UNIFORM ASYMPTOTIC JOINT NORMALITY OF SAMPLE QUANTILES IN CENSORED CASES

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Summary

The asymptotic joint distribution of an increasing number of sample quantiles as the sample size increases, when the underlying sample is censored, is shown to be asymptotically uniformly (or type $(\boldsymbol{B})_d$) normally distributed under fairly general conditions. The discussions for uncensored cases have been given by [4].

1. Introduction

Let $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a random sample of size n from a univariate, real and continuous distribution with pdf. $f_n(x)$ and cdf. $F_n(x)$. Further, let $l_{ni}=n_i/(n+1)$, $s_{ni}=F_n^{-1}(l_{ni})$ and $f_{ni}=f_n(s_{ni})$, $i=1,\cdots,k$, where $n_1<n_2<\cdots<n_k$, $k=k(n)$ and $n_i=n_i(n)$ are allowable to depend on n. Then, it has been shown in [4] that under certain general regularity conditions for $f_n(x)$, the joint distribution of the k sample quantiles, $X_{n(k)}=(X_{n1},\cdots,X_{nk})'$, is asymptotically equivalent $(\boldsymbol{B})_d$ to a k-dimensional normal distribution with mean vector $s_{n(k)}=(s_{n1},\cdots,s_{nk})'$ and dispersion matrix $S_{n(k)}=||l_{ni}(1-l_{nj})/(f_{ni}f_{nj})||/(n+2)$, $1\leqq i\leqq j\leqq k$, provided that $k^2/\min_{1\leqq i\leqq k+1}(n_i-n_{i-1})\to 0$ as $n\to\infty$.

There sometimes occur the situations, for example, in life testing, where we have to treat censored data and make statistical inferences by using them. In this respect it is worth while considering the asymptotic joint normality of sample quantiles for censored samples. In this article, it is shown that under some mild conditions, the asymptotic joint $(\boldsymbol{B})_d$ normality of sample quantiles, discussed in [4], can be extended to the cases of Type I and Type II censored samples.

In Section 2, we state necessary notations and assumptions. In Sections 3 and 4 the Type I censored cases are treated. In Section 3, the cases of uniform distributions are discussed, which play a fundamental role in discussions of subsequent sections. Section 4 concerns general cases of *unequal basic distributions*. Finally, Type II censored cases are treated in Section 5.

2. Preliminaries

Let, for each positive integer n, a_n and b_n be the left censorship point and the right one, respectively; they are preassigned extended real numbers such that $F_n(a_n)<F_n(b_n)$, for each n. In the first place we shall consider the typical Type I doubly censoring in which case the variates X_{ni}'s such that $X_{ni}\leqq a_n$ and $b_n\leqq X_{ni}$ are censored for each n.

Suppose that $X_{n,S+1}<X_{n,S+2}<\cdots<X_{n,S+T}$ are doubly censored ordered variates, where, for each n, $S=S(n)$ is the number of observations in the left censored portion and $T=T(n)$ is the number of the variates observed in the uncensored portion. As in [3], it should be noted that both S and T are random variables, whose joint probabilities for possible integers $s=s(n)$ and $t=t(n)$ $(0\leqq s\leqq n-t,\ 0\leqq t\leqq n)$ are given by

$$p_n(s, t)=\frac{n!}{s!t!(n-s-t)!}(\alpha_n)^s(\eta_n)^t(1-\beta_n)^{n-s-t}, \tag{2.1}$$

where we put $\alpha_n=F(a_n)$, $\beta_n=F(b_n)$ and $\eta_n=\beta_n-\alpha_n>0$.

Let, for each n, $((S, T): X_{n(k)}^{S,T})$ be a mixed $(k+2)$-dimensional random vector (cf. [6]) such that $X_{n(k)}^{s,t}=(X_{nn_1},\cdots, X_{nn_k})'$ given $(S, T)=(s, t)$ is the joint random variable of k sample quantiles $(a_n<)\ X_{nn_1}<\cdots<X_{nn_k}\ (<b_n)$, where the ranks n_i, $i=1,\cdots,k$, are so chosen that $s<n_1<n_2<\cdots<n_k<s+t$. Thus, $X_{n(k)}^{s,t}$ is regarded as a k-dimensional continuous conditional random vector under the condition $(S, T)=(s, t)$. Let, further, for each n, $((S, T): Y_{n(k)}^{S,T})$ be a mixed $(k+2)$-dimensional random vector such that $Y_{n(k)}^{s,t}=(Y_{n1},\cdots, Y_{nk})'$ given $(S, T)=(s, t)$ is a certain k-dimensional *normal* random vector. Note that the number k generally depends not only on n but also on t, since, in effect, k must be less than t, for each n. In the following discussions, however, to avoid complexity, we shall restrict the cases where k may depend only on n, but not on t, and it is assumed to satisfy the following

ASSUMPTION I-0. For every n, $k(n)<n(\eta_n-\varepsilon_n)$, where ε_n is a non-negative constant such that $\varepsilon_n<\eta_n$ and $n\varepsilon_n^2\to\infty$.

Under this assumption it is easily seen by [5] that $\Pr\{T>k\}\to 1$ holds as $n\to\infty$. We now define

DEFINITION 2.1. Under Assumption I-0, the joint variable of $k=k(n)$ sample quantiles from the Type I (doubly) censored sample $X_{n(k)}^{S,T}$, or precisely $((S, T): X_{n(k)}^{S,T})$ is said to be asymptotically $(\boldsymbol{B})_d$ normal, and is denoted by

$$X_{n(k)}^{S,T}\sim Y_{n(k)}^{S,T}\ (\boldsymbol{B})_d\,, \qquad (n\to\infty)\,, \tag{2.2}$$

if it holds that

$$\sup_{\substack{K_n \in I_{(2)} \\ E \in \boldsymbol{B}_{(k)}}} \left| \sum_{(s,t) \in K_n} \{p_n(s, t) \cdot (\mathrm{P}^{X^{s,t}_{n(k)}}(E) - \mathrm{P}^{Y^{s,t}_{n(k)}}(E))\} \right| \to 0 , \quad (n \to \infty) , \tag{2.3}$$

where $I_{(2)} = \{(i, j): i, j \text{ are non-negative integers}\}$ and $\boldsymbol{B}_{(k)}$ denotes the usual Borel field of subsets of k-dimensional Euclidean space $R_{(k)}$.

Here, we state a sufficient condition for (2.3), which gives us a criterion of the asymptotic $(\boldsymbol{B})_d$ normality of $X^{S,T}_{n(k)}$:

LEMMA 2.1. *The condition*

$$\sum_{(s,t) \in K_n} \{p_n(s, t) \cdot I(X^{s,t}_{n(k)} : Y^{s,t}_{n(k)})\} \to 0 , \qquad (n \to \infty) , \tag{2.4}$$

implies (2.2), *where* $I(X^{s,t}_{n(k)} : Y^{s,t}_{n(k)})$ *stands for Kullback-Leiblers' mean information for conditional variables* $X^{S,T}_{n(k)}$ *and* $Y^{S,T}_{n(k)}$ *given* $(S, T) = (s, t)$ *and* $K_n \stackrel{d}{=} \{(s, t): 0 \leqq s \leqq n-t,\ 0 < k < t \leqq n\}$.

The proof of this lemma can be done by a similar way of Lemma 1.3 in [3], and will be omitted.

The following assumptions will be referred to in the later section dealing with Type I censored samples from general unequal basic distributions.

ASSUMPTION I-1. For every n, $D_{\mathrm{I}}(f_n) = \{x: f_n(x) > 0 \text{ and } a_n < x < b_n\}$ is an open interval on the real line.

ASSUMPTION I-2. For every n, $f_n(x)$ is differentiable once over the interval $D_{\mathrm{I}}(f_n)$,

and for some distributions, we put

ASSUMPTION I-3. For some fixed positive numbers M_1 and M_2,

$$\inf_{x \in D_{\mathrm{I}}(f_n)} f_n(x) \geqq M_1 \qquad \text{and} \qquad \sup_{x \in D_{\mathrm{I}}(f_n)} |f_n'(x)| \leqq M_2 ,$$

uniformly in n.

In the second place, we shall consider the setup of the asymptotic joint $(\boldsymbol{B})_d$ normality of an increasing number of sample quantiles in Type II censored cases. Let us consider the situation such that the first p and the last q variables are censored from the whole ordered sample $X_{n1} < \cdots < X_{nn}$. Then we have a Type II doubly censored sample $X_{n,p+1} < X_{n,p+2} < \cdots < X_{n,n-q}$ of size $n' \stackrel{d}{=} n-p-q$, where p and q may depend on n.

Suppose that $X_{n(k)}^{p,q}=(X_{nn_1},\cdots,X_{nn_k})'$ is k sample quantiles from the Type II censored sample where $p<n_1<n_2<\cdots<n_k<n-q+1$ and k may depend on n. Further, let $Y_{n(k)}^{p,q}=(Y_{n1},\cdots,Y_{nk})'$ be a certain k-dimensional *normal* random vector. Under the above setup we define the following

DEFINITION 2.2. The joint variable of $k=k(n)$ sample quantiles from a Type II (doubly) censored sample, $X_{n(k)}^{p,q}$, is said to be asymptotically $(\boldsymbol{B})_d$ normal, if it holds that

$$\sup_{E\in\boldsymbol{B}_{(k)}} |\mathrm{P}^{X_{n(k)}^{p,q}}(E)-\mathrm{P}^{Y_{n(k)}^{p,q}}(E)|\rightarrow 0\,, \qquad (n\rightarrow\infty)\,. \tag{2.5}$$

Analogously to Lemma 2.1, we may state

LEMMA 2.2. *The condition* $I(X_{n(k)}^{p,q}:Y_{n(k)}^{p,q})\rightarrow 0$, $(n\rightarrow\infty)$ *implies* (2.5).

The assumptions below will be needed for treating Type II censoring cases for unequal basic distributions in Section 5.

ASSUMPTION II-1. For every n, $D_{\mathrm{II}}(f_n)=\{x: f_n(x)>0\}$ is an open interval on the real line.

ASSUMPTION II-2. For every n, $f_n(x)$ is differentiable once over the interval $D_{\mathrm{II}}(f_n)$,

and for some distributions we put

ASSUMPTION II-3. For some fixed positive numbers M_1 and M_2,

$$\inf_{x\in D_{\mathrm{II}}(f_n)} f_n(x)\geqq M_1 \qquad \text{and} \qquad \sup_{x\in D_{\mathrm{II}}(f_n)} |f_n'(x)|\leqq M_2\,,$$

uniformly in n.

3. The case of uniform distribution

Let $U_{n1}<U_{n2}<\cdots<U_{nn}$ be order statistics of a random sample of size n from the uniform distribution over $(0,1)$. Suppose that, for each n, we have a censored sample $(\alpha_n<)\ U_{n,S+1}<U_{n,S+2}<\cdots<U_{n,S+T}$ $(<\beta_n)$ by Type I doubly censoring at α_n and β_n, which are assumed to be satisfied with Assumption I-0 in the preceding section.

Now let the joint variable of $k=k(n)$ ordered variates $(\alpha_n<)\ U_{nn_1}<\cdots<U_{nn_k}\ (<\beta_n)$ among the above censored sample, say $U_{n(k)}^{S,T}=(U_{nn_1},\cdots,U_{nn_k})'$. Then, for each n the conditional pdf. of $U_{n(k)}^{S,T}$ given $(S,T)=(s,t)$ is

$$h_n(x_{(k)}\,|\,s,t)=\frac{t!}{\eta_n^t\prod_{i=1}^{k+1}(d_{ni}^{st}!)}\prod_{i=1}^{k+1}(x_i-x_{i-1})^{d_{ni}^{st}}\,, \tag{3.1}$$

for $\alpha_n = x_0 < x_1 < \cdots < x_k < x_{k+1} = \beta_n$, where $d_{ni}^{st} = n_i - n_{i-1} - 1$, $i = 1, \cdots, k+1$, with the conventions $n_0 = s$, $n_{k+1} = s+t+1$ and $x_{(k)} = (x_1, \cdots, x_k)'$.

Here, for each n, making the transformation

$$(3.2) \qquad (x_{(k)} - \alpha_n \cdot 1_{(k)})/\eta_n = z_{(k)} ,$$

where $z_{(k)} = (z_1, \cdots, z_k)'$ and $1_{(k)} = (\underbrace{1, \cdots, 1}_{k})'$, then the pdf. of the transformed variables of $U_{n(k)}^{s,t}$, say $\boldsymbol{U}_{n(k)}^{s,t} = (\boldsymbol{U}_{nn_1}, \cdots, \boldsymbol{U}_{nn_k})'$, is

$$(3.3) \qquad h_n^*(z_{(k)} \mid s, t) = \frac{t!}{\prod_{i=1}^{k+1} (d_{ni}^{st}!)} \prod_{i=1}^{k+1} (z_i - z_{i-1})^{d_{ni}^{st}} ,$$

for $0 = z_0 < z_1 < \cdots < z_k < z_{k+1} = 1$, which is the joint pdf. of k order statistics from the uniform distribution over (0, 1). Then, it is easy to see

$$(3.4) \qquad \mathcal{E}h_n^*[\boldsymbol{U}_{nn_i}] = (n_i - s)/(t+1) \overset{d}{=} l_{ni}^{st} , \qquad 1 \leqq i \leqq k ,$$

and

$$(3.5) \qquad \mathrm{Cov}h_n^* [\boldsymbol{U}_{nn_i}, \boldsymbol{U}_{nn_j}] = l_{ni}^{st}(1 - l_{nj}^{st})/(t+2) , \qquad 1 \leqq i \leqq j \leqq k .$$

Hence,

$$(3.6) \qquad \mathcal{E}h_n[U_{nn_i}] = \eta_n l_{ni}^{st} + \alpha_n \overset{d}{=} \mu_{ni}^{st} , \qquad 1 \leqq i \leqq k ,$$

and

$$(3.7) \qquad \mathrm{Cov}h_n [U_{nn_i}, U_{nn_j}] = (\mu_{ni}^{st} - \alpha_n)(\beta_n - \mu_{nj}^{st})/(t+2) , \qquad 1 \leqq i \leqq j \leqq k .$$

Therefore, the mean vector and the dispersion matrix of $U_{n(k)}^{s,t}$ under the condition $(S, T) = (s, t)$ are given respectively by

$$(3.8) \qquad \mu_{n(k)}^{s,t} = (\mu_{n1}^{st}, \mu_{n2}^{st}, \cdots, \mu_{nk}^{st})'$$

and

$$(3.9) \qquad M_{n(k)}^{s,t} = \| (\mu_{ni}^{st} - \alpha_n)(\beta_n - \mu_{nj}^{st}) \| /(t+2) , \qquad 1 \leqq i \leqq j \leqq k .$$

Let $((S, T) : Z_{n(k)}^{S,T})$ be a mixed $(k+2)$-dimensional random vector such that its continuous conditional random vector $Z_{n(k)}^{s,t} = (Z_{n1}, \cdots, Z_{nk})'$ given $(S, T) = (s, t)$ is a k-dimensional *normal* random vector whose mean vector and dispersion matrix given $(S, T) = (s, t)$ are defined by (3.8) and (3.9), respectively. Then, for each n the conditional pdf. of $Z_{n(k)}^{S,T}$ given $(S, T) = (s, t)$ is, for $-\infty < x_i < \infty$, $i = 1, \cdots, k$,

$$(3.10) \qquad q_n(x_{(k)} \mid s, t) = (2\pi)^{-k/2} | M_{n(k)}^{s,t} |^{-1/2} \times \exp \left[\frac{-1}{2} (x_{(k)} - \mu_{n(k)}^{s,t})' (M_{n(k)}^{s,t})^{-1} (x_{(k)} - \mu_{n(k)}^{s,t}) \right] .$$

Using the transformation (3.2) again, the pdf. of the transformed k-

dimensional *normal* random vector, say $Z_{n(k)}^{s,t}=(Z_{n1},\cdots,Z_{nk})'$, is given by

$$(3.11)\qquad q_n^*(z_{(k)}|s,t)=(2\pi)^{-k/2}|L_{n(k)}^{s,t}|^{-1/2} \times\exp\left[\frac{-1}{2}(z_{(k)}-l_{n(k)}^{s,t})'(L_{n(k)}^{s,t})^{-1}(z_{(k)}-l_{n(k)}^{s,t})\right],$$

where

$$(3.12)\qquad l_{n(k)}^{s,t}=(l_{n1}^{st},l_{n2}^{st},\cdots,l_{nk}^{st})'$$

and

$$(3.13)\qquad L_{n(k)}^{s,t}=M_{n(k)}^{s,t}/\eta_n^2=||l_{ni}^{st}(1-l_{nj}^{st})||/(t+2)\,,\qquad 1\leqq i\leqq j\leqq k\,,$$

which, of course, coinside with the mean vector and the covariance matrix of $U_{n(k)}^{s,t}$.

Now, we shall prove the following

THEOREM 3.1. *Under Assumption* I-0, *if the condition*

$$(3.14)\qquad k(n)\sum_{(s,t)\in K_n}\left\{p_n(s,t)\Big/\min_{1\leqq i\leqq k+1}d_{ni}^{st}\right\}\to 0\,,\qquad (n\to\infty)$$

is satisfied, then it holds that

$$(3.15)\qquad U_{n(k)}^{S,T}\sim Z_{n(k)}^{S,T}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,.$$

PROOF. Since, for every n, $I(U_{n(k)}^{s,t}:Z_{n(k)}^{s,t})=I(\boldsymbol{U}_{n(k)}^{s,t}:\boldsymbol{Z}_{n(k)}^{s,t})$, it suffices to show that

$$(3.16)\qquad \sum_{(s,t)\in K_n}\{p_n(s,t)I(\boldsymbol{U}_{n(k)}^{s,t}:\boldsymbol{Z}_{n(k)}^{s,t})\}\stackrel{d}{=}I(\boldsymbol{U}_{n(k)}^{S,T}:\boldsymbol{Z}_{n(k)}^{S,T})\to 0\,,\qquad (n\to\infty)\,.$$

Using Stirling's formula, we have

$$(3.17)\qquad I(\boldsymbol{U}_{n(k)}^{s,t}:\boldsymbol{Z}_{n(k)}^{s,t})=\mathcal{E}h_n^*[\log[h_n^*(\boldsymbol{U}_{n(k)}^{S,T}|s,t)/q_n^*(\boldsymbol{U}_{n(k)}^{S,T}|s,t)]]$$
$$=\frac{k}{2}\log\left(1-\frac{2}{t+2}\right)+\frac{k+1}{2}\log\left(1-\frac{1}{t+1}\right)$$
$$+\frac{1}{2}\sum_{i=1}^{k+1}\log\left(1+\frac{1}{d_{ni}^{st}}\right)+\frac{k}{2t}+\left(1-\frac{k}{t}\right)A_n(t)$$
$$-\sum_{i=1}^{k+1}A_n(d_{ni}^{st})+\frac{c(t)}{t}+\sum_{i=1}^{k+1}\frac{c_i(t)}{d_{ni}^{st}}\,,$$

where $c(t)=O(1)$, $\max\{c_i(t);\ i=1,\cdots,k+1\}=O(1)$ as $t\to\infty$ and

$$A_n(m)=\sum_{r=1}^{\infty}\frac{a_{r+1}}{(m+1)\cdots(m+r)}$$

for any integer $m\geqq 2$, with

$$a_r=\frac{1}{r}\int_0^1 z(1-z)(2-z)\cdots(r-1-z)dz\ , \qquad (r\geqq 2)\ .$$

Since the K-L information is always non-negative, we may delete non-positive terms from the right-hand side of (3.17), which gives us

$$I(\boldsymbol{U}_{n(k)}^{s,t}:\boldsymbol{Z}_{n(k)}^{s,t})<\frac{k+1}{2d_n^{st}}+\frac{k}{2t}+A_n(t)+\left(\frac{1}{t}+\frac{k+1}{d_n^{st}}\right)c\ , \tag{3.18}$$

where $d_n^{st}=\min\{d_{ni}^{st}:\ i=1,\cdots,k+1\}$, and c is a positive constant.

Taking the expectation of both sides of (3.17) with respect to S and T, we have

$$\begin{aligned}I(\boldsymbol{U}_{n(k)}^{S,T}:\boldsymbol{Z}_{n(k)}^{S,T})<(c+1/2)(k+1)&\sum_{(s,t)\in K_n}\{p_n(s,t)/d_n^{st}\}\\&+(c+k/2)\sum_{t\in K_n'}\{p_n(t)/t\}+\sum_{t\in K_n'}\{p_n(t)A_n(t)\}\ ,\end{aligned} \tag{3.19}$$

where K_n is the same in (2.4), $K_n'=\{t:\ 0<k<t\leqq n\}$ and $p_n(t)$ is the marginal probability function of T.

Under Assumption I-0, we define the following set

$$E_{n,\varepsilon}=\{t:\ n(\eta_n-\varepsilon_n)\leqq t\leqq n(\eta_n+\varepsilon_n)\}\quad\text{and}\quad E_{n,\varepsilon}^c=K_n'-E_{n,\varepsilon}\ ,$$

where ε_n is a non-negative constant satisfying the assumption for each n. Then, by [5], it holds that

$$\mathrm{P}_T(E_{n,\varepsilon}^c)\leqq 2e^{-2n\varepsilon_n^2} \tag{3.20}$$

for any given n.

Since, for every n, the series $A_n(t)$ is absolutely convergent for $t\geqq 2$ and further the sequence $\{A_n(t),\ t=2,3,\cdots\}$ is monotone decreasing, then

$$\sum_{t\in E_{n,\varepsilon}}\{p_n(t)A_n(t)\}\to 0\ , \qquad (n\to\infty)\ ,$$

and because of the boundedness of $A_n(t)$ $(0<A_n(t)<1/12$, for $t\geqq 2)$,

$$\sum_{t\in E_{n,\varepsilon}^c}\{p_n(t)A_n(t)\}<(1/6)e^{-2n\varepsilon_n^2}\to 0\ , \qquad (n\to\infty)\ .$$

Thus, the condition (3.14) implies (3.16), which completes the proof of the theorem.

Instead of $Z_{n(k)}^{S,T}$ in Theorem 3.1 we introduce another mixed random vector $((S,T):\bar{Z}_{n(k)}^{S,T})$ such that its continuous conditional random vector $\bar{Z}_{n(k)}^{s,t}=(\bar{Z}_{n1},\cdots,\bar{Z}_{nk})'$ given $(S,T)=(s,t)$ is a k-dimensional *normal* random vector whose mean vector and dispersion matrix are, respectively,

$$\bar{\mu}_{n(k)}^{s,t}=(\bar{\mu}_{n1}^{st},\bar{\mu}_{n2}^{st},\cdots,\bar{\mu}_{nk}^{st})' \tag{3.21}$$

and

$$\bar{M}^{s,t}_{n(k)} = \|(\bar{\mu}^{st}_{ni} - \alpha_n)(\beta_n - \bar{\mu}^{st}_{nj})\|/t\ , \qquad 1 \leqq i \leqq j \leqq k\ , \tag{3.22}$$

where $\bar{\mu}^{st}_{ni} = \eta_n\{(n_i - s)/t\} + \alpha_n$, $i = 1, \cdots, k$. It is easily verified that $\bar{Z}^{s,t}_{n(k)} \sim Z^{s,t}_{n(k)}\ (\boldsymbol{B})_d$, as $n \to \infty$, provided the condition (3.14) (see Lemma 2.4 of [4]). Hence, from the above theorem, we may state the following

THEOREM 3.2. *Under Assumption* I-0, *if the condition* (3.14) *is satisfied, then it holds that*

$$U^{S,T}_{n(k)} \sim \bar{Z}^{S,T}_{n(k)}\ (\boldsymbol{B})_d\ , \qquad (n \to \infty)\ . \tag{3.23}$$

By means of these theorems, we immediately have the following

COROLLARY 3.1. (a) *If k is fixed independently of n, then the condition*

$$\sum_{(s,t) \in K_n} \left\{ p_n(s, t) \Big/ \min_{1 \leqq i \leqq k+1} d^{st}_{ni} \right\} \to 0\ , \qquad (n \to \infty)\ , \tag{3.24}$$

implies (3.15) *and* (3.23).

(b) *Let U^{ST}_{nm} is the mth order statistic, where, under the condition $(S, T) = (s, t)$, $m = m(n; s, t)$ and $s < m < s+t$. Then, in the sense of* (2.3) *U^{ST}_{nm} is asymptotically $(\boldsymbol{B})_d$ normally distributed according to*

$$N\left(\eta_n \cdot \frac{m-s}{t+1} + \alpha_n,\ \eta_n^2 \cdot \frac{(m-s)(s+t+1-m)}{(t+2)(t+1)^2}\right)$$

and/or

$$N\left(\eta_n \cdot \frac{m-s}{t} + \alpha_n,\ \eta_n^2 \cdot \frac{(m-s)(s+t-m)}{t^3}\right)$$

provided that the condition

$$\sum_{(s,t) \in K_n} [p_n(s, t)/\min\{(m-s), (s+t+1-m)\}] \to 0\ , \qquad (n \to \infty)\ . \tag{3.25}$$

From a practical point of view it is interesting to consider the situations in which a set of $k = k(n)$ spacings is given first, and then the corresponding sample quantiles are chosen.

Suppose that, for each n, a positive integer $k = k(n)$ is given, and the ranks of k sample quantiles are defined by

$$n_i = n^{s,t}_i \stackrel{d}{=} s + m^{st}_{ni}\ , \qquad i = 1, \cdots, k\ , \tag{3.26}$$

where m^{st}_{ni}'s satisfy the relations $1 \leqq m^{st}_{n1} < \cdots < m^{st}_{nk} \leqq t$ and the condition corresponding to (3.14), namely,

$$k(n) \sum_{(s,t) \in K_n} \left\{ p_n(s, t) \Big/ \min_{1 \leqq i \leqq k+1} (m^{st}_{ni} - m^{st}_{ni-1}) \right\} \to 0\ , \qquad (n \to \infty)\ , \tag{3.27}$$

As for m_{ni}^{st}'s, we shall consider only the following typical cases:

Case A. $m_{ni}^{st} \stackrel{d}{=} [t\lambda_{ni}^{st}]+1$, $i=1, \cdots, k$, for a given set of spacings $0=\lambda_{n0}^{st}<\lambda_{n1}^{st}<\cdots<\lambda_{nk}^{st}<\lambda_{nk+1}^{st}=1$, where [ ] denotes the Gauss' symbol.

Specifically, when λ_{ni}^{st}, $i=1,\cdots,k$, are independent of s, let us define

Case B. $m_{ni}^{st} \stackrel{d}{=} [t\lambda_{ni}^{t}]+1$, $i=1, \cdots, k$, for a given set of spacings $0=\lambda_{n0}^{t}<\lambda_{n1}^{t}<\cdots<\lambda_{nk}^{t}<\lambda_{nk+1}^{t}=1$.

Further, when λ_{ni}^{t}, $i=1,\cdots,k$, are independent of t, let us define

Case C. $m_{ni}^{st} \stackrel{d}{=} [t\lambda_{ni}]+1$, $i=1, \cdots, k$, for a given set of spacings $0=\lambda_{n0}<\lambda_{n1}<\cdots<\lambda_{nk}<\lambda_{nk+1}=1$.

Moreover, if k and $\lambda_{ni}=\lambda_i$, $i=1,\cdots,k$, are fixed independently of n, let us put

Case D. $m_{ni}^{st} \stackrel{d}{=} [t\lambda_{i}]+1$, $i=1, \cdots, k$, for a given set of spacings $0=\lambda_{0}<\lambda_{1}<\cdots<\lambda_{k}<\lambda_{k+1}=1$.

Let, for each n, $((S, T): \tilde{Z}_{n(k)}^{S,T})$ be a mixed $(k+2)$-dimensional random vector such that its continuous conditional random vector $\tilde{Z}_{n(k)}^{s,t}=(\tilde{Z}_{n1}, \cdots, \tilde{Z}_{nk})'$ given $(S, T)=(s, t)$ is a k-dimensional *normal* random vector distributed according to $N(\xi_{n(k)}^{s,t}, (1/t)\cdot \Xi_{n(k)}^{s,t})$, where

$$\xi_{n(k)}^{s,t}=(\xi_{n1}^{st}, \xi_{n2}^{st}, \cdots, \xi_{nk}^{st})' \tag{3.28}$$

and

$$\Xi_{n(k)}^{s,t}=\|(\xi_{ni}^{st}-\alpha_n)(\beta_n-\xi_{nj}^{st})\| , \qquad 1\leqq i\leqq j\leqq k , \tag{3.29}$$

with $\xi_{ni}^{st}=\eta_n\lambda_{ni}^{st}+\alpha_n$, $i=1,\cdots,k$. Define $Z_{n(k)}^{s,t}$ as the same *normal* random vector in Theorem 3.1 with n_i's given according to Case A. Then, it is seen that $\tilde{Z}_{n(k)}^{s,t} \sim Z_{n(k)}^{s,t}$ $(\boldsymbol{B})_d$, $(n\rightarrow\infty)$, provided the condition (3.27). (See Lemma 2.4 of [4]). Thus, we may state the following theorem, in which the condition (3.27) is restated in terms of spacings:

THEOREM 3.3. *Under Assumption* I-0, *if the condition*

$$k(n) \sum_{(s,t)\in K_n} \left[p_n(s, t)\Big/\left\{t\cdot \min_{1\leqq i\leqq k+1} (\lambda_{ni}^{st}-\lambda_{ni-1}^{st})\right\}\right]\rightarrow 0 , \qquad (n\rightarrow\infty) , \tag{3.30}$$

is satisfied, then it holds that

$$U_{n(k)}^{S,T} \sim \tilde{Z}_{n(k)}^{S,T} \ (\boldsymbol{B})_d , \qquad (n\rightarrow\infty) . \tag{3.31}$$

Under Case B, the following corollary is immediate.

COROLLARY 3.2. *Under Assumption* I-0, *if the condition*

$$(3.32)\qquad k(n)\sum_{t\in K_n}\left[p_n(t)\Big/\left\{t\cdot\min_{1\le i\le k+1}(\lambda_{ni}^t-\lambda_{ni-1}^t)\right\}\right]\to 0\,,\qquad (n\to\infty)\,,$$

is satisfied, then it holds that

$$(3.33)\qquad U_{n(k)}^{S,T}\sim\tilde{Z}_{n(k)}^{T}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,,$$

where $\tilde{Z}_{n(k)}^{T}$ *is a mixed* $(k+1)$*-dimensional random vector such that its continuous conditional random vector* $\tilde{Z}_{n(k)}^{t}$ *given* $T=t$ *being distributed according to* $N(\xi_{n(k)}^t, (1/t)\cdot\Xi_{n(k)}^t)$, *where* $\xi_{n(k)}^t$ *and* $\Xi_{n(k)}^t$ *are given by striking out all s from* (3.28) *and* (3.29), *respectively.*

Furthermore, let $\tilde{Z}_{n(k)}=(\tilde{Z}_{n1},\cdots,\tilde{Z}_{nk})'$ be a k-dimensional *normal* random vector being distributed according to $N(\xi_{n(k)}, 1/(n\eta_n)\cdot\Xi_{n(k)})$, where $\xi_{n(k)}$ and $\Xi_{n(k)}$ are given by deleting all s and t from (3.28) and (3.29), respectively. Then, under Case C, the following is an immediate consequence of the above results.

THEOREM 3.4. *Under Assumption* I-0, *if the condition*

$$(3.34)\qquad k(n)\Big/\left\{n\eta_n\cdot\min_{1\le i\le k+1}(\lambda_{ni}-\lambda_{ni-1})\right\}\to 0\,,\qquad (n\to\infty)\,,$$

is satisfied, then it holds that

$$(3.35)\qquad U_{n(k)}^{S,T}\sim\tilde{Z}_{n(k)}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,.$$

For Case D, we may state the following

COROLLARY 3.3. *If k is fixed independently of n, then in the sense of* (2.3) $U_{n(k)}^{S,T}$ *is asymptotically* $(\boldsymbol{B})_d$ *normally distributed according to* $N(\xi_{(k)}, 1/(n\eta)\Xi_{(k)})$, *where* $\xi_{(k)}$ *and* $\Xi_{(k)}$ *are given by deleting the suffices n, s and t from* (3.28) *and* (3.29), *respectively.*

4. The case of general distributions

In this section, we shall be concerned with the asymptotic joint $(\boldsymbol{B})_d$ normality of $k(n)$ sample quantiles from Type I censored samples in the case of *unequal basic distributions*. Let, for each positive integer n, $X_{n1}<X_{n2}<\cdots<X_{nn}$ be order statistics of a random sample of size n from a continuous distribution over the real line, whose pdf. and cdf. are given by $f_n(x)$ and $F_n(x)$, respectively.

Censoring at preassigned extended real numbers a_n and b_n such that $a_n<b_n$, for each n, we have doubly censored ordered variates $(a_n<)\ X_{n,S+1}<X_{n,S+2}<\cdots<X_{n,S+T}\ (<b_n)$, where $S=S(n)$ and $T=T(n)$ stand for the number of variates in the left censored portion and those in the uncensored portion, respectively. Under $(S,T)=(s,t)$ for each n, $X_{n,s+1}<X_{n,s+2}<$

$\cdots < X_{n,s+t}$ are regarded as order statistics based on a random sample of size t drawn from a doubly truncated distribution whose pdf. and cdf. are given by

$$(4.1) \qquad g_n(x) = \begin{cases} f_n(x)/\eta_n , & \text{if } a_n \leqq x < b_n , \\ 0 , & \text{otherwise} , \end{cases}$$

and

$$(4.2) \qquad G_n(x) = \begin{cases} 0 , & \text{if } x < a_n , \\ (F_n(x) - \alpha_n)/\eta_n , & \text{if } a_n \leqq x < b_n , \\ 1 , & \text{otherwise} , \end{cases}$$

respectively, where α_n and η_n are the same as in (2.1).

Put $G_n(x_{ni}) = U_{ni}$, $i = s+1, \cdots, s+t$. Then we have ordered variates $U_{n,s+1} < \cdots < U_{n,s+t}$ from the uniform distribution over (0, 1), for each n. This fact suggests us to extend the results in preceding section to general cases by making use of Ikeda's asymptotic equivalence theory (see references [1], [2]). Under Assumption I-1 in Section 2, it is assured that, for every n, $G_n^{-1}(u)$ is a measurable and one-to-one transformation from the interval (0, 1) onto the interval $D(g_n) \stackrel{d}{=} \{x : g_n(x) > 0\}$.

Now, let, for each n, $((S, T) : X_{n(k)}^{S,T})$ be a mixed $(k+2)$-dimensional random vector such that its continuous conditional random vector $X_{n(k)}^{s,t} = (X_{nn_1}, \cdots, X_{nn_k})'$ given $(S, T) = (s, t)$ is the joint random variable of k sample quantiles of orders $s < n_1 < n_2 < \cdots < n_k < s+t$, where $k = k(n)$ and $n_i = n_i^{s,t}(n)$. We define

$$(4.3) \qquad G_n^{-1}(l_{ni}^{st}) \stackrel{d}{=} \omega_{ni}^{st} \quad \text{and} \quad f_n(\omega_{ni}^{st}) \stackrel{d}{=} f_{ni}^{st} , \qquad i = 1, \cdots, k ,$$

where $l_{ni}^{st} \in (0, 1)$ is the same as in (3.4).

Assume that, for each n, $((S, T) : Y_{n(k)}^{S,T})$ is a mixed $(k+2)$-dimensional random vector such that $Y_{n(k)}^{s,t} = (Y_{n1}, \cdots, Y_{nk})'$ given $(S, T) = (s, t)$ is a k-dimensional *normal* random vector whose mean vector and dispersion matrix are given by

$$(4.4) \qquad \omega_{n(k)}^{s,t} = (\omega_{n1}^{st}, \omega_{n2}^{st}, \cdots, \omega_{nk}^{st})'$$

and

$$(4.5) \qquad \Omega_{n(k)}^{s,t} = \| (\mu_{ni}^{st} - \alpha_n)(\beta_n - \mu_{nj}^{st}) / (f_{ni}^{st} f_{nj}^{st}) \| / (t+2) , \qquad 1 \leqq i \leqq j \leqq k ,$$

respectively. Further, for each n, let us define another mixed $(k+2)$-dimensional random vector $((S, T) : V_{n(k)}^{S,T})$ such that its continuous conditional random vector given $(S, T) = (s, t)$ is $V_{n(k)}^{s,t} \stackrel{d}{=} (V_{n1}, \cdots, V_{nk})'$ with $V_{ni} = G_n(Y_{ni})$, $i = 1, \cdots, k$. Then $V_{n(k)}^{s,t}$ is distributed over the closure of a k-dimensional open cube $Q_k=\prod_{j=1}^{k} I_j$ with $I_j=(0,1)$ for all j, and is discontinuous on the boundary of this set unless $D(g_n)=(-\infty, \infty)$ for each n. It is obvious that, over the domain Q_k, the conditional variable $V_{n(k)}^{s,t}$ given $(S, T)=(s, t)$ is absolutely continuous with respect to the Lebesgue measure over $R_{(k)}$, and that has the density

$$(4.6)\qquad p_n^*(z_{(k)}\,|\,s, t)=(2\pi)^{-k/2}|\Omega_{n(k)}^{s,t}|^{-1/2}\left\{\prod_{i=1}^{k} g_n(G_n^{-1}(z^i)\right\}^{-1} \times\exp\left[\frac{-1}{2}(G_n^{-1}(z_{(k)})-\omega_{n(k)}^{s,t})'(\Omega_{n(k)}^{s,t})^{-1}(G_n^{-1}(z_{(k)})-\omega_{n(k)}^{s,t})\right], \qquad (z_{(k)}\in Q_k),$$

where $G_n^{-1}(z_{(k)})=(G_n^{-1}(z_1),\cdots, G_n^{-1}(z_k))'$.

Since, for every n,

$$\delta_{n1}^{st}\stackrel{d}{=}\{d_{n1}^{st}l_{n1}^{st}(1-l_{n1}^{st})/(t+2)\}^{1/2}<l_{n1}^{st}$$

and

$$\delta_{nk}^{st}\stackrel{d}{=}\{d_{nk+1}^{st}l_{nk}^{st}(1-l_{nk}^{st})/(t+2)\}^{1/2}<1-l_{nk}^{st},$$

then the set

$$(4.7)\qquad Q_{n(k)}^{s,t}=\{z_{(k)}:\ 0<l_{n1}^{st}-\delta_{n1}^{st}<z_1<\cdots<z_k<l_{nk}^{st}+\delta_{nk}^{st}<1\}$$

is well defined for every n. Under the condition (3.14), it is easily verified by Chebycheff inequality that

$$(4.8)\qquad \sum_{(s,t)\in K_n}\{p_n(s, t)\mathrm{P}^{U_{n(k)}^{s,t}}(Q_{n(k)}^{s,t})\}\rightarrow 1, \qquad (n\rightarrow\infty).$$

Therefore, by Theorem 3.1, we have

$$(4.9)\qquad \sum_{(s,t)\in K_n}\{p_n(s, t)\mathrm{P}^{Z_{n(k)}^{s,t}}(Q_{n(k)}^{s,t})\}\rightarrow 1, \qquad (n\rightarrow\infty).$$

Hence, if the condition

$$(4.10)\qquad \sum_{(s,t)\in K_n}\{p_n(s, t)I^*(Z_{n(k)}^{s,t}: V_{n(k)}^{s,t})\} \stackrel{d}{=}\sum_{(s,t)\in K_n}\left\{p_n(s, t)\int_{Q_{n(k)}^{s,t}} q_n^*\log(q_n^*/p_n^*)dz_{(k)}\right\}\rightarrow 0, \qquad (n\rightarrow\infty)$$

is satisfied, in addition to (3.14), then it holds that

$$(4.11)\qquad Z_{n(k)}^{S,T}\sim V_{n(k)}^{S,T}\ (\boldsymbol{B})_d, \qquad (n\rightarrow\infty),$$

and consequently, $X_{n(k)}^{S,T}\sim Y_{n(k)}^{S,T}\ (\boldsymbol{B})_d$, as $n\rightarrow\infty$ by a little modification of Lemma 2.2 and Lemma 2.3 in [4].

Now, let us derive conditions under which the condition (4.10) is satisfied. Under Assumption I-2 in Section 2, by (3.10) and (4.6), it is seen that

$$\begin{aligned}\text{(4.12)}\qquad &\log\{q_n^*(z_{(k)}\,|\,s,t)/p_n^*(z_{(k)}\,|\,s,t)\}\\ &=\sum_{i=1}^{k}\varphi_n(z_{ni}^{**})(z_i-l_{ni}^{st})-(1/2)(w_{n(k)}^{s,t})'(L_{n(k)}^{s,t})^{-1}(z_{(k)}-l_{n(k)}^{s,t})\\ &\quad+(1/8)(w_{n(k)}^{s,t})'(L_{n(k)}^{s,t})^{-1}w_{n(k)}^{s,t}\,,\end{aligned}$$

for every n, where

$$\text{(4.13)}\qquad \begin{aligned}&\varphi_n(z)=g_n'(G_n^{-1}(z))/g_n^2(G_n^{-1}(z))\,, && (0<z<1)\,,\\ &\phi_n(z;l)=g_n(G_n^{-1}(l))/g_n(G_n^{-1}(z))\,, && (0<z,l<1)\,,\end{aligned}$$

and

$$\text{(4.14)}\qquad \begin{aligned}&w_{n(k)}^{s,t}=(w_{n1}^{st},w_{n2}^{st},\cdots,w_{nk}^{st})'\\ &w_{ni}^{st}=\varphi_n(z_{ni}^{*})\phi(z_{ni}^{*};l_{ni}^{st})(z_i-l_{ni}^{st})^2\,, \qquad i=1,\cdots,k\,,\end{aligned}$$

and z_{ni}^{**} and z_{ni}^{*} are some functions of z_i which lie between z_i and l_{ni}^{st} and will be denoted by $z_{ni}^{**}, z_{ni}^{*}\in((z_i,l_{ni}^{st}))$, for each i.

Under the situation mentioned above, we obtain the following theorem, whose proof is carried out analogously to that of Theorem 4.1 of [4].

THEOREM 4.1. *Under Assumptions* I-0, I-1 *and* I-2, *assume that for some positive constant M and some positive integer N, the condition*

$$\text{(4.15)}\qquad \sum_{(s,t)\in K_n}\{p_n(s,t)\Phi_n(s,t)\}\leqq M\,,$$

with

$$\text{(4.16)}\quad \Phi_n(s,t)=\sup_{z_{(k)}\in Q_{n(k)}^{s,t}}\ \max_{1\leqq i\leqq k}\ \sup_{z_i^*\in((z_i,l_{ni}^{st}))}\ \max\{|\varphi_n(z_i^*)|,\phi_n(z_i^*;l_{ni}^{st})\}\,,$$

is satisfied uniformly in $n\geqq N$. Then, the condition

$$\text{(4.17)}\qquad k(n)^2\sum_{(s,t)\in K_n}\left\{p_n(s,t)\Big/\min_{1\leqq i\leqq k+1}d_{ni}^{st}\right\}\to 0\,,\qquad (n\to\infty)\,,$$

implies that

$$\text{(4.18)}\qquad X_{n(k)}^{S,T}\sim Y_{n(k)}^{S,T}\ (\boldsymbol{B})_d\,,\qquad (n\to\infty)\,.$$

The condition (4.15) is very complicated and is difficult to check the validity of it. However, if Assumption I-3 is satisfied, $|\varphi_n(z)|$ and $\phi_n(z;l)$ are uniformly bounded for all z, l and n, and hence the condition (4.15) is automatically satisfied. Therefore, the following corollary is an immediate consequence of the above theorem.

COROLLARY 4.1. *Under Assumptions* I-0, I-1, I-2 *and* I-3, *the condition* (4.17) *implies* (4.18).

In the next place, we shall consider the situation where spacings are chosen in advance and the corresponding sample quantiles subsequently. In this case we can proceed with our discussion as in the latter half of the preceding section, so that we shall state only the results briefly.

Here, let us begin with Case A in the preceding section. Let, for each n, $((S, T): \tilde{Y}^{S,T}_{n(k)})$ be a mixed $(k+2)$-dimensional random vector whose continuous conditional random vector $\tilde{Y}^{s,t}_{n(k)}=(Y_{n1},\cdots,Y_{nk})'$ given $(S,T)=(s,t)$ is a k-dimensional *normal* random vector distributed as $N(\zeta^{s,t}_{n(k)}, (1/t)\cdot\Sigma^{s,t}_{n(k)})$, where

$$\zeta^{s,t}_{n(k)}=(\zeta^{st}_{n1},\zeta^{st}_{n2},\cdots,\zeta^{st}_{nk})' \tag{4.19}$$

and

$$\Sigma^{s,t}_{n(k)}=||(\zeta^{st}_{ni}-\alpha_n)(\beta_n-\zeta^{st}_{nj})/(\tilde{f}^{st}_{ni}\tilde{f}^{st}_{nj})||\ , \qquad 1\leqq i\leqq j\leqq k\ , \tag{4.20}$$

with the following notations

$$\zeta^{st}_{ni}=G^{-1}_n(\lambda^{st}_{ni}) \quad\text{and}\quad \tilde{f}^{st}_{ni}=f_n(\zeta^{st}_{ni})\ , \qquad i=1,\cdots,k\ . \tag{4.21}$$

Then, we have the following theorem very similar to Theorem 4.1.

THEOREM 4.2. *Under Assumptions* I-0, I-1 *and* I-2, *assume that for some positive constant M and some positive integer N, the condition*

$$\sum_{(s,t)\in K_n}\{p_n(s,t)\Psi_n(s,t)\}\leqq M\ , \tag{4.22}$$

with

$$\Psi_n(s,t)=\sup_{z_{(k)}\in\tilde{Q}^{s,t}_{n(k)}}\ \max_{1\leqq i\leqq k}\ \sup_{z^*_i\in((z_i,\lambda^{st}_{ni}))}\max\{|\varphi_n(z^*_i)|,\phi_n(z^*_i;\lambda^{st}_{ni})\}\ , \tag{4.23}$$

is satisfied uniformly in $n\geqq N$, where

$$\tilde{Q}^{s,t}_{n(k)}=\{z_{(k)}:\ 0<\lambda^{st}_{n1}-\tilde{\delta}^{st}_{n1}<z_1<\cdots<z_k<\lambda^{st}_{nk}+\tilde{\delta}^{st}_{nk}<1\}\ , \tag{4.24}$$

with $\tilde{\delta}^{st}_{n1}$ and $\tilde{\delta}^{st}_{nk}$ defined analogously to δ^{st}_{n1} and δ^{st}_{nk} in (4.7) *by replacing l^{st}_{ni}'s by λ^{st}_{ni}'s. Then, the condition*

$$k(n)^2\sum_{(s,t)\in K_n}\left[p_n(s,t)\Big/\left\{t\cdot\min_{1\leqq i\leqq k+1}(\lambda^{st}_{ni}-\lambda^{st}_{ni-1})\right\}\right]\to 0\ , \qquad (n\to\infty) \tag{4.25}$$

implies that

$$X^{S,T}_{n(k)}\sim\tilde{Y}^{S,T}_{n(k)}\ (\boldsymbol{B})_d\ , \qquad (n\to\infty)\ . \tag{4.26}$$

For Case B, we immediately obtain an analogous result to the above.

Next, let $\tilde{Y}_{n(k)}=(\tilde{Y}_{n1},\cdots,\tilde{Y}_{nk})'$ be a k-dimensional *normal* random vector distributed according to $N(\zeta_{n(k)}, 1/(n\eta_n)\cdot\Sigma_{n(k)})$, where $\zeta_{n(k)}$ and $\Sigma_{n(k)}$ are formally defined by deleting all s and t from (4.19) and (4.20), respectively. Then, for Case C, we may state the following

THEOREM 4.3. *Under Assumptions* I-0, I-1 *and* I-2, *suppose that, for some positive constant M and some positive integer N, the condition*

$$(4.27) \qquad \sum_{(s,t)\in K_n} \{p_n(s,t)\Psi_n^0(s,t)\} \leqq M ,$$

with

$$(4.28) \quad \Psi_n^0(s,t)=\sup_{z_{(k)}\in\tilde{Q}_{n(k)}} \max_{1\leqq i\leqq k} \sup_{z_i^*\in((z_i,\lambda_{ni}))} \max\{|\varphi_n(z_i^*)|, \phi_n(z_i^*;\lambda_{ni})\} ,$$

is satisfied uniformly in $n\geqq N$, where

$$(4.29) \qquad \tilde{Q}_{n(k)}=\{z_{(k)}:\ 0<\lambda_{n1}-\tilde{\delta}_{n1}<z_1<\cdots<z_k<\lambda_{nk}+\tilde{\delta}_{nk}<1\} ,$$

is defined analogously to (4.24). *Then, the condition*

$$(4.30) \quad k(n)^2 \sum_{(s,t)\in K_n} \left[p_n(s,t)\Big/\left\{t\cdot\min_{1\leqq i\leqq k+1}(\lambda_{ni}-\lambda_{ni-1})\right\}\right]\to 0 , \qquad (n\to\infty)$$

implies that

$$(4.31) \qquad X_{n(k)}^{S,T}\sim\tilde{Y}_{n(k)}\ (\boldsymbol{B})_d , \qquad (n\to\infty) .$$

By this theorem we immediately have the following

COROLLARY 4.2. *Under Assumptions* I-0, I-1, I-2 *and* I-3, *the condition* (4.25) *implies* (4.26), *and that* (4.30) *implies* (4.31).

Remark. In particular, if, for all n, $f_n(x)=f(x)$ together with $a_n=a$ and $b_n=b$, we have the Type I censored sample from an *equal* basic distribution. In such a case, we can straightforwardly obtain the corresponding results of this and the preceding sections.

5. The case of Type II censored samples

In this final section we shall consider the asymptotic joint $(\boldsymbol{B})_d$ normality of increasing number of sample quantiles from Type II censored one, which has been defined at the latter half in Section 2. It is easily seen that the analysis used in the uncensored cases [4] is directly applicable to Type II censored ones, so that we will only summarize main results here, which are expected to be useful to statistical inferences based on Type II censored sample.

First, the case of uniform distribution over $(0, 1)$ will be stated. Let $U_{n1} < U_{n2} < \cdots < U_{nn}$ be order statistics of a random sample of size n from a uniform distribution over $(0, 1)$, and let $U_{n,p+1} < U_{n,p+2} < \cdots < U_{n,n-q}$ be a Type II doubly censored sample of size n' $(=n-p-q)$, where p and q stand for the numbers of censored observations on left and on right, respectively, and both may depend on n. Designate $k=k(n)$ sample quantiles of the above censored sample by $U_{n(k)}^{p,q}=(U_{nn_1}, \cdots, U_{nn_k})'$ with the rank orders $(n_0=)\ p<n_1<\cdots<n_k<n-q+1\ (=n_{k+1})$.

Now, let $Z_{n(k)}^{p,q}=(Z_{n1}, \cdots, Z_{nk})'$ be a k-dimensional *normal* random vector distributed as $N(l_{n(k)}^{p,q}, L_{n(k)}^{p,q})$, where

$$l_{n(k)}^{p,q}=(l_{n1}^{pq}, l_{n2}^{pq}, \cdots, l_{nk}^{pq})' , \tag{5.1}$$

with $l_{ni}^{pq}=n_i/(n'+1)$, $i=1, \cdots, k$; and

$$L_{n(k)}^{p,q}=||l_{ni}^{pq}(1-l_{nj}^{pq})||/(n'+2) , \qquad 1\leqq i\leqq j\leqq k . \tag{5.2}$$

Further, let $\bar{Z}_{n(k)}^{p,q}=(\bar{Z}_{n1}, \cdots, \bar{Z}_{nk})'$ be a k-dimensional *normal* random vector distributed as $N(\bar{l}_{n(k)}^{p,q}, \bar{L}_{n(k)}^{p,q})$, where

$$\bar{l}_{n(k)}^{p,q}=(\bar{l}_{n1}^{pq}, \bar{l}_{n2}^{pq}, \cdots, \bar{l}_{nk}^{pq})' , \tag{5.3}$$

and

$$\bar{L}_{n(k)}^{p,q}=||\bar{l}_{ni}^{pq}(1-\bar{l}_{nj}^{pq})||/n' , \qquad 1\leqq i\leqq j\leqq k , \tag{5.4}$$

with $\bar{l}_{ni}^{pq}=n_i/n'$, $i=1, \cdots, k$.

Put $d_{ni}^{pq}=n_i-n_{i-1}-1$, $i=1, \cdots, k$, then we immediately obtain the following theorem by virtue of Theorem 3.1 in [4].

THEOREM 5.1. (a) *If the condition*

$$k(n)\Big/\min_{1\leqq i\leqq k+1} d_{ni}^{pq} \to 0 , \qquad (n\to\infty) , \tag{5.5}$$

is satisfied, then

$$U_{n(k)}^{p,q} \sim Z_{n(k)}^{p,q} \sim \bar{Z}_{n(k)}^{p,q}\ (\boldsymbol{B})_d , \qquad (n\to\infty) . \tag{5.6}$$

(b) *If k is fixed independently of n, then the condition*

$$\min_{1\leqq i\leqq k+1} d_{ni}^{pq} \to \infty , \qquad (n\to\infty) , \tag{5.7}$$

implies (5.6).

(c) *Let U_{nm} be the mth order statistics such that $p<m\leqq n-q$, then U_{nm} is asymptotically $(\boldsymbol{B})_d$ normally distributed according to*

$$N\left(\frac{m-p}{n-p-q+1}, \frac{(m-p)(n-p-q-m+1)}{(n-p-q+2)(n-p-q+1)^2}\right)$$

and/or

$$N\left(\frac{m-p}{n-p-q}, \frac{(m-p)(n-p-q-m)}{(n-p-q)^3}\right),$$

provided that $m-p\to\infty$ *and* $n-p-q-m\to\infty$, *as* $n\to\infty$.

Next, we shall treat the cases of general distributions. Let, as in Section 2, $X_{n(k)}^{p,q}=(X_{nn_1},\cdots,X_{nn_k})'$ be k sample quantiles satisfying with the relations $(n_0=)\ p<n_1<\cdots<n_k<n-q+1\ (=n_{k+1})$ from a certain Type II censored sample, and let $Y_{n(k)}^{p,q}$ be a k-dimensional *normal* random vector distributed as $N(s_{n(k)}^{p,q}, S_{n(k)}^{p,q})$, where

$$s_{n(k)}^{p,q}=(s_{n1}^{pq}, s_{n2}^{pq},\cdots, s_{nk}^{pq})' \tag{5.8}$$

and

$$S_{n(k)}^{p,q}=\|\, l_{ni}^{pq}(1-l_{nj}^{pq})/(f_{ni}^{pq}f_{nj}^{pq})\,\|/(n'+2)\ , \qquad 1\leqq i\leqq j\leqq k\ , \tag{5.9}$$

with $l_{ni}^{pq}=n_i/(n'+1)$, $s_{ni}^{pq}=F_n^{-1}(l_{ni}^{pq})$ and $f_{ni}^{pq}=f_n(s_{ni}^{pq})$, $i=1,\cdots,k$.

We also define the set

$$Q_{n(k)}^{p,q}=\{z_{(k)}:\ 0<l_{n1}^{pq}-\delta_{n1}^{pq}<z_1<\cdots<z_k<l_{nk}^{pq}+\delta_{nk}^{pq}<1\}\ , \tag{5.10}$$

where δ_{ni}^{pq}, $i=1,k$, are analogously given to δ_{ni}^{st}, $i=1,k$ in (4.7). Further, put

$$\begin{aligned} &\varphi_n(z)=f_n'(F_n^{-1}(z))/f_n^2(F_n^{-1}(z))\ , && (0<z<1)\ ,\\ &\phi_n(z;l)=f_n(F_n^{-1}(l))/f_n(F_n^{-1}(z))\ , && (0<z, l<1)\ . \end{aligned} \tag{5.11}$$

The following theorem is concerned with the case where $\varphi_n(z)$ and $\phi_n(z;l)$ are uniformly bounded over $Q_{n(k)}^{p,q}$:

THEOREM 5.2. (a) *Under Assumptions* II-1 *and* II-2, *assume that, for some positive constant M and some positive integer N, the condition*

$$\sup_{z_{(k)}\in Q_{n(k)}^{p,q}}\ \max_{1\leqq i\leqq k}\ \sup_{z_i^*\in((z_i,\, l_{ni}^{pq}))}\ \max\{|\varphi_n(z_i^*)|, \phi_n(z_i^*; l_{ni}^{pq})\}\leqq M\ , \tag{5.12}$$

is satisfied uniformly in $n\geqq N$. *Then, the condition*

$$k(n)^2\Big/\min_{1\leqq i\leqq k+1} d_{ni}^{pq}\to 0\ , \qquad (n\to\infty)\ , \tag{5.13}$$

implies that

$$X_{n(k)}^{p,q}\sim Y_{n(k)}^{p,q}\ (\boldsymbol{B})_d\ , \qquad (n\to\infty)\ . \tag{5.14}$$

(b) *Under Assumptions* II-1, II-2 *and* II-3, *the condition* (5.13) *implies* (5.14).

Immediately, in the case of equal basic distributions similar theorems to the above can be obtained. Further, it is also possible to obtain parallel results for cases where the spacings l_{ni}^{pq}'s are chosen first and the corresponding sample quantiles subsequently, which should be referred to Section 4 in [4].

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REFERENCES

[1] Ikeda, S. (1963). Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence, *Ann. Inst. Statist. Math.*, **15**, 87-116.

[2] Ikeda, S. (1968). Asymptotic equivalence of real probability distributions, *Ann. Inst. Statist. Math.*, **20**, 339-416.

[3] Ikeda, S. and Matsunawa, T. (1970). On asymptotic independence of order statistics, *Ann. Inst. Statist. Math.*, **22**, 435-449.

[4] Ikeda, S. and Matsunawa, T. (1972) On the uniform asymptotic joint normality of sample quantiles, *Ann. Inst. Statist. Math.*, **24**, 33-52.

[5] Okamoto, M. (1958). Some inequalities relating to the partial sum of binomial probabilities, *Ann. Inst. Statist. Math.*, **10**, 29-35.

[6] Wilks, S. S. (1962). *Mathematical Statistics*, 2nd ed., p. 63, John Wiley & Sons, Inc.