1),\ i=1,2,\cdots,p\\ 0,\ (p-i+1)/2+k_i,\ i=1,2,\cdots,p\end{matrix}\right.\right)\right],\\ &\qquad\qquad 0<q\leqq 2\lambda_p\,,\end{aligned} \tag{3.18}$$

the existence of G-function follows from ([4], p. 1068) since $\lambda^*<1$.

4. Exact distribution of the LRC for problem (b)

We first obtain the moments of the LR statistic for a more general problem of testing H_2': $\Sigma=\begin{pmatrix}\sigma_1^2I_{p_1} & 0\\ 0 & \sigma_2^2I_{p_2}\end{pmatrix}$, σ_1^2, σ_2^2 unknown against the alternative $A_2'\neq H_2'$. The LRC for this problem is based on the statistic

$$\lambda'=[|S|/\{p_1^{-1}\operatorname{tr}S_{11}\}^{p_1}\{p_2^{-1}\operatorname{tr}S_{22}\}^{p_2}]^{n/2}\,, \tag{4.1}$$

where $S=\begin{array}{cc} \begin{pmatrix} S_{11} & S_{12} \\ S'_{12} & S_{22} \end{pmatrix} & \begin{array}{c} p_1 \\ p_2 \end{array} \\ \begin{array}{cc} p_1 & p_2 \end{array} & \end{array}$, $p_1+p_2=p$ and $n=N-1$; N is the size of the sample taken from $N(\boldsymbol{\mu}, \Sigma)$, $\Sigma>0$. Without loss of generality let $p_2 \geqq p_1$. It is well known that $S_{11}-S_{12}S_{22}^{-1}S'_{12}$ and (S_{12}, S_{22}) are independently distributed. Then, it is easy to find the joint distribution of S_{11}, S_{22} and $T=S_{11}^{-1/2}S_{12}S_{22}^{-1}S'_{12}(S_{11}^{-1/2})'$ where $S_{11}=(S_{11}^{1/2})(S_{11}^{1/2})'$. This joint pdf of T, S_{11} and S_{22} can be written as

$$(4.2) \qquad C_1|S_{11}|^{(n-p_1-1)/2}|S_{22}|^{(n-p_2-1)/2}\exp\left[-\frac{1}{2}\operatorname{tr}\Sigma_{1\cdot2}^{-1}S_{11}-\frac{1}{2}\operatorname{tr}\Sigma_{2\cdot1}^{-1}S_{22}\right]$$
$$\times|I-T|^{(n-p_2-p_1-1)/2}|T|^{(p_2-p_1-1)/2}$$
$$\times\sum_{k=0}^{\infty}\sum_{\kappa}C_\kappa\left(\frac{1}{4}(S_{11}^{1/2})'\Sigma_{1\cdot2}^{-1}\beta S_{22}\beta'\Sigma_{1\cdot2}^{-1}(S_{11})^{1/2}T\right)\Big/k!\left(\frac{1}{2}p_2\right)_\kappa$$

where $\Sigma_{1\cdot2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma'_{12}$, $\Sigma_{2\cdot1}=\Sigma_{22}-\Sigma'_{12}\Sigma_{11}^{-1}\Sigma_{12}$, $\beta=\Sigma_{12}\Sigma_{22}^{-1}$, $\Sigma=\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_{22} \end{pmatrix}\begin{array}{c} p_1 \\ p_2 \end{array}$ and

$$(4.3) \qquad C_1^{-1}=2^{n(p_1+p_2)/2}\Gamma_{p_1}\left(\frac{1}{2}p_2\right)\Gamma_{p_1}\left(\frac{n-p_2}{2}\right)\Gamma_{p_2}\left(\frac{1}{2}n\right)|\Sigma_{1\cdot2}|^{n/2}|\Sigma_{22}|^{n/2}.$$

With these notations, λ in (4.1) can be written as

$$(4.4) \qquad \lambda=\left\{\left(\frac{|S_{11}|}{((1/p_1)\operatorname{tr}S_{11})^{p_1}}\right)\left(\frac{|S_{22}|}{((1/p_2)\operatorname{tr}S_{22})^{p_2}}\right)|I-T|\right\}^{n/2}.$$

Then, it is easy to see that

$$(4.5) \quad \mathrm{E}(\lambda^h)=C_2\int_{S_{11}>0}\int_{S_{22}>0}\left\{\left(\frac{|S_{11}|}{((1/p_1)\operatorname{tr}S_{11})^{p_1}}\right)\left(\frac{|S_{22}|}{((1/p_2)\operatorname{tr}S_{22})^{p_2}}\right)\right\}^{nh/2}$$
$$\times|S_{11}|^{(n-p_1-1)/2}|S_{22}|^{(n-p_2-1)/2}\exp\left[-\frac{1}{2}\operatorname{tr}\Sigma_{1\cdot2}^{-1}S_{11}-\frac{1}{2}\operatorname{tr}\Sigma_{2\cdot1}^{-1}S_{22}\right]$$
$$\times\sum_{k=0}^{\infty}\sum_{\kappa}C_\kappa\left(\frac{1}{4}\Sigma_{1\cdot2}^{-1}\beta S_{22}\beta'\Sigma_{1\cdot2}^{-1}S_{11}\right)\Big/k!\left(\frac{1}{2}n(1+h)\right)_\kappa$$

where $C_2^{-1}=2^{n(p_1+p_2)/2}\Gamma_{p_1}(n(1+h)/2)\Gamma_{p_2}(n/2)\{\Gamma_{p_1}((n-p_2)/2)/\Gamma_{p_1}((n(1+h)-p_2)/2)\}$ $\times|\Sigma_{1\cdot2}|^{n/2}|\Sigma_{22}|^{n/2}$.

Particular case. Let $p_1=1$. Then H'_2 and A'_2 becomes H_2 and A_2 respectively with $p_2=p-1$ (see Lemma 2). In this case

$$|S_{11}|/((1/p_1)\operatorname{tr}S_{11})^{p_1}=1.$$

Hence the hth moment for the LR statistic λ for testing H_2 against A_2 can be obtained from (4.5) and is given by

$$(4.6)\quad \mathrm{E}(\lambda^h)=C_3\int_{S_{22}>0}\left(\frac{|S_{22}|}{((1/p_2)\operatorname{tr}S_{22})^{p_2}}\right)^{nh/2}|S_{22}|^{(n-p_2-1)/2}\exp\left[-\frac{1}{2}\operatorname{tr}\Sigma_{2\cdot1}^{-1}S_{22}\right]$$
$$\times\sum_{j=0}^{\infty}\left(\frac{1}{2}\sigma_{1\cdot2}^{-1}\beta S_{22}\beta'\right)^j\Gamma\left(\frac{1}{2}n+j\right)\Big/j!\,\Gamma\left(\frac{n(1+h)}{2}+j\right)$$

where, $p_2=p-1$, $\sigma_{1\cdot2}=\sigma_{11}^{-1}\beta\Sigma_{22}^{-1}\beta'$ and C_3 is given by

$$C_3^{-1}=2^{p_2n/2}\Gamma_{p_2}\left(\frac{1}{2}n\right)|\Sigma_{22}|^{n/2}\Gamma\left(\frac{n-p_2}{2}\right)\Big/\Gamma\left(\frac{n(1+h)-p_2}{2}\right).$$

Using the arguments similar to Section 3, we have, for $0<q\leqq2\lambda_q^*$, where λ_q^* is the smallest characteristic root of $\Sigma_{2\cdot1}$

$$(4.7)\quad \int_{S_{22}>0}\left(\frac{|S_{22}|}{((1/p_2)\operatorname{tr}S_{22})^{p_2}}\right)^{nh/2}|S_{22}|^{(n-p_2-1)/2}\left(\frac{1}{2}\sigma_{1\cdot2}^{-1}\beta S_{22}\beta'\right)^j$$
$$\times\exp\left\{-\frac{1}{2}\operatorname{tr}\Sigma_{2\cdot1}^{-1}S_{22}\right\}dS_{22}$$
$$=(2q)^{p_2n/2}p_2^{np_2h/2}\Gamma_{p_2}\left(\frac{n(1+h)}{2}\right)\Gamma\left(\frac{n(1+h)}{2}+k\right)\left\{\Gamma\left(\frac{n(1+h)}{2}\right)\right\}^{-1}$$
$$\times\sum_{k=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{\kappa}a_\delta(j)\frac{(n(1+h)/2)_\kappa}{k!}C_\kappa(I-q\Sigma_{2\cdot1}^{-1})$$
$$\times\left\{\Gamma\left(\frac{1}{2}p_2n+j+\delta+k\right)\Big/\Gamma\left(\frac{1}{2}np_2(1+h)+j+\delta+k\right)\right\}$$

where

$$(4.8)\quad \sum_{\delta=0}^{\infty}a_\delta(j)x^\delta=\left[\frac{1}{\sigma_{1\cdot2}}\beta(I-x(I-q\Sigma_{2\cdot1}^{-1}))^{-1}\beta'\right]^j.$$

Hence, we get

$$(4.9)\quad \mathrm{E}(\lambda^h)=|q\Sigma_{2\cdot1}^{-1}|^{n/2}(1-\rho^2)^{n/2}p_2^{np_2h/2}\Gamma_{p_2-1}\left(\frac{n(1+h)-1}{2}\right)\left\{\Gamma_{p_2-1}\left(\frac{n-1}{2}\right)\right\}^{-1}$$
$$\times\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\Gamma(n/2+j)}{j!\,\Gamma(n/2)}a_\delta(j)\frac{(n(1+h)/2)_\kappa}{k!}C_\kappa(I-q\Sigma_{2\cdot1}^{-1})$$
$$\times\frac{\Gamma(p_2n/2+j+\delta+k)}{\Gamma(p_2n(1+h)/2+j+\delta+k)},\qquad 0<q\leqq2\lambda_q^*$$

where $a_\delta(j)$ is defined by (4.8), $p_2=p-1$, $1-\rho^2=\sigma_{1\cdot2}/\sigma_{11}$, and $0<q<\infty$.

Let $p=2$ and $q=\sigma_{2\cdot1}$. Then (4.9) can be written as

$$(4.10)\quad \mathrm{E}(\lambda^h)=(1-\rho^2)^{n/2}\sum_{j=0}^{\infty}\frac{\Gamma(n/2+j)}{j!\,\Gamma(n/2)}(\rho^2)^j\frac{\Gamma(n/2+j)}{\Gamma(n(1+h)/2+j)},$$

which is obvious from (4.5) after integrating over S_{11} and S_{22}.

Let $\lambda^*=\lambda^{2/n}$. Then from (4.9) we get

$$(4.11)\quad \mathrm{E}\,(\lambda^*)^h=|q\Sigma_{2\cdot 1}^{-1}|^{n/2}(1-\rho^2)^{n/2}p_2^{p_2h}\Gamma_{p_2-1}\left(\frac{n+2h-1}{2}\right)\left\{\Gamma_{p_2-1}\left(\frac{n-1}{2}\right)\right\}$$
$$\times\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}\left[\frac{\Gamma(n/2+j)}{j!\,\Gamma(n/2)}a_\delta(j)\frac{(n/2+h)_\kappa}{k!}C_\kappa(I-q\Sigma_{2\cdot 1}^{-1})\right.$$
$$\left.\times\frac{\Gamma(p_2n/2+j+\delta+k)}{\Gamma(p_2n/2+p_2h+j+\delta+k)}\right].$$

Hence the pdf of λ^* is given by

$$(4.12)\quad p(\lambda^*)=\frac{|q\Sigma_{2\cdot 1}^{-1}|^{n/2}(1-\rho^2)^{n/2}\pi^{p_2(p_2-1)/4}}{p^{(np_2-1)/2}2^{-(p_2-1)/2}\Gamma_{p_2-1}((n-1)/2)}(\lambda^*)^{(n-p_2-2)/2}$$
$$\times\sum_{j=0}^{\infty}\sum_{\delta=0}^{\infty}\sum_{k=0}^{\infty}\sum_{\kappa}\left[\frac{\Gamma(n/2+j)}{j!\,\Gamma(n/2)}a_\delta(j)\frac{C_\kappa(I-q\Sigma_{2\cdot 1}^{-1})}{k!}\right.$$
$$\times\frac{\Gamma(p_2n/2+j+\delta+k)}{p^{j+\delta+k}}$$
$$\left.\times G_{p+1,p}^{p,\ 0}\left(\lambda^*\middle|\begin{matrix}p_2/2,\ p_2/2+p_2^{-1}(j+k+\delta)+p_2^{-1}(i-1),\ i=1,\cdots,p_2\\(p_2-i+1)/2+k_i,\ i=1,\cdots,p_2\end{matrix}\right)\right]$$

where $p_2=p-1$ and $0<q\leqq 2\lambda_p^*$; the existence of this G-function follows from [4], p. 1068, since $p+1\geqq 1$ and $p+1>p$.

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Appendix

The H-functions are defined by

$$\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty}\frac{\prod_{i=1}^{m}\Gamma(b_i-\beta_is)\prod_{i=1}^{n}\Gamma(1-a_i+\alpha_is)}{\prod_{m+1}^{q}\Gamma(1-b_i+\beta_is)\prod_{n+1}^{p}\Gamma(a_i-\alpha_is)}\lambda^s ds$$
$$=H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1,\alpha_1),(a_2,\alpha_2),\cdots,(a_p,\alpha_p)\\(b_1,\beta_1),(b_2,\beta_2),\cdots,(b_q,\beta_q)\end{matrix}\right].$$

Now we give some special cases of this function.

Case 1. If $\beta_j=\alpha_i=\gamma$ (say) for all i and j, then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1,\gamma),\cdots,(a_p,\gamma)\\(b_1,\gamma),\cdots,(b_q,\gamma)\end{matrix}\right]=\left(\frac{1}{\gamma}\right)G_{p,q}^{m,n}\left[\lambda^{1/\gamma}\middle|\begin{matrix}a_1,\cdots,a_p\\b_1,\cdots,b_q\end{matrix}\right].$$

Case 2. If $\beta_1=\beta_2=\cdots=\beta_q=\beta$ (say), then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\(b_1,\beta),\ \cdots,(b_q,\beta)\end{matrix}\right]=\left(\frac{1}{\beta}\right)H_{p,q}^{m,n}\left[\lambda^{1/\beta}\middle|\begin{matrix}(a_1,\alpha_1/\beta),\cdots,(a_p,\alpha_p/\beta)\\(b_1,1),\cdots,(b_q,1)\end{matrix}\right].$$

Case 3. Let $\alpha_1=\alpha_2=\cdots=\alpha_p=d$ be a positive integer, then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1, d),\cdots,(a_p, d)\\(b_1, \beta_1),\cdots,(b_q, \beta_q)\end{matrix}\right]$$

$$=(2\pi)^{\{(p-2n)(d-1)\}/2}d^{p/2-\sum_{i=1}^{p}a_i}H_{pd,q}^{m,nd}\left[\lambda d^{pd}\middle|\begin{matrix}\Delta(a_1/d, 1),\cdots,\Delta(a_p/d, 1)\\(b_1, \beta_1),\cdots,(b_q, \beta_q)\end{matrix}\right],$$

where $\Delta(a_i/d, 1)=\{((a_i+d-1)/d, 1), ((a_i+d-2)/d, 1),\cdots,(a_i/d, 1)\}$.

Case 4. Let $\alpha_1=\alpha_2=\cdots=\alpha_p=\alpha$ and $\beta_1=\beta_2=\cdots=\beta_q=\beta$ be positive integers. Then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1, \alpha),\cdots,(a_p, \alpha)\\(b_1, \beta),\cdots,(b_q, \beta)\end{matrix}\right]$$

$$=2\pi^{\{(p-2n)(\alpha-1)\}/2-\{(q-2m)(\beta-1)\}/2}\alpha^{p/2-\sum_{i=1}^{p}a_i}\beta^{-q/2+\sum_{i=1}^{p}b_i}$$

$$\times G_{\alpha p,\beta q}^{m\beta,n\alpha}\left[\frac{\lambda\alpha^{\alpha p}}{\beta^{\beta q}}\middle|\begin{matrix}a_i+\alpha-j,\ i=1, 2,\cdots,p,\ j=1, 2,\cdots,\alpha\\b_i+\beta-j,\ i=1, 2,\cdots,q,\ j=1, 2,\cdots,\beta\end{matrix}\right].$$

Case 5. Let b_1 be the least number. Then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1, \alpha_1),\cdots,(a_p, \alpha_p)\\(b_1, \beta_1),\cdots,(b_p, \beta_p)\end{matrix}\right]$$

$$=\frac{1}{\beta_1}\lambda^{b_1/\beta_1}H_{p,q}^{m,n}\left[\lambda^{1/\beta_1}\middle|\begin{matrix}(a_1-b_1\alpha_1/\beta_1, \alpha_1/\beta_1),\cdots,(a_p-b_1\alpha_p/\beta, \alpha_p/\beta_1)\\(0, 1), (b_2-b_1\beta_2/\beta_1, \beta_2/\beta_1),\cdots,(b_q-b_1\beta_q/\beta_1, \beta_q/\beta_1)\end{matrix}\right].$$

Case 6. Let $\alpha_1=\cdots=\alpha_p=\alpha$ (say), $\beta_1=\cdots=\beta_p=\beta$ (say) and $\alpha/\beta=d$ (say) where d is an integer. Then

$$H_{p,q}^{m,n}\left[\lambda\middle|\begin{matrix}(a_1, \alpha_1),\cdots,(a_p, \alpha_p)\\(b_1, \beta_1),\cdots,(b_p, \beta_p)\end{matrix}\right]$$

$$=(1/\beta)H_{p,q}^{m,n}\left[\lambda^{1/\beta}\middle|\begin{matrix}(a_1, d),\cdots,(a_p, d)\\(b_1, 1),\cdots,(b_q, 1)\end{matrix}\right]$$

$$=(2\pi)^{\{(p-2n)(d-1)\}/2}d^{p/2-\sum_{i=1}^{p}a_i}(1/\beta)$$

$$\times H_{pd,q}^{m,nd}\left[\lambda^{1/\beta}d^{pd}\middle|\begin{matrix}\Delta(a_1/d, 1),\cdots,\Delta(a_p/d, 1)\\(b_1, 1),\cdots,(b_q, 1)\end{matrix}\right]$$

$$=(2\pi)^{\{(p-2n)(d-1)\}/2}d^{p/2-\sum_{i=1}^{p}a_i}(1/\beta)$$

$$\times G_{pd,q}^{m,nd}\left[\lambda^{1/\beta}d^{pd}\middle|\begin{matrix}(a_i+d-j)/2,\ i=1,\cdots,p,\ j=1,\cdots,d\\b_1,\cdots,b_q\end{matrix}\right].$$

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