ON KOLMOGOROV-SMIRNOV-TYPE TESTS FOR SYMMETRY*

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Summary

For a set of independent but not necessarily identically distributed random variables, a simple Kolmogorov-Smirnov-type test is proposed for testing the hypothesis of symmetry (about a common and specified point). The exact and asymptotic (null hypothesis) distributions of some allied statistics are obtained, and the Bahadur-efficiency of the test is studied.

1. Introduction

Let $\{X_i\}$ be a sequence of independent real valued random variables with continuous distribution functions (df) $\{F_i(x)\}$, all defined on $(-\infty, \infty)$ and not necessarily identical. Based on a sample $(X_1, \cdots, X_n)$, we want to test the null hypothesis (H_0) that all the df $F_1, \cdots, F_n$ are symmetric around their respective (specified) medians. Without any loss of generality, we may take all these medians to be equal to 0, and thus, frame H_0 as

$$(1.1) \qquad H_0 : F_i(x)+F_i(-x)=1 \qquad \text{for all } x\geqq 0, \text{ and } i=1, \cdots, n .$$

Let $c(u)$ be equal to 0 or 1 according as $u<0$ or $\geqq 0$, and let

$$(1.2) \quad F_n^*(x)=n^{-1}\sum_{i=1}^{n} c(x-X_i) , \quad \bar{F}_{(n)}(x)=n^{-1}\sum_{i=1}^{n} F_i(x) , \qquad -\infty<x<\infty .$$

Thus, F_n^* is the *empirical* df and it estimates unbiasedly the average df $\bar{F}_{(n)}$.

In testing the null hypothesis (1.2), we are interested in the following alternative hypotheses:

$$(1.3) \qquad \begin{aligned} H_1 &: \sup_{x\geqq 0}\,[\bar{F}_{(n)}(x)+\bar{F}_{(n)}(-x)]>1 , \\ H_2 &: \inf_{x\geqq 0}\,[\bar{F}_{(n)}(x)+\bar{F}_{(n)}(-x)]<1 ; \end{aligned}$$

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$$H_3 = H_1 \cup H_2 : \sup_{x \geq 0} |\bar{F}_{(n)}(x) + \bar{F}_{(n)}(-x) - 1| > 0 \ . \tag{1.4}$$

When $F_1 \equiv \cdots \equiv F_n \equiv F$, H_1 means that $F(x) > 1 - F(-x)$, at least for some $x \geqq 0$, or writing X for a random variable following the df $F(x)$, $-X$ is neither identically distributed as nor stochastically smaller than X in the usual sense. In this case, H_3 means that $F(x) \not\equiv 1 - F(-x)$ and covers all types of departure from symmetry. In many practical problems, though it may be unwise to impose the restriction that $F_1 \equiv \cdots \equiv F_n \equiv F$, it may not be unreasonable to assume that when (1.1) does not hold, $F_1, \cdots, F_n$ have a common pattern of skewness. For example, let $F_i(x) = F_i^0(x - m_i)$ and $F_i^0 \in \mathcal{F}_0$, $i = 1, \cdots, n$, where $\mathcal{F}_0 = \{F : F(x) + F(-x) = 1$, for all $x \geqq 0\}$, and the m_i are location parameters. If the m_i have all the same sign, $F_1, \cdots, F_n$ are either all positively or all negatively skew, no matter whatever be their forms. Thus, (1.3) and (1.4) cover such situations. However, unlike the one-sample location problem, we are not confining ourselves to translation alternatives only. Thus, let $F_i^0(x)$ be a strictly increasing df belonging to $\mathcal{F}_0$, and $F_i(x)$ be $F_i^0(v_i(x))$, where $v_i(x)$, $i = 1, \cdots, n$ are strictly increasing continuous functions which are not everywhere odd, i.e., for which $v_i(x) + v_i(-x) = 0$ does not hold for all $x \geqq 0$. Then, if the v_i resemble each other in the sense that there are points ($x \geqq 0$) at which $v_i(x) > -v_i(-x)$, for all $i = 1, \cdots, n$, H_1 will hold. A similar case holds for H_2. We also note that unlike the classical one-sample goodness of fit problem (where the Kolmogorov test applies), our hypothesis is not a simple one (as under (1.1), the true df's $F_1, \cdots, F_n$ remain unspecified).

For testing the null hypothesis, we keep in mind (1.3) and (1.4), and replacing the average df $\bar{F}_{(n)}$ by the empirical df F_n^*, consider the following Kolmogorov-Smirnov type statistics:

$$\begin{aligned} D_n^+ &= \sup_{x \geq 0} [F_n^*(x) + F_n^*(-x-) - 1] \ , \\ D_n^- &= \sup_{x \geq 0} [1 - F_n^*(x) - F_n^*(-x-)] \ ; \end{aligned} \tag{1.5}$$

$$D_n = \max [D_n^+, D_n^-] = \sup_{x \geq 0} |F_n^*(x) + F_n^*(-x-) - 1| \ . \tag{1.6}$$

Note that F_n^* is a step-function, and hence, to avoid some complications in the distribution theory, we have taken $F_n^*(-x-)$ for $F_n^*(-x)$, $x \geqq 0$.

The small sample null distributions of D_n^+, D_n^- and D_n are deduced in Section 2, and tabulated too, for $n \leqq 16$. Section 3 deals with asymptotic null distributions of these statistics. Section 4 is concerned with the non-null distribution theory. The last section is devoted to the study of the Bahadur-efficiency of the test based on D_n with respect to the sign test.

2. Exact null distributions: An application of the random walk model

Since F_n^* is a step function assuming the values i/n, $i=1,\cdots,n$, the process $\{n[F_n^*(x)+F_n^*(-x-)-1];\ x\geqq 0\}$ can only assume the integral values between $-n$ and n. Thus, the permissible values of nD_n^+, nD_n^- and nD_n are the integers $0,1,\cdots,n$, but not all of these are admissible. We denote by $\boldsymbol{F}_n=(F_1,\cdots,F_n)$, and let

$$\mathcal{F}_n^0=\{\boldsymbol{F}_n : F_i\in\mathcal{F}_0,\ i=1,\cdots,n\}\ . \tag{2.1}$$

Then, we have the following.

THEOREM 2.1. *For every* $\boldsymbol{F}_n\in\mathcal{F}_n^0$ [*i.e., under* (1.1)] *and* $k=1,\cdots,n$,

$$\mathrm{P}\{nD_n^+\geqq k\}=\mathrm{P}\{nD_n^-\geqq k\}=\sum_{j=k}^{n}(k/j)\,\mathrm{P}\{N_j=j-k\}\ , \tag{2.2}$$

where

$$\mathrm{P}\{N_j=j-k\}=\begin{cases}2^{-j}\binom{j}{r}\ , & j-k=2r,\ r=0,1,2,\cdots\ ,\\ 0\ , & j-k=2r+1,\ j\geqq k\geqq 1\ ;\end{cases} \tag{2.3}$$

and for every $k\geqq 1$,

$$\mathrm{P}\{nD_n\geqq k\}=2\sum_{j=0}^{u}(-1)^j\,\mathrm{P}\{nD_n^+\geqq(2j+1)k\}\ , \tag{2.4}$$

where $u=[n/2k]-1$, $k\geqq 1$, *and* (2.4) *is equal to one for* $k=1$.

PROOF. Let $Y_1\geqq\cdots\geqq Y_n$ be the ordered values of $|X_1|,\cdots,|X_n|$, arranged in descending order of magnitude. Let $t_{n,i}=\bar{F}_{(n)}(-Y_i)$, for $1\leqq i\leqq n$, so that $0\leqq t_{n,1}\leqq t_{n,2}\leqq\cdots\leqq t_{n,n}\leqq\bar{F}_{(n)}(0)=1/2$ (as $\boldsymbol{F}_n\in\mathcal{F}_n^0\Rightarrow\bar{F}_{(n)}\in\mathcal{F}_0$). Since, $F_1,\cdots,F_n$ are symmetric and continuous, ties among $|X_1|,\cdots,|X_n|$, and hence, among $t_{n,1},\cdots,t_{n,n}$ can be neglected in probability. Thus, $0<t_{n,1}<\cdots<t_{n,n}<1/2$, in probability. Define then $V_n(t)=n^{1/2}[G_n^*(t)-t]$, $0<t<1$, where $G_n^*(t)=n^{-1}\sum_{i=1}^{n}c(t-\bar{F}_{(n)}(X_i))$, and let

$$V_n^*(t)=V_n(t-)+V_n(1-t)\ ,\qquad 0\leqq t\leqq 1/2\ . \tag{2.5}$$

For $t\leqq t_{n,1}$, $n^{1/2}V_n^*(t)=0$. At $t=t_{n,1}+$, $n^{1/2}V_n^*(t)$ is either $+1$ or -1, depending upon whether the random variable X_i associated with Y_n has negative or positive sign. The process $n^{1/2}V_n^*(t)$ continues to have the same value until $t=t_{n,2}+$, where it makes another jump of $+1$ or -1, depending on whether the X_i associated with Y_{n-1} is negative or not. And thus the process continues. Hence, on $I=(0,1/2)$, $n^{1/2}V_n^*(t)$ makes n jumps (at $t_{n,1},\cdots,t_{n,n}$) and each jump is either $+1$ or -1.

Let $p_{ij}=\mathrm{P}\{Y_{n-i+1}=|X_j|\}$, $i, j=1,\cdots,n$, $\left(\text{thus } \sum_{j=1}^{n} p_{ij}=1,\ i=1,\cdots,n\right)$. Since, for $\boldsymbol{F}_n \in \mathcal{F}_n^0$, the df of X_i is symmetric about 0, $1\leqq i\leqq n$,

$$\text{(2.6)} \qquad \begin{aligned} &\mathrm{P}\{Y_{n-i+1} \text{ corresponds to a positive } X_j\} \\ &\quad =\sum_{j=1}^{n} p_{ij}\cdot \mathrm{P}\{X_j>0 \,|\, |X_j|=Y_{n-i+1}\}=(1/2)\sum_{j=1}^{n} p_{ij}=1/2\ , \end{aligned}$$

as the distribution of sign X_i is independent of $|X_i|$ when $F_i \in \mathcal{F}_0$, $i=1,\cdots,n$. Thus, the jumps ($+1$ or -1) at $t_{n,i}$ are both equally likely with probability 1/2. Moreover, for $\boldsymbol{F}_n \in \mathcal{F}_n^0$, the vector (sign $X_1,\cdots,$ sign X_n) is distributed independently of $(|X_1|,\cdots,|X_n|)$ and sign $X_1,\cdots,$ sign X_n are also mutually stochastically independent. Hence, the jumps of $n^{1/2}V_n^*(t)$ at $t_{n,1},\cdots,t_{n,n}$ are mutually independent. Finally, the values of $nD_n^+(=\sup_{t\in I} n^{1/2}V_n^*(t))$, $nD_n^-(=\sup_{t\in I}[-n^{1/2}V_n^*(t)])$ and $nD_n(=\sup_{t\in I}|n^{1/2}V_n^*(t)|)$ are independent of the particular realization of $\boldsymbol{t}_n=(t_{n,1},\cdots,t_{n,n})\in I$. Hence, we conclude that (i) the distribution of nD_n^+ (or nD_n^-) (under H_0) is the same as that of the maximum positive (or negative) displacement in n steps of a symmetric random walk starting from the origin, and (ii) the distribution of nD_n agrees with that of the corresponding maximum absolute displacement. Thus, (2.2) follows directly from Theorem 1 (Section 8) of Takacs ([9], p. 24). Using an alternative standard expression given in Uspensky ([10], p. 149), (2.2) can also be written as

$$\text{(2.7)} \qquad 2^{-(n-1)}\sum_{t=0}^{s}\binom{n}{t}-\delta_k\binom{n}{s}2^{-n}\ ,$$

where $s=[(n-k)/2]$ and δ_k is 0 or 1 according as $n-k$ is odd or even.

For the proof of (2.4), we have on writing $Q^+(a, n)$ (or $Q(a, n)$) for the probability that a particle starting a symmetric random walk at the origin with the absorbing barrier at a (or barriers at $\pm a$), $a>0$, will be absorbed at the barrier in course of time n,

$$\text{(2.8)} \qquad \mathrm{P}\{nD_n\leqq k'\}=1-2Q(k'+1, n)\ ;$$

$$\text{(2.9)} \qquad \mathrm{P}\{nD_n^+\leqq k'\}=1-Q^+(k'+1, n)\ .$$

Also, from Uspensky ([10], p. 156), we obtain that

$$\text{(2.10)} \qquad \begin{aligned} Q(k'+1, n)=&\,Q^+(k'+1, n)-Q^+(3k'+3, n)+Q^+(5k'+5, n) \\ &-\cdots+(-1)^u Q^+((2u+1)k', u)\ ; \\ &\qquad u=[(n/2k')-1]\ . \end{aligned}$$

Then, (2.4) readily follows from (2.8)-(2.10) and (2.2), by letting $k'=k-1$. Q.E.D.

We may remark that (2.2) and (2.4) are not affected by the het-

Table 1 Table for the values of $P\{nD_n^+ \geq k\} = P\{nD_n^- \geq k\}$ for $k \leq n \leq 16$

n \ $k \to$	0	1	2	3	4	5	6	7	8	9	10*	11	12	13	14	15
2	1	.500	.250													
3	1	.625	.250	.125												
4	1	.625	.375	.125	.063											
5	1	.688	.375	.219	.063	.031										
6	1	.688	.453	.219	.125	.031	.016									
7	1	.727	.453	.289	.125	.072	.016	.008								
8	1	.727	.508	.289	.235	.072	.039	.008	.004							
9	1	.752	.508	.344	.235	.111	.039	.022	.004	.002						
10	1	.752	.549	.344	.282	.111	.065	.022	.012	.002	.0010					
11	1	.772	.549	.388	.282	.147	.065	.039	.012	.006	.0010	.0005				
12	1	.772	.581	.388	.317	.147	.092	.039	.023	.006	.0034	.0005	.0002			
13	1	.789	.581	.423	.317	.181	.092	.057	.023	.013	.0034	.0018	.0002	.0001		
14	1	.789	.607	.423	.329	.181	.119	.057	.035	.013	.0074	.0018	.0009	.0001	.00005	
15	1	.803	.607	.436	.329	.211	.119	.077	.035	.019	.0074	.0032	.0009	.0004	.00005	.00003
16	1	.803	.629	.436	.340	.211	.143	.077	.049	.019	.0106	.0032	.0023	.0004	.00026	.00003

* Values are correct to 4 decimal places for $k \geq 10$, and three decimal places for $k \leq 9$.

Table 2 Table for the values of $P\{nD_n \geq k\}$ for $1 \leq k \leq n \leq 16$

n \ k	1	2	3	4	5	6	7	8	9	10*	11	12	13	14	15
2	1	.500													
3	1	.500	.250												
4	1	.750	.250	.125											
5	1	.750	.438	.125	.063										
6	1	.875	.438	.250	.063	.031									
7	1	.875	.598	.250	.143	.031	.015								
8	1	.938	.598	.470	.143	.078	.015	.008							
9	1	.938	.684	.470	.221	.078	.043	.008	.004						
10	1	.970	.684	.563	.221	.131	.043	.023	.004	.0020					
11	1	.970	.764	.563	.294	.131	.077	.023	.013	.0020	.0010				
12	1	.984	.764	.633	.294	.185	.077	.045	.013	.0068	.0010	.0005			
13	1	.984	.820	.633	.362	.185	.115	.045	.026	.0068	.0036	.0005	.0003		
14	1	.992	.820	.656	.362	.237	.115	.070	.026	.0148	.0036	.0018	.0003	.0001	
15	1	.992	.825	.656	.423	.237	.154	.070	.037	.0148	.0064	.0018	.0008	.0001	—
16	1	.994	.825	.678	.423	.286	.154	.098	.037	.0212	.0064	.0046	.0008	.0003	.0001

* Correct to 4 decimal places for $k \geq 10$ and up to three decimal places for $k \leq 9$.

erogeneity of the F_i, so long as (1.1) holds. In the particular case of $F_1=\cdots=F_n=F\in\mathcal{F}_0$, the proof of the theorem simplifies considerably.

The probabilities in (2.2) and (2.4) are computed for $n\leqq 16$, and presented in Tables 1 and 2.

3. Asymptotic distribution theory under the null hypothesis

Here we consider certain asymptotic expressions for (a) $\mathrm{P}\{n^{1/2}D_n^+\geqq y\}$, $\mathrm{P}\{n^{1/2}D_n^-\geqq y\}$ and $\mathrm{P}\{n^{1/2}D_n\geqq y\}$ and (b) $\mathrm{P}\{D_n^+\geqq y\}$, $\mathrm{P}\{D_n^-\geqq y\}$ and $\mathrm{P}\{D_n\geqq y\}$, where y $(0<y<\infty)$ is fixed. For this, let

$$\Phi(y)=(2\pi)^{-1/2}\int_{-\infty}^{y}[\exp(-t^2/2)]dt\ , \qquad -\infty<y<\infty\ . \tag{3.1}$$

Then, we have the following theorem.

THEOREM 3.1. *For every fixed y $(0<y<\infty)$, under H_0 (i.e., $\forall F_n\in\mathcal{F}_n^0$),*

$$\lim_{n\to\infty}\mathrm{P}\{n^{1/2}D_n^+\geqq y\}=\lim_{n\to\infty}\mathrm{P}\{n^{1/2}D_n^-\geqq y\}=2\Phi(-y)\ ; \tag{3.2}$$

$$\lim_{n\to\infty}\mathrm{P}\{n^{1/2}D_n\geqq y\}=4\left[\sum_{k=1}^{\infty}(-1)^{k-1}\Phi(-(2k-1)y)\right]\ . \tag{3.3}$$

PROOF. Let r_n be the number of successes in n independent Bernoullian trials with probability 1/2. Then, by (2.2) and (2.7),

$$\begin{aligned}\mathrm{P}\{n^{1/2}D_n^+\geqq y\}&=\mathrm{P}\{n^{1/2}D_n^-\geqq y\}\\&=2\,\mathrm{P}\{r_n\leqq s_n\}-\delta_k\,\mathrm{P}\{r_n=s_n\}\\&=2\,\mathrm{P}\{n^{-1/2}(2r_n-n)\leqq n^{-1/2}(2s_n-n)\}-\delta_k\,\mathrm{P}\{r_n=s_n\}\ ,\end{aligned} \tag{3.4}$$

where $s_n=[n/2-n^{1/2}y/2]$, so that $n^{-1/2}(2s_n-n)\to -y$, as $n\to\infty$. Also, by the DeMoivre-Laplace theorem, the right hand side of (3.4) tends to $\Phi(-y)$ as $n\to\infty$. Hence, (3.2) follows from (3.4). A similar proof applies to (3.3). Q.E.D.

Remark. By standard arguments [such as in Feller ([5], p. 230)], one could have approximated the random walk of Section 2 by a Brownian movement process, and then used the well-known results on the maximum (or absolute maximum) displacement of such a process to provide alternative proofs of (3.2) and (3.3).

For every ε: $0<\varepsilon<1/2$, let us now define

$$\rho(\varepsilon)=(1+2\varepsilon)^{-(1/2+\varepsilon)}(1-2\varepsilon)^{-(1/2-\varepsilon)}\ ; \qquad \rho(\varepsilon)=0 \text{ for } \varepsilon\geqq 1/2\ . \tag{3.5}$$

It is then easy to verify that $\rho(\varepsilon)$ is strictly $\downarrow$ in ε: $0<\varepsilon<1/2$, with $\rho(0)=1$ and $\lim_{\varepsilon\to 1/2}\rho(\varepsilon)=1/2$. Hence for any $\lambda>1$

$$\rho(\lambda\varepsilon)/\rho(\varepsilon)<1\ , \qquad \text{for all } 0<\varepsilon\leqq\lambda/2\ . \tag{3.6}$$

THEOREM 3.2. *Under H_0, for every ε: $0<\varepsilon<1$,*

$$\mathrm{P}\{D_n^+\geqq\varepsilon\}=\mathrm{P}\{D_n^-\geqq\varepsilon\}\leqq 2[\rho(\varepsilon/2)]^n\ , \tag{3.7}$$

$$\lim_{n\to\infty}[n^{-1}\log \mathrm{P}\{D_n^+\geqq\varepsilon\}]=\log\rho(\varepsilon/2)\ ; \tag{3.8}$$

$$\mathrm{P}\{D_n\geqq\varepsilon\}\leqq 4[\rho(\varepsilon/2)]^n\ , \quad \textit{and} \quad \lim_{n\to\infty}[n^{-1}\log \mathrm{P}\{D_n\geqq\varepsilon\}]=\log\rho(\varepsilon/2)\ . \tag{3.9}$$

PROOF. By (2.2) and (3.4), $\mathrm{P}\{D_n^+\geqq\varepsilon\}=\mathrm{P}\{D_n^-\geqq\varepsilon\}\leqq 2\,\mathrm{P}\{r_n\leqq s_n^*\}$, where $s_n^*=[n(1-\varepsilon)/2]$. Since, r_n is a sum of independent and bounded valued random variables, (3.7) follows from the Theorem 1 of Hoeffding [6], and (3.8) follows from Lemma 1 of Abrahamson [1], attributed to Bahadur and Rao [3]. Also, noting that for every $\varepsilon>0$ and $n\geqq1$,

$$\mathrm{P}\{D_n^+\geqq\varepsilon\}\leqq\mathrm{P}\{D_n\geqq\varepsilon\}\leqq\mathrm{P}\{D_n^+\geqq\varepsilon\}+\mathrm{P}\{D_n^-\geqq\varepsilon\}\ , \tag{3.10}$$

(3.9) follows readily from (3.7) and (3.8). Q.E.D.

4. Asymptotic non-null distribution theory

Let us define for every n ($\geqq1$),

$$\begin{aligned}\delta_n^+&=\sup_{x\geqq0}[\bar F_{(n)}(x)+\bar F_{(n)}(-x)-1]\ ,\\ \delta_n^-&=\sup_{x\geqq0}[1-\bar F_{(n)}(x)-\bar F_{(n)}(-x)]\ ;\end{aligned} \tag{4.1}$$

$$\delta_n=\max(\delta_n^+,\delta_n^-)=\sup_{x\geqq0}|\bar F_{(n)}(x)+\bar F_{(n)}(-x)-1|\ . \tag{4.2}$$

Now, by the same proof as in the Glivenko-Cantelli Theorem, $\limsup_n \{\sup_x|\bar F_n^*(x)-\bar F_{(n)}(x)|\}=0$ a.s. (almost surely). Hence, by (1.5), (1.6), (4.1) and (4.2), as $n\to\infty$,

$$D_n^+-\delta_n^+\ , \quad D_n^--\delta_n^- \quad \text{and} \quad D_n-\delta_n \qquad \text{all tend to 0 a.s.} \tag{4.3}$$

Thus, for every $\varepsilon>0$,

$$\lim_{n\to\infty}\mathrm{P}\{D_n^+>\delta_n^++\varepsilon\}=0\ , \qquad \lim_{n\to\infty}\mathrm{P}\{D_n^+<\delta_n^+-\varepsilon\}=0\ , \tag{4.4}$$

and similar results hold for D_n^- and D_n. In the same fashion as in Theorem 3.2, we shall now provide certain exponential rates of convergence to (4.4).

For every n ($\geqq1$) and x ($\geqq0$), we let

$$a_n^+(x)=\delta_n^+-[\bar F_{(n)}(x)+\bar F_{(n)}(-x)-1] \qquad (\geqq0)\ , \tag{4.5}$$

(4.6) $g_x(X_i)=c(x-X_i)+c(-x-X_i-)-F_i(x)-F_i(-x)$, $i=1,\cdots,n$;

(4.7) $\psi_{n,\varepsilon}^{(i)}(t,x)=\{\exp[-t(a_n^+(x)+\varepsilon)]\}\,\mathrm{E}\{\exp[tg_x(X_i)]\}$, $i=1,\cdots,n$,

(4.8) $\psi_{n,\varepsilon}^*(t,x)=\left[\prod_{i=1}^n \psi_{n,\varepsilon}^{(i)}(t,x)\right]^{1/n}$;

(4.9) $\rho_n^+(\varepsilon,x)=\psi_{n,\varepsilon}^*(t_n,x)=\inf_{t>0}\psi_{n,\varepsilon}^*(t,x)$;

(4.10) $\rho_n^+(\varepsilon)=\sup_{x\geq 0}\rho_n^+(\varepsilon,x)$.

We may remark that $g_x(X_i)$, $i=1,\cdots,n$, are all bounded random variables for all $x\geqq 0$, $\mathrm{E}\,g_x(X_i)=0$, and $\psi_{n,\varepsilon}^*(t,x)<\infty$ for all $x\geqq 0$, $t>0$. Hence, if we assume that

(4.11) $$\inf_n n^{-1}\sum_{i=1}^n \mathrm{var}\,[g_x(X_i)]\geqq B(x)>0 \qquad \text{for every } x\geqq 0 ,$$

it readily follows that

(4.12) $$\sup_n \rho_n^+(\varepsilon)<1 \qquad \text{for every } \varepsilon>0 .$$

We shall also assume that the average df $\bar{F}_{(n)}$ is non-degenerate and uniformly (in n) continuous, so that for every $\eta_1>0$, there exists a η_2 (>0), such that

(4.13) $$|\bar{F}_{(n)}(y)-\bar{F}_{(n)}(x)|<\eta_1 \qquad \text{for all } |y-x|<\eta_2, \text{ and } n .$$

(4.11) and (4.13) are satisfied, for example, when $X_1,\cdots,X_n$ are from c ($\geqq 1$) different homogeneous distributions $F_1,\cdots,F_c$, such as in the paired comparisons models, considered by Puri and Sen [7], and others. Sen [8] has also considered some other models for which (4.11) and (4.13) hold. We may add that (4.11) insures the applicability of the central limit theorem for $\{g_x(X_i),\ i=1,\cdots,n\}$, $x\geqq 0$, as will be needed in the sequel. For homogeneous df's, $\bar{F}_{(n)}=F$, and we do not need (4.11) and (4.13). But, in the heterogeneous case, without (4.13), we allow the possibility of having the entire variation of $\bar{F}_{(n)}$ in an arbitralily small interval, so that we may have $\mathrm{P}\{D_n^+>\delta_n^++\varepsilon\}$ either equal to 0 or converging to 0 faster than an exponential rate.

THEOREM 4.1. *Under* (4.11) *and* (4.13), *for every* $\varepsilon>0$,

(4.14) $$\limsup_n |n^{-1}\log \mathrm{P}\{D_n^+>\delta_n^++\varepsilon\}-\log \rho_n^+(\varepsilon)|=0 .$$

PROOF. By (1.2), (1.5) and (4.1), for every $\varepsilon>0$,

(4.15) $$P_{n,\varepsilon}^+=\mathrm{P}\{D_n^+>\delta_n^++\varepsilon\}$$
$$=\mathrm{P}\left\{n^{-1}\sum_{i=1}^n g_x(X_i)>a_n^+(x)+\varepsilon,\ \text{for some } x\geqq 0\right\} .$$

Now, on using the basic transformation in Section 3 of Feller [4], Lemma 2 of Bahadur and Ranga Rao [3] readily extends to the case of non-identically distributed random variables, so that for every $x \geqq 0$,

$$\text{(4.16)} \qquad \mathrm{P}\left\{n^{-1}\sum_{i=1}^{n} g_x(X_i) > a_n^+(x) + \varepsilon\right\} = [\rho_n^+(\varepsilon, x)]^n I_n(x) ,$$

where under (4.11), for every $x \geqq 0$,

$$\text{(4.17)} \qquad n^{-1}\log I_n(x) = o(1) .$$

By (4.15) and (4.16),

$$\text{(4.18)} \qquad P_{n,\varepsilon}^+ \geqq [\rho_n^+(\varepsilon, x)]^n I_n(x) , \qquad \text{for every } x \geqq 0 ,$$

and hence, on taking the supremum (over x) and using (4.17), we have

$$\text{(4.19)} \qquad \liminf_n \, [n^{-1}\log P_{n,\varepsilon}^+ - \log \rho_n^+(\varepsilon)] \geqq 0 .$$

Thus, it suffices to show that

$$\text{(4.20)} \qquad \limsup_n \, [n^{-1}\log P_{n,\varepsilon}^+ - \log \rho_n^+(\varepsilon)] \leqq 0 .$$

Now, by (4.13), for every η: $0<\eta<\varepsilon$, we can choose a set of $(m+1)$ points (where $m=m(\eta)$), $x_0, x_1, \cdots, x_m$, where $0=x_0<x_1<\cdots<x_{m-1}<x_m=+\infty$, such that for all n,

$$\text{(4.21)} \qquad \bar{F}_{(n)}(x_i) + \bar{F}_{(n)}(-x_i) - \bar{F}_{(n)}(x_{i-1}) - \bar{F}_{(n)}(-x_{i-1}) < \eta , \qquad i=1,\cdots, m .$$

Hence, it is easy to show that

$$\text{(4.22)} \qquad \left|\sup_x \left[n^{-1}\sum_{i=1}^{n} g_x(X_i) - a_n^+(x)\right] - \max_{0\leqq j\leqq m}\left[n^{-1}\sum_{i=1}^{m} g_{x_j}(X_i) - a_n^+(x_j)\right]\right| < \eta .$$

Consequently, using (4.15) for each x_j and replacing ε by $\varepsilon-\eta$, we obtain that

$$\begin{aligned}\text{(4.23)} \qquad n^{-1}\log P_{n,\varepsilon}^+ &\leqq n^{-1}\log \mathrm{P}\left\{\max_{0\leqq j\leqq m}\left[n^{-1}\sum_{i=1}^{n} g_{x_j}(X_i) - a_n^+(x_j)\right] > \varepsilon-\eta\right\} \\ &\leqq n^{-1}\log\left\{\sum_{j=0}^{m}\mathrm{P}\left\{\left[n^{-1}\sum_{i=1}^{n} g_{x_j}(X_i)\right] > a_n^+(x_j)+\varepsilon-\eta\right\}\right\} \\ &\leqq n^{-1}\log\left\{\sum_{j=0}^{m}[\rho_n^+(\varepsilon-\eta, x_j)]^n I_n(x_j)\right\} \\ &\leqq \log \rho_n^+(\varepsilon-\eta) + n^{-1}\log m + o(1) .\end{aligned}$$

Now, it is easy to show that $\rho_n^+(\varepsilon)$ is left continuous in ε (uniformly in n, under (4.11) and (4.13)). Hence, we complete the proof of (4.20) from (4.23) by letting η to be arbitrarily small. Q.E.D.

In (4.7) through (4.10), on replacing $g_x(X_i)$ by $-g_x(X_i)$, $i=1,\cdots,n$,

$x \geqq 0$, we define $\rho_n^-(\varepsilon)$ is an analogous way. Also, let

$$\rho_n(\varepsilon) = \max [\rho_n^+(\varepsilon), \rho_n^-(\varepsilon)] . \tag{4.24}$$

Then, proceeding as in Theorem 4.1, we have the following.

THEOREM 4.2. *Under* (4.11) *and* (4.13), *for every* $\varepsilon > 0$,

$$\limsup_n | n^{-1} \log \mathrm{P}\{D_n^- > \delta_n^- + \varepsilon\} - \log \rho_n^-(\varepsilon) | = 0 , \tag{4.25}$$

$$\limsup_n | n^{-1} \log \mathrm{P}\{D_n > \delta_n + \varepsilon\} - \log \rho_n(\varepsilon) | = 0 . \tag{4.26}$$

Analogous to Theorem 3.1, we may consider the asymptotic distribution of $n^{1/2}D_n^+$ (or $n^{1/2}D_n^-$ or $n^{1/2}D_n$). This, however, requires $n^{1/2}\delta_n$ to be bounded as $n \to \infty$, and moreover, that $\bar{F}_{(n)}$ weakly converges to a df $\bar{F}$, as $n \to \infty$, and

$$\lim_{n\to\infty} n^{1/2}[\bar{F}_{(n)}(x) + \bar{F}_{(n)}(-x) - 1] = h(x) = h^*(\bar{F}(x)) , \qquad \text{for all } x \geqq 0 , \tag{4.27}$$

where $\bar{F}$ is symmetric about 0. In this case, if we define the process $V_n^*(t)$ as in (2.5), then under (4.27), it follows that $\mathrm{E}\, V_n^*(t)$ converges to $h^*(1-t)$ as $n \to \infty$, while the covariance structure remains the same as in the null hypothesis case, treated in Section 2. Consequently, the distribution of $n^{1/2}D_n^+$ asymptotically reduces to that of the maximum positive displacement of a Gaussian function $Y(t)$ $t \in [0, 1/2]$, where $\mathrm{E}\, Y(t) = h^*(1-t)$ and $\mathrm{Cov}\,[Y(s), Y(t)] = 2(\min[s, t])$. Thus, for specific nature of the drift $h^*(1-t)$ (such as linear in t etc.,), existing results on Brownian motion processes can be utilized for the study of the asymptotic distribution of $n^{1/2}D_n^+$, and similarly for $n^{1/2}D_n^-$ or $n^{1/2}D_n$. In this respect, the situation is similar to that of the one-sample Kolmogorov or the two-sample Kolmogorov-Smirnov goodness of fit test which involves the same problem but with a Brownian bridge instead of a Brownian motion. The authors feel that much more work in this general area needs to be accomplished before a systematic presentation of the allied asymptotic distribution theory can be made.

5. Exact Bahadur-efficiencies for D_n and the sign statistics

Following Abrahamson [1], but without restricting ourselves to the case of identical distributions, we briefly sketch the Bahadur-efficiency of two sequences of statistics, when, in particular, we are interested in the hypothesis of symmetry, as considered in Section 1. Let $\mathcal{F}_1$ be the class of all continuous df's on the real line, not symmetric about 0. Thus, if we let

(5.1) $$\delta(F)=\sup_{x\geq 0}|F(x)+F(-x)-1| ,$$

then $\delta(F)=0$, $\forall F\in\mathcal{F}_0$, while $\delta(F)>0$, for any $F\in\mathcal{F}_1$.

Consider now two sequences $\{T_1^{(1)}\}$ and $\{T_n^{(2)}\}$ of non-negative real valued statistics, satisfying the following four conditions:

(1) there exists a non-degenerate and continuous df $\Psi_i(x)$, such that for all $\boldsymbol{F}_n\in\mathcal{F}_n^0$ and real r $(0<r<\infty)$,

(5.2) $$\lim_{n\to\infty} \mathrm{P}_{\boldsymbol{F}_n}\{T_n^{(i)}<r\}=\Psi_i(r) ,$$

(2) there exists a non-negative function l_i on $[0, \infty]$ such that (i) $l_i(z)>0$ for all $z\in(0, \infty)$, and (ii) whenever $\{u_n\}$ is a sequence of real numbers for which $n^{-1}u_n^2\to z\in(0, \infty)$, we have

(5.3) $$-\lim_{n\to\infty}(2/n)\log \mathrm{P}_{\boldsymbol{F}_n}\{T_n^{(i)}\geqq u_n\}=l_i(z) ,$$

uniformly in $\boldsymbol{F}_n\in\mathcal{F}_n^0$,

(3) for every $\boldsymbol{F}_n$ not necessarily belonging to $\mathcal{F}_n^0$,

(5.4) $$|n^{-1/2}T_n^{(i)}-b_i(\bar{F}_{(n)})|\to 0 \text{ a.s.} , \qquad \text{as } n\to\infty,\ i=1, 2 ,$$

and finally, (4) the average df $\bar{F}_{(n)}$ converges to a continuous df $\bar{F}$, such that as $n\to\infty$,

(5.5) $$b_i(\bar{F}_{(n)})\to b_i(\bar{F}) \quad (>0 \text{ whenever } \bar{F}\notin\mathcal{F}_0) , \qquad i=1, 2 .$$

The last assumption, needed only for the heterogeneous case, appears to be necessary for justifying the existence of a limit implicit in the definition of the asymptotic efficiency, and will be clear in the definition (5.6).

We now define the exact asymptotic efficiency of $T_n^{(1)}$ with respect to $T_n^{(2)}$ as equal to

(5.6) $$\begin{aligned} e_{1,2}^{(1)}&=\lim_{n\to\infty}[l_1(b_1^2(\bar{F}_{(n)}))/l_2(b_2^2(\bar{F}_{(n)}))] \\ &=[l_1(b_1^2(\bar{F}))/l_2(b_2^2(\bar{F}))]=e_{1,2}^{(1)}(\bar{F}) , \qquad \text{say} , \end{aligned}$$

and with the metric $\delta(F)$, defined by (5.1), the limit

(5.7) $$e_{1,2}^{(2)}(\bar{F})=\lim_{\delta(\bar{F})\to 0} e_{1,2}^{(1)}(\bar{F}) \qquad \text{(assumed to exist)}$$

is defined the exact asymptotic limiting efficiency, both defined after Bahadur [2], as further interpreted in Abrahamson [1].

Let now $T_n^{(1)}=n^{1/2}D_n$. Under H_0 in (1.5), the distribution of $T_n^{(1)}$ is independent of $\boldsymbol{F}_n$, and by (3.3), we have

(5.8) $$\Psi_1(r)=1-4\sum_{k=1}^{\infty}(-1)^{k-1}\Phi(-(2k-1)r) , \qquad 0<r<\infty,\ \forall \boldsymbol{F}_n\in\mathcal{F}_n^0 .$$

Further, using Theorem 3.2, (3.5) and some standard computations we obtain that for $\{u_n\}$ for which $u_n^2/n \to z \in (0, 1)$,

$$-\lim_{n\to\infty} (2/n) \log \mathrm{P}\{T_n^{(1)} \geqq u_n\} = \sum_{k=1}^{\infty} z^k/k(2k-1)\ , \qquad \forall F_n \in \mathcal{F}_n^0\ . \tag{5.9}$$

Finally, by the Glivenko-Cantelli theorem, $\limsup_{n\to\infty} \sup_x |F_n^*(x) - \bar{F}_{(n)}(x)| = 0$, a.s., and hence, by (1.6), (5.1) and noting that $T_n^{(1)} = n^{1/2} D_n$,

$$|n^{-1/2} T_n^{(1)} - \delta(\bar{F}_{(n)})| \to 0 \text{ a.s.}\ , \qquad \text{as } n \to \infty\ , \tag{5.10}$$

and as $\delta(F)$ is a bounded and continuous functional of F,

$$\bar{F}_{(n)} \to \bar{F} \text{ (weakly)} \Rightarrow \delta(\bar{F}_{(n)}) \to \delta(\bar{F})\ , \qquad \text{as } n \to \infty\ . \tag{5.11}$$

So, for D_n all the four conditions are satisfied.

Let us now consider the sign statistic S_n, defined by

$$S_n = n^{-1/2}(2r_n - n)\ ; \qquad r_n = \sum_{i=1}^{n} c(X_i)\ , \tag{5.12}$$

where $c(u)$ is defined after (1.1). If we then let $T_n^{(2)} = |S_n|$, we have

$$\Psi_2(r) = \Phi(r) - \Phi(-r)\ , \qquad 0 \leqq r < \infty,\ \forall F_n \in \mathcal{F}_n^0\ . \tag{5.13}$$

Also, using Lemma 1 of Abrahamson [1] and some standard computations, we have, parallel to (5.9),

$$-\lim_{n\to\infty} (2/n) \log \mathrm{P}\{T_n^{(2)} \geqq u_n\} = \sum_{k=1}^{\infty} z^k/k(2k-1)\ , \qquad \forall F_n \in \mathcal{F}_n^0\ . \tag{5.14}$$

Finally, by the Borel strong law of large numbers, as $n \to \infty$,

$$n^{-1/2} T_n^{(2)} = n^{-1}(2r_n - n) \sim \delta_0(\bar{F}_{(n)}) = 2\bar{F}_{(n)}(0) - 1\ , \qquad \text{a.s.}\ , \tag{5.15}$$

where obviously,

$$\bar{F}_{(n)} \to \bar{F} \text{ (weakly)} \Rightarrow \delta_0(\bar{F}_{(n)}) \to \delta_0(\bar{F}) \qquad \text{as } n \to \infty\ . \tag{5.16}$$

Hence, the conditions are also satisfied for the sign statistic. Thus, the asymptotic efficiencies of D_n with respect to S_n, as defined by (5.6) and (5.7), are equal to

$$e^{(1)}(\bar{F}) = \left[\sum_{k=1}^{\infty} \{\delta(\bar{F})\}^{2k}/k(2k-1)\right] \Big/ \left[\sum_{k=1}^{\infty} \{\delta_0(\bar{F})\}^{2k}/k(2k-1)\right]\ , \tag{5.17}$$

$$e^{(2)}(\bar{F}) = \lim_{\delta(F)\to 0} \{\delta(\bar{F})/\delta_0(\bar{F})\}^2\ . \tag{5.18}$$

Now, note that by (5.1) and (5.15), $\delta(\bar{F}) \geqq \delta_0(\bar{F})$, $\forall \bar{F} \in \mathcal{F}_0 \cup \mathcal{F}_1$. Hence, from (5.17) and (5.18) we arrive at the following:

$$e^{(1)}(\bar{F}) \geqq e^{(2)}(\bar{F}) \geqq 1\ , \qquad \text{for all } \bar{F}\ . \tag{5.19}$$

Thus, the proposed test is at least as efficient (asymptotically) as the sign-test for all $\bar{F}$. In particular, if $\bar{F}(x)$ ($\in \mathscr{F}_0$) is symmetric and unimodal, and we are interested only in shift alternatives, then $\delta(\bar{F})=\delta_0(\bar{F})$, so that in (5.19) the equality signs hold; the conclusion is not necessarily true when $\bar{F}(x)$ is not strictly unimodal [viz., the uniform df]. On the other hand, for certain specific type of asymmetry (of $\bar{F}$), $\delta_0(\bar{F})$ may be exactly or nearly equal to zero, but $\delta(\bar{F})$ can still be positive, making (5.17) or (5.18) either ∞ or indefinitely large.

For other tests for symmetry, the Bahadur efficiency of D_n may be computed in a similar way; for brevity the details are omitted.

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Note added in the proof. For the particular case of iidrv, Theorem 2.1 has been obtained earlier by C. C. Butler [*Ann. Math. Statist.*, **40**, 2209-2210]. However, all the other results deduced here are new and also applicable in his case.