THE ASYMPTOTIC NON-NULL DISTRIBUTION OF THE F-STATISTIC FOR TESTING A PARTIAL NULL-HYPOTHESIS IN A RANDOMIZED PBIB DESIGN WITH m ASSOCIATE CLASSES UNDER THE NEYMAN MODEL[1)]

JUNJIRO OGAWA AND SADAO IKEDA[2)]

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Summary

As the final result of a series of our investigations [5]–[8], [10], [11] we present in this article a probability distribution—a non-central F-distribution—which is asymptotically equivalent in the sense of type $(M)_d$ to the power function of the F-statistic for testing a partial null-hypothesis in the analysis of variance of a randomized PBIB design with m associate classes under the Neyman model which is a linear model taking both technical and unit errors into account. Thus this seems to be the final answer for which we have been after from the very beginning of our investigation.

1. Introduction

We are concerned with the power function of the F-statistic occurring in the analysis of variance of a PBIB design with m associate classes, where there are v treatments with an association of m associate classes being defined among them, b blocks of size k each, r replications of each treatment, and the number of incidence of any pair of treatments is λ_u if they are uth associates. The randomization procedure is applied in allocating k treatments to the k units in each block, independently from block to block. As for terminologies and notations which will be used in this article, reference should be made to the papers [9]–[11].

Let us take the special labeling of the whole $n=vr=bk$ experimental units in such a way that the ith unit in the pth block bears the number $f=(p-1)k+i$. We will fix this labeling throughout the

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2) S. Ikeda was formerly with Department of Statistics, Nihon Univ. Japan and now with Department of Economics, Soka University, Japan.

present paper.

Let Φ and Ψ be the incidence matrices of the treatments and blocks respectively and put $B=\Psi\Psi'$ and $N=\Phi'\Psi$, where N is the incidence matrix of the design under consideration. Furthermore, let $\boldsymbol{\tau}=(\tau_1,\cdots,\tau_v)'$ and $\boldsymbol{\beta}=(\beta_1,\cdots,\beta_b)'$ be the treatment-effects and block-effects being subjected to the restrictions

$$\sum_{\alpha=1}^{v}\tau_\alpha=0 \quad \text{and} \quad \sum_{p=1}^{b}\beta_p=0$$

respectively, and let $\boldsymbol{\pi}=(\pi_1,\cdots,\pi_n)'$ be the unit-error vector being subjected to the restrictions

$$\sum_{i=1}^{k}\pi_i^{(p)}=0\ , \qquad p=1,\cdots,b$$

where $\pi_f=\pi_i^{(p)}$ if $f=(p-1)k+i$. In the matrix notation, one can write as $\Psi'\boldsymbol{\pi}=\mathbf{0}$.

The Neyman model assuming no interaction between treatments and experimental units is given by

$$\boldsymbol{x}=\gamma\mathbf{1}+\Phi\boldsymbol{\tau}+\Psi\boldsymbol{\beta}+\boldsymbol{\pi}+\boldsymbol{e}\ , \tag{1.1}$$

where $\boldsymbol{x}=(x_1,\cdots,x_n)'$ is the observation vector, γ is the general mean, $\mathbf{1}=(1,\cdots,1)'$ and $\boldsymbol{e}=(e_1,\cdots,e_n)'$ stands for the technical-error vector, which is assumed to be distributed as $N(0,\sigma^2I_n)$ with unknown variances σ^2. This Neyman model includes the Fisher model and the normal regression model as its special cases. In fact,

The Fisher model: $\boldsymbol{x}=\gamma\mathbf{1}+\Phi\boldsymbol{\tau}+\Psi\boldsymbol{\beta}+\boldsymbol{\pi}$,

Normal Regression model: $\boldsymbol{x}=\gamma\mathbf{1}+\Phi\boldsymbol{\tau}+\Psi\boldsymbol{\beta}+\boldsymbol{e}$.

The null-hypothesis to be tested is

$$H_{0(h)}A_u^{\#}\boldsymbol{\tau}=\mathbf{0}\ , \qquad u=1,\cdots,h\ , \tag{1.2}$$

where h is any given integer such that $1\leqq h\leqq m$, and $A_0^{\#}=G_v/v$, $A_1^{\#}$, $\cdots$, $A_m^{\#}$ are $m+1$ mutually orthogonal idempotent matrices of the association algebra. This null-hypothesis is called a 'partial' null-hypothesis and it reduces to the 'total' null-hypothesis H_0: $\boldsymbol{\tau}=\mathbf{0}$ if $h=m$.

To test the null-hypothesis $H_{0(h)}$, one uses the F-statistic given by

$$F=\frac{n-b-v+1}{\bar{\alpha}}\frac{S_{t(h)}^2}{S_e^2}\ , \tag{1.3}$$

where $\bar{\alpha}=\alpha_1+\cdots+\alpha_h$, α_u being the rank of the matrix $A_u^{\#}$, $u=1,\cdots,m$ and

(1.4)
$$S^2_{t(h)}=\boldsymbol{x}'\Big(\sum_{u=1}^{h} V^\sharp_u\Big)\boldsymbol{x} ,$$
$$S^2_e=\boldsymbol{x}'\Big(I_n-\frac{1}{k}B-\sum_{u=1}^{m} V^\sharp_u\Big)\boldsymbol{x}$$

i.e., $S^2_{t(h)}$ and S^2_e are the sums of squares due to treatments adjusted by blocks and due to errors respectively. Here we have put

(1.5)
$$V^\sharp_u=\Big(I_n-\frac{1}{k}B\Big)\Phi(c_uA^\sharp_u)\Phi'\Big(I_n-\frac{1}{k}B\Big) , \qquad u=1,\cdots,m$$

and

(1.6)
$$c_u=\frac{k}{rk-\rho_u} , \qquad u=1,\cdots,m$$

where ρ_u, $u=0,1,\cdots,m$ are the characteristic roots of NN', with respective multiplicities $\alpha_0=1$, α_u, $u=1,\cdots,m$. It is known that

$$\sum_{u=1}^{m}\alpha_u=v-1 , \qquad \sum_{u=0}^{m}A^\sharp_u=I_v$$

and

$$NN'=rkA^\sharp_0+\rho_1A^\sharp_1+\cdots+\rho_mA^\sharp_m .$$

If $h=m$, the above F-statistic reduces to the usual

(1.7)
$$F=\frac{n-b-v+1}{v-1}\frac{S^2_t}{S^2_e} .$$

In the previous paper [11], the asymptotic null-distribution of the F-statistic given by (1.3) was discussed rigorously from the point of view of the theory of the asymptotic equivalence which had been developed by one of the authors [1], [4]. It was shown that the null-distribution of the F-statistic after the randomization is asymptotically equivalent in the sense of type $(M)_d$ to the usual central F-distribution, which is to be obtained under the normal regression model without the unit-errors. One may say that one can get rid of the unit-errors (nuisance parameters) asymptotically by means of the randomization procedure.

Based upon the same stand point as in the null case, we show that a non-central F-distribution which is to be obtained under the normal regression model is asymptotically equivalent in the sense of type $(M)_d$ to the non-null distribution of the F-statistic given by (1.3) under the Neyman model.

In the following section, the non-null distribution of the F-statistic before the randomization is derived, and that involves the conditioning random variables $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ as parameters. Then, in Section 3, the asymptotic behavior of the permutation distribution of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ due to the randomization procedure is discussed, where it is shown that $(\xi/(\Delta+T), \bar{\eta}/(\bar{T}/(\Delta+T)), \bar{\bar{\eta}}/(\bar{\bar{T}}/(\Delta+T)))$ converges in probability to $(1, 1, 1)$ in a certain limiting process under consideration. Sections 4 and 5 are devoted to the derivation of the asymptotic power function of the F-statistic based upon a theorem of the theory of the asymptotic equivalence [1], [4]. In Section 4 we show that the conditions of the theorem are satisfied in our present case. Then, in Section 5, a non-central F-distribution where non-centrality parameter involves $\Delta=\boldsymbol{\pi}'\boldsymbol{\pi}$ and $\bar{T}$ is derived in the first place and this is shown to be asymptotically equivalent in the sense of type $(M)_a$ to the power function of the F-statistic (1.3). In the second place, then, we can show that a non-central F-distribution whose non-centrality parameter being dependent only on $\bar{T}$ and that is to be obtained under the normal regression model is asymptotically equivalent in the sense of type $(M)_a$ to the non-central F-distribution involving Δ and $\bar{T}$ above mentioned.

2. The non-null distribution of the F-statistic before the randomization

Since $S^2_{t(h)}$ becomes

$$S^2_{t(h)}=(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})'\left(\sum_{u=1}^{h} V_u^{\sharp}\right)(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})+2(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})'\left(\sum_{u=1}^{h} V_u^{\sharp}\right)\boldsymbol{e}+\boldsymbol{e}'\left(\sum_{u=1}^{h} V_u^{\sharp}\right)\boldsymbol{e}$$

under the Neyman model (1.1), and the matrix $\sum_{u=1}^{h} V_u^{\sharp}$ is of rank $\bar{\alpha}$, the non-null distribution of the variate

$$\bar{\chi}_1^2=S^2_{t(h)}/\sigma^2 \tag{2.1}$$

before the randomization is the non-central chi-square distribution of degrees of freedom $\bar{\alpha}$, with the non-centrality parameter $\bar{\delta}_1/\sigma^2$, where

$$\bar{\delta}_1=(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})'\left(\sum_{u=1}^{h} V_u^{\sharp}\right)(\Phi\boldsymbol{\tau}+\boldsymbol{\pi}) . \tag{2.2}$$

Hence its probability element is given by

$$\exp\left(-\frac{\bar{\delta}_1}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\frac{(\bar{\delta}_1/2\sigma^2)^{\mu}}{\mu!}\frac{(\bar{\chi}_1^2/2)^{\bar{\alpha}/2+\mu-1}}{\Gamma(\bar{\alpha}/2+\mu)}\exp\left(-\frac{\bar{\chi}_1^2}{2}\right)d\left(\frac{\bar{\chi}_1^2}{2}\right) . \tag{2.3}$$

Similarly, the non-null distribution of the variate

$$\chi_2^2=S_e^2/\sigma^2 \tag{2.4}$$

before the randomization is seen to be the non-central chi-square distribution of degrees of freedom $n-b-v+1$, and with the non-centrality parameter δ_2/σ^2, where

$$\delta_2=(\Phi\tau+\pi)'\left(I_n-\frac{1}{k}B-\sum_{u=1}^{m}V_u^{\ddagger}\right)(\Phi\tau+\pi) . \tag{2.5}$$

Hence its probability element is given by

$$\exp\left(-\frac{\delta_2}{2\sigma^2}\right)\sum_{\nu=0}^{\infty}\frac{(\delta_2/2\sigma_2)^{\nu}}{\nu!}\frac{(\chi_2^2/2)^{(n-b-v+1)/2+\nu-1}}{\Gamma((n-b-v+1)/2+\nu)}\exp\left(-\frac{\chi_2^2}{2}\right)d\left(\frac{\chi_2^2}{2}\right) . \tag{2.6}$$

Since the two variates $\bar{\chi}_1^2$ and χ_2^2 are stochastically independent before the randomization, the non-null distribution of the F given by (1.3) is seen to be a non-central F-distribution with degrees of freedom $(\bar{\alpha}, n-b-v+1)$. The probability element of the F-statistic before the randomization is given by

$$\begin{aligned}
&\exp\left(-\frac{\bar{\delta}_1+\delta_2}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\bar{\delta}_1/2\sigma^2)^{\mu}}{\mu!}\frac{(\delta_2/2\sigma^2)^{\nu}}{\nu!}\\
&\quad\cdot\frac{\Gamma((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\\
&\quad\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)\\
&=\exp\left(-\frac{\xi}{2\sigma^2}\right)\sum_{l=0}^{\infty}\frac{(\xi/2\sigma^2)^{l}}{l!}\sum_{\mu+\nu+\gamma=l}\frac{l!}{\mu!\nu!\gamma!}\bar{\eta}^{\mu}\bar{\bar{\eta}}^{\gamma}(1-\bar{\eta}-\bar{\bar{\eta}})^{\nu}\\
&\quad\cdot\frac{\Gamma((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\\
&\quad\cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\bar{\alpha}})/2+\mu+\nu)}d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right) ,
\end{aligned} \tag{2.7}$$

where we have put

$$\begin{aligned}
&\bar{\bar{\alpha}}=v-1-\bar{\alpha} ,\\
&\bar{\bar{\delta}}_1=\delta_1-\bar{\delta}_1 \qquad \text{with} \qquad \delta_1=(\Phi\tau+\pi)'\left(\sum_{u=1}^{m}V_u^{\ddagger}\right)(\Phi\tau+\pi) ,\\
&\xi=\delta_1+\delta_2 , \qquad \bar{\eta}=\bar{\delta}_1/(\delta_1+\delta_2) , \qquad \bar{\bar{\eta}}=\bar{\bar{\delta}}_1/(\delta_1+\delta_2) .
\end{aligned} \tag{2.8}$$

It should be remarked that in the case of $h=m$, the probability element of the F-statistic (1.7) before the randomization is given by

$$\begin{aligned}
&\exp\left(-\frac{\xi}{2\sigma^2}\right)\sum_{l=0}^{\infty}\frac{(\xi/2\sigma^2)^{l}}{l!}\sum_{\mu+\nu=l}\frac{l!}{\mu!\nu!}\eta^{\mu}(1-\eta)^{\nu}\\
&\quad\cdot\frac{\Gamma((n-b)/2+\mu+\nu)}{\Gamma((v-1)/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\left(\frac{v-1}{n-b-v+1}F\right)^{(v-1)/2+\mu-1}
\end{aligned} \tag{2.9}$$

$$\cdot\left(1+\frac{v-1}{n-b-v+1}F\right)^{-((n-b)/2+\mu+\nu)}d\left(\frac{v-1}{n-b-v+1}F\right),$$

where

$$\eta=\delta_1/(\delta_1+\delta_2)\ . \tag{2.10}$$

The probability element of the power function of the F-statistic (1.3) after the randomization should be obtained by taking the mathematical expectation of (2.7) with respect to the permutation distribution of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ due to the randomization.

3. Asymptotic behavior of the permutation distribution of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ due to the randomization procedure

Let us denote the permutation associated with the randomization within the pth block by

$$\sigma_p=\begin{pmatrix} 1 & 2\cdots\cdots k \\ \sigma_p(1) & \sigma_p(2)\cdots\sigma_p(k)\end{pmatrix},$$

and let us put

$$U_\sigma=\begin{Vmatrix} S_{\sigma_1} & & & 0 \\ & S_{\sigma_2} & & \\ & & \ddots & \\ 0 & & & S_{\sigma_b}\end{Vmatrix}, \tag{3.1}$$

where S_{σ_p} is a $k\times k$ permutation matrix corresponding to the permutation σ_p, $p=1,\cdots,b$.

The incidence matrix of the treatments becomes a random variable through the randomization and that takes one of $(k!)^b$ values $\{U_\sigma\Phi\}$ with equal probability $1/(k!)^b$, where Φ is any one fixed incidence matrix of treatments.

Now, since

$$\bar{\delta}_1=(\Phi\boldsymbol{\tau})'\left(\sum_{u=1}^{h}V_u^\sharp\right)(\Phi\boldsymbol{\tau})+2(\Phi\boldsymbol{\tau})'\left(\sum_{u=1}^{h}V_u^\sharp\right)\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=1}^{h}V_u^\sharp\right)\boldsymbol{\pi}\ ,$$

and

$$\begin{aligned}(\Phi\boldsymbol{\tau})'\left(\sum_{u=1}^{h}V_u^\sharp\right)(\Phi\boldsymbol{\tau})&=\boldsymbol{\tau}'\Phi'\left(I_n-\frac{1}{k}B\right)\Phi\left(\sum_{u=1}^{h}c_uA_u^\sharp\right)\Phi'\left(I_n-\frac{1}{k}B\right)\Phi\boldsymbol{\tau}\\&=\boldsymbol{\tau}'\left(rI_v-\frac{1}{k}NN'\right)\left(\sum_{u=1}^{h}c_uA_u^\sharp\right)\left(rI_v-\frac{1}{k}NN'\right)\boldsymbol{\tau}\\&=\boldsymbol{\tau}'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^\sharp\right)\left(\sum_{u=1}^{h}c_uA_u^\sharp\right)\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^\sharp\right)\boldsymbol{\tau}\end{aligned}$$

$$=\boldsymbol{\tau}'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\boldsymbol{\tau}$$

$$=\left(\sum_{u=1}^{h}A_u^{*}\boldsymbol{\tau}\right)'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\left(\sum_{u=1}^{h}A_u^{*}\boldsymbol{\tau}\right),$$

$$(\Phi\boldsymbol{\tau})'\left(\sum_{u=1}^{h}V_u^{*}\right)\boldsymbol{\pi}=\boldsymbol{\tau}'\Phi'\left(I_n-\frac{1}{k}B\right)\Phi\left(\sum_{u=1}^{h}c_uA_u^{*}\right)\Phi'\left(I_n-\frac{1}{k}B\right)\boldsymbol{\pi}$$

$$=\boldsymbol{\tau}'(rI_v-NN')\left(\sum_{u=1}^{h}c_uA_u^{*}\right)\Phi'\boldsymbol{\pi}$$

$$=\boldsymbol{\tau}'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\left(\sum_{u=1}^{h}c_uA_u^{*}\right)\Phi'\boldsymbol{\pi}$$

$$=\boldsymbol{\tau}'\left(\sum_{u=1}^{h}A_u^{*}\right)\Phi'\boldsymbol{\pi},$$

one gets

$$\bar{\delta}_1=\bar{T}+2\left(\sum_{u=1}^{h}A_u^{*}\boldsymbol{\tau}\right)'\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=1}^{h}V_u^{*}\right)\boldsymbol{\pi},\tag{3.2}$$

where

$$\bar{T}=\left(\sum_{u=1}^{h}A_u^{*}\boldsymbol{\tau}\right)'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\left(\sum_{u=1}^{h}A_u^{*}\boldsymbol{\tau}\right).\tag{3.3}$$

In a similar manner, one also gets

$$\bar{\bar{\delta}}_1=\bar{\bar{T}}+2\left(\sum_{u=h+1}^{m}A_u^{*}\boldsymbol{\tau}\right)'\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=h+1}^{m}V_u^{*}\right)\boldsymbol{\pi},\tag{3.4}$$

where

$$\bar{\bar{T}}=\left(\sum_{u=h+1}^{m}A_u^{*}\boldsymbol{\tau}\right)'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\left(\sum_{u=h+1}^{m}A_u^{*}\boldsymbol{\tau}\right).\tag{3.5}$$

Here it should be noted that

$$T=\bar{T}+\bar{\bar{T}},$$

where

$$T=\boldsymbol{\tau}'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{*}\right)\boldsymbol{\tau}.\tag{3.6}$$

From (3.2) and (3.4), it follows that

$$\delta_1=\bar{\delta}_1+\bar{\bar{\delta}}_1=T+2\boldsymbol{\tau}'\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=1}^{m}V_u^{*}\right)\boldsymbol{\pi}.\tag{3.7}$$

Since

$$\delta_1+\delta_2=(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})'\left(I_n-\frac{1}{k}B\right)(\Phi\boldsymbol{\tau}+\boldsymbol{\pi})$$

$$=\boldsymbol{\tau}'\left(rI_n-\frac{1}{k}NN'\right)\boldsymbol{\tau}+2\boldsymbol{\tau}'\Phi'\boldsymbol{\pi}$$

$$=\boldsymbol{\tau}'\left(\sum_{u=1}^{m}\frac{1}{c_u}A_u^{\sharp}\right)\boldsymbol{\tau}'+2\boldsymbol{\tau}'\Phi'\boldsymbol{\pi}\ ,$$

it is also seen that

$$\delta_2=\boldsymbol{\pi}'\boldsymbol{\pi}-\boldsymbol{\pi}'\left(\sum_{u=1}^{m}V_u^{\sharp}\right)\boldsymbol{\pi}=\Delta-\boldsymbol{\pi}'\left(\sum_{u=1}^{m}V_u^{\sharp}\right)\boldsymbol{\pi}\ . \tag{3.8}$$

From (2.8), (2.10), (3.2), (3.4), (3.7) and (3.8), it follows that

$$\begin{aligned}
\xi&=\Delta+T+2\boldsymbol{\tau}'\Phi'\boldsymbol{\pi}\ ,\\
\eta&=\left\{T+2\boldsymbol{\tau}'\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=1}^{m}V_u^{\sharp}\right)\boldsymbol{\pi}\right\}\Big/\xi\ ,\\
\bar{\eta}&=\left\{\bar{T}+2\boldsymbol{\tau}'\left(\sum_{u=1}^{h}A_u^{\sharp}\right)\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=1}^{h}V_u^{\sharp}\right)\boldsymbol{\pi}\right\}\Big/\xi\ ,\\
\bar{\bar{\eta}}&=\left\{\bar{\bar{T}}+2\boldsymbol{\tau}'\left(\sum_{u=h+1}^{m}A_u^{\sharp}\right)\Phi'\boldsymbol{\pi}+\boldsymbol{\pi}'\left(\sum_{u=h+1}^{m}V_u^{\sharp}\right)\boldsymbol{\pi}\right\}\Big/\xi\ .
\end{aligned} \tag{3.9}$$

These quantities become random variables through the permutation distribution of the incidence matrix Φ of the treatments due to the randomization. However, it should be noted that T, $\bar{T}$, $\bar{\bar{T}}$ and Δ are constant parameters.

For the sake of the notational simplicity, let us put

$$X=\boldsymbol{\tau}'\Phi_\sigma'\boldsymbol{\pi}\ ,\qquad \bar{X}=\boldsymbol{\tau}'\left(\sum_{u=1}^{h}A_u^{\sharp}\right)\Phi_\sigma'\boldsymbol{\pi}\ ,\qquad \bar{\bar{X}}=\boldsymbol{\tau}'\left(\sum_{u=h+1}^{m}A_u^{\sharp}\right)\Phi_\sigma'\boldsymbol{\pi}\ , \tag{3.10}$$

and

$$Y=\boldsymbol{\pi}'\left(\sum_{u=1}^{m}V_u^{\sharp}\right)_\sigma\boldsymbol{\pi}\ ,\qquad \bar{Y}=\boldsymbol{\pi}'\left(\sum_{u=1}^{h}V_u^{\sharp}\right)_\sigma\boldsymbol{\pi}\ ,\qquad \bar{\bar{Y}}=\boldsymbol{\pi}'\left(\sum_{u=h+1}^{m}V_u^{\sharp}\right)_\sigma\boldsymbol{\pi}\ , \tag{3.11}$$

where we have put

$$\Phi_\sigma=U_\sigma\Phi\ ,\qquad \left(\sum_{u=1}^{m}V_u^{\sharp}\right)_\sigma=\left(I_n-\frac{1}{k}B\right)\Phi_\sigma\left(\sum_{u=1}^{m}c_uA_u^{\sharp}\right)\Phi_\sigma'\left(I_n-\frac{1}{k}B\right)\ .$$

Then the variates in (3.9) can be written as

$$\begin{aligned}
\xi&=\Delta+T+2X\ ,\\
\eta&=(T+2X+Y)/(\Delta+T+2X)\ ,\\
\bar{\eta}&=(\bar{T}+2\bar{X}+\bar{Y})/(\Delta+T+2X)\ ,\\
\bar{\bar{\eta}}&=(\bar{\bar{T}}+2\bar{\bar{X}}+\bar{\bar{Y}})/(\Delta+T+2X)\ .
\end{aligned} \tag{3.12}$$

Now, as in the previous paper [11], we consider the limiting process such that

(3.13) $b \to \infty$ whereas v, k, n_i, p^i_{jk} $(i, j, k=0, 1, \cdots, m)$ are kept fixed,

and denote this limiting process simply by $b \to \infty$. Under this limiting process, r and at least one λ_u must tend to infinity with the same order of magnitude as b. Suppose that one can find non-negative numbers ω_u such that

$$(rk-\rho_u)/b \to \omega_u \qquad \text{as } b \to \infty,\ u=0, 1, \cdots, m\ , \tag{3.14}$$

where $\omega_0 = r/b = k/v$. Furthermore, we assume that the following uniformity conditions of unit errors are satisfied:

$$\bar{\Delta} \equiv \frac{1}{b} \sum_{p=1}^{b} \Delta_p \to \Delta_0 \quad \text{and} \quad \frac{1}{b} \sum_{p=1}^{b} |\Delta_p - \bar{\Delta}|^{1+\delta} \to 0 \qquad \text{as } b \to \infty \tag{3.15}$$

where $\Delta_p = \sum_{i=1}^{k} \pi_i^{(p)^2}$ and Δ_0 and δ are some positive constants.

Under such situation, we have shown in the previous paper [11] that the permutation distribution of the variates $((k-1)\bar{Y}/\Delta_0,\ (k-1)\bar{\bar{Y}}/\Delta_0)$ converges in law to that of the mutually independent chi-square variates $(\chi^2_{\bar{a}}, \chi^2_{\bar{\bar{a}}})$ under the limiting process (3.14), provided the conditions (3.14) and (3.15) are satisfied.

We will consider the asymptotic behavior of the variates $\bar{X}$, $\bar{\bar{X}}$ and X.

From (3.1), one can see that

$$\Phi'_\sigma \boldsymbol{\pi} = \Phi' U'_\sigma \boldsymbol{\pi} = \Phi' \begin{bmatrix} S'_{\sigma_1}\boldsymbol{\pi}^{(1)} \\ S'_{\sigma_2}\boldsymbol{\pi}^{(2)} \\ \vdots \\ S'_{\sigma_b}\boldsymbol{\pi}^{(b)} \end{bmatrix}, \tag{3.16}$$

where $\boldsymbol{\pi}^{(p)} = (\pi_1^{(p)}, \cdots, \pi_k^{(p)})'$. Let us put

$$\Pi^\sigma_{\alpha p} \equiv \sum_{i=1}^{k} \zeta_{\alpha(p-1)k+i} \pi^{(p)}_{\sigma_p(i)}\ , \qquad \alpha=1, \cdots, v$$

and

$$\boldsymbol{\Pi}^\sigma_p = \begin{bmatrix} \Pi^\sigma_{1p} \\ \vdots \\ \Pi^\sigma_{vp} \end{bmatrix}, \qquad p=1, \cdots, b\ . \tag{3.17}$$

Furthermore let

$$\bar{\boldsymbol{\tau}} = \begin{bmatrix} \bar{\tau}_1 \\ \vdots \\ \bar{\tau}_v \end{bmatrix} = \sum_{u=1}^{h} A^\sharp_u \boldsymbol{\tau}\ , \qquad \bar{\bar{\boldsymbol{\tau}}} = \begin{bmatrix} \bar{\bar{\tau}}_1 \\ \vdots \\ \bar{\bar{\tau}}_v \end{bmatrix} = \sum_{u=h+1}^{m} A^\sharp_u \boldsymbol{\tau}\ . \tag{3.18}$$

Then, by (3.10), (3.16), (3.17) and (3.18), it follows that

$$\bar{X}=\sum_{p=1}^{b}\bar{\boldsymbol{\tau}}'\boldsymbol{\Pi}_p^{\sigma}, \qquad \bar{\bar{X}}=\sum_{p=1}^{b}\bar{\bar{\boldsymbol{\tau}}}'\boldsymbol{\Pi}_p^{\sigma}, \qquad X=\sum_{p=1}^{b}\boldsymbol{\tau}\boldsymbol{\Pi}_p^{\sigma}. \tag{3.19}$$

It is noted that the variates $\boldsymbol{\Pi}_p^{\sigma}$, $p=1,\cdots,b$ given by (3.17) form a stochastically independent set of v-dimensional random vectors under the permutation distribution due to the randomization. Furthermore, one can notice that

$$\bar{\boldsymbol{\tau}}'\bar{\bar{\boldsymbol{\tau}}}=0, \qquad \bar{\boldsymbol{\tau}}'A_u^{*}\bar{\bar{\boldsymbol{\tau}}}=0, \quad u=1,\cdots,m,$$

and consequently $\bar{\boldsymbol{\tau}}'NN'\bar{\bar{\boldsymbol{\tau}}}=0$.

Since

$$\mathrm{E}(\pi_{\sigma_p(1)}^{(p)})=0, \quad \mathrm{E}(\pi_{\sigma_p(i)}^{(p)2})=\frac{1}{k}\Delta_p, \quad \mathrm{E}(\pi_{\sigma_p(i)}^{(p)}\pi_{\sigma_p(j)}^{(p)})=\frac{-1}{k(k-1)}\Delta_p, \quad (i\neq j),$$

where E denotes the mathematical expectation with respect to the permutation distribution due to the randomization, it is seen that

$$\begin{aligned} &\mathrm{E}(\boldsymbol{\Pi}_p^{\sigma})=0, \qquad p=1,\cdots,b, \\ &\mathrm{E}(\boldsymbol{\Pi}_p^{\sigma}\boldsymbol{\Pi}_p^{\sigma\prime})=\frac{\Delta_p}{k(k-1)}\Lambda_p, \qquad p=1,\cdots,b, \end{aligned} \tag{3.20}$$

where

$$\Lambda_p=\begin{Vmatrix} (k-1)n_{1p} & -n_{1p}n_{2p} & \cdots & -n_{1p}n_{vp} \\ -n_{1p}n_{2p} & (k-1)n_{2p} & \cdots & -n_{2p}n_{vp} \\ \cdots & \cdots & \cdots & \cdots \\ -n_{1p}n_{vp} & -n_{2p}n_{vp} & \cdots & (k-1)n_{vp} \end{Vmatrix}. \tag{3.21}$$

Notice that

$$\sum_{p=1}^{b}\Lambda_p=rkI_v-NN'. \tag{3.22}$$

From (3.19) and (3.20), it follows that

$$\mathrm{E}(\bar{X})=\mathrm{E}(\bar{\bar{X}})=\mathrm{E}(X)=0. \tag{3.23}$$

The variance-covariance matrix of $(\bar{X},\bar{\bar{X}})$ is seen by (3.20) to be

$$\begin{aligned} D(\bar{X},\bar{\bar{X}})&=\frac{1}{k(k-1)}\sum_{p=1}^{b}\Delta_p\begin{Vmatrix}\bar{\boldsymbol{\tau}}'\Lambda_p\bar{\boldsymbol{\tau}} & \bar{\boldsymbol{\tau}}'\Lambda_p\bar{\bar{\boldsymbol{\tau}}} \\ \bar{\bar{\boldsymbol{\tau}}}'\Lambda_p\bar{\boldsymbol{\tau}} & \bar{\bar{\boldsymbol{\tau}}}'\Lambda_p\bar{\bar{\boldsymbol{\tau}}}\end{Vmatrix} \\ &=\frac{\bar{\Delta}}{k-1}\begin{Vmatrix}\bar{T} & 0 \\ 0 & \bar{\bar{T}}\end{Vmatrix}+\frac{1}{k(k-1)}\sum_{p=1}^{b}(\Delta_p-\bar{\Delta})\begin{Vmatrix}\bar{\boldsymbol{\tau}}'\Lambda_p\bar{\boldsymbol{\tau}} & \bar{\boldsymbol{\tau}}'\Lambda_p\bar{\bar{\boldsymbol{\tau}}} \\ \bar{\bar{\boldsymbol{\tau}}}'\Lambda_p\bar{\boldsymbol{\tau}} & \bar{\bar{\boldsymbol{\tau}}}'\Lambda_p\bar{\bar{\boldsymbol{\tau}}}\end{Vmatrix}, \end{aligned} \tag{3.24}$$

where $\bar{T}$ and $\bar{\bar{T}}$ are defined by (3.3) and (3.5) respectively. If we put

$$(3.25)\qquad \begin{aligned} T_0 &= \bar{\boldsymbol{\tau}}' \sum_{u=1}^{m} (\omega_u/k) A_u^{*} \\ \bar{T}_0 &= \bar{\bar{\boldsymbol{\tau}}}' \sum_{u=1}^{h} (\omega_u/k) A_u^{*} \\ \bar{\bar{T}}_0 &= \boldsymbol{\tau}' \sum_{u=h+1}^{m} (\omega_u/k) A_u^{*} \end{aligned}$$

then, under the conditions (3.14) and (3.15) one obtains

$$\frac{\bar{\Delta}}{b(k-1)} \begin{Vmatrix} \bar{T} & 0 \\ 0 & \bar{\bar{T}} \end{Vmatrix} \rightarrow \frac{\Delta_0}{k-1} \begin{Vmatrix} \bar{T}_0 & 0 \\ 0 & \bar{\bar{T}}_0 \end{Vmatrix}, \qquad \text{as } b \rightarrow \infty .$$

From (3.21), by using the Hölder inequality, it follows that

$$\left| \frac{1}{b} \sum_{p=1}^{b} (\Delta_p - \bar{\Delta}) \bar{\boldsymbol{\tau}}' \Lambda_p \bar{\boldsymbol{\tau}} \right| \leqq \left(\frac{1}{b} \sum_{p=1}^{b} |\Delta_p - \bar{\Delta}|^{\mu} \right)^{1/\mu} \left(\frac{1}{b} \sum_{p=1}^{b} |\bar{\boldsymbol{\tau}}' \Lambda_p \bar{\boldsymbol{\tau}}|^{\eta} \right)^{1/\eta}$$

for any given $\mu, \eta > 1$ such that $1/\mu + 1/\eta = 1$. But, since

$$\begin{aligned} |\bar{\boldsymbol{\tau}}' \Lambda_p \bar{\boldsymbol{\tau}}| &= \left| (k-1) \sum_{\alpha=1}^{v} n_{\alpha p} \bar{\tau}_\alpha^2 - \sum_{\alpha \neq \beta} n_{\alpha p} n_{\beta p} \bar{\tau}_\alpha \bar{\tau}_\beta \right| \\ &\leqq (k-1) \left(\sum_{\alpha=1}^{v} n_{\alpha p} |\bar{\tau}_\alpha| \right)^2 \leqq k^2 (k-1) \bar{\tau}_*^2 , \qquad p = 1, \cdots, b , \end{aligned}$$

where $\bar{\tau}_* = \max_{1 \leqq \alpha \leqq v} |\bar{\tau}_\alpha|$, one has

$$\left(\frac{1}{b} \sum_{p=1}^{b} |\bar{\boldsymbol{\tau}}' \Lambda_p \bar{\boldsymbol{\tau}}|^{\eta} \right)^{1/\eta} \leqq k^2 (k-1) \bar{\tau}_*^2 .$$

Since $\bar{\tau}_*$ depends only on the parameters of the association under consideration, it is bounded independently of b. Thus, if one chooses μ so close to unity that $1 < \mu \leqq 1 + \delta$, it follows that

$$\frac{1}{b} \sum_{p=1}^{b} (\Delta_p - \bar{\Delta}) \bar{\boldsymbol{\tau}}' \Lambda_p \bar{\boldsymbol{\tau}} \rightarrow 0 \qquad \text{as } b \rightarrow \infty ,$$

provided the conditions (3.15) are satisfied. In a similar manner, one can show that the remaining elements of the second matrix of the right-hand side of (3.24) tend to zero as $b \rightarrow \infty$. Hence

$$(3.26)\qquad \frac{1}{b} D(\bar{X}, \bar{\bar{X}}) \rightarrow \frac{\Delta_0}{k-1} \begin{Vmatrix} \bar{T}_0 & 0 \\ 0 & \bar{\bar{T}}_0 \end{Vmatrix} \qquad \text{as } b \rightarrow \infty$$

under the conditions (3.14) and (3.15). In particular

$$(3.27)\qquad \frac{1}{b} \operatorname{Var}(X) \rightarrow \frac{\Delta_0}{k-1} T_0 \qquad \text{as } b \rightarrow \infty .$$

Now we are going to show that

$$(3.28)\qquad \left(\frac{\xi}{\Delta+T}, \frac{\bar{\eta}}{\bar{T}/(\Delta+T)}, \frac{\bar{\bar{\eta}}}{\bar{\bar{T}}/(\Delta+T)}\right)\to(1, 1, 1) \qquad \text{in prob.},$$

and hence

$$(3.29)\qquad \left(\frac{\xi}{\Delta+T}, \frac{\eta}{T/(\Delta+T)}\right)\to(1, 1) \qquad \text{in prob.}.$$

To prove (3.28) and hence (3.29), it will be sufficient to show that

$$(3.30)\qquad \begin{aligned} \eta&=(T+2X+Y)/\xi\to T_0/(\Delta_0+T_0) &&\text{in prob.},\\ \bar{\eta}&=(\bar{T}+2\bar{X}+\bar{Y})/\xi\to \bar{T}_0/(\Delta_0+T_0) &&\text{in prob.},\\ \bar{\bar{\eta}}&=(\bar{\bar{T}}+2\bar{\bar{X}}+\bar{\bar{Y}})/\xi\to \bar{\bar{T}}_0/(\Delta_0+T_0) &&\text{in prob.},\end{aligned}$$

as $b\to\infty$. Indeed, since $\xi=\Delta+T+2X$ and we have already shown that $X/b\to 0$ in prob., it is clear that $\xi/b\to\Delta_0+T_0$ in prob. as $b\to\infty$. Since

$$\eta=\frac{T+2X+Y}{\Delta+T+2X}=1-\frac{\Delta-Y}{\Delta+T+2X},$$

and we have seen in the previous paper [11] that $(n-b)Y$ is asymptotically equivalent in the sense of type $(M)_d$ to χ^2_{v-1}. Hence $Y/b\to 0$. Thus one can see that

$$\eta\to 1-\frac{\Delta_0}{\Delta_0+T_0}=\frac{T_0}{\Delta_0+T_0} \qquad \text{in prob.}.$$

4. Validity of certain conditions in a theorem from the theory of asymptotic equivalence

In this section, we will show that certain conditions in a theorem —Theorem 6.3 of [4]—on the asymptotic equivalence of two probability distributions are satisfied in our present situation. The proof which is given in this section will be useful in the next section too.

From (2.7), the conditional p.d.f. of the F-statistic (1.3), given ξ, $\bar{\eta}$ and $\bar{\bar{\eta}}$, is expressed as

$$(4.1)\qquad \begin{aligned} &p_b(F\mid\xi,\bar{\eta},\bar{\bar{\eta}})\\ &=\exp\left(\frac{-\xi}{2\sigma^2}\right)\sum_{l=0}^{\infty}\frac{(\xi/2\sigma^2)^l}{l!}\sum_{\mu+\nu+\gamma=l}\frac{l!}{\mu!\nu!\gamma!}\bar{\eta}^{\mu}\bar{\bar{\eta}}^{\gamma}(1-\bar{\eta}-\bar{\bar{\eta}})^{\nu}\\ &\quad\cdot\frac{\bar{\alpha}}{n-b-v+1}\,\frac{\Gamma((n-b-\bar{\alpha})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\\ &\quad\cdot\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\alpha})/2+\mu+\nu)},\end{aligned}$$

and hence, the conditional c.d.f. of the F given ξ, $\bar{\eta}$ and $\bar{\bar{\eta}}$ is given by

$$P_b(F\,|\,\xi, \bar{\eta}, \bar{\bar{\eta}})=\int_0^F p_b(F\,|\,\xi, \bar{\eta}, \bar{\bar{\eta}})dF\,. \tag{4.2}$$

Let us put

$$Q_b(F\,|\,\xi, \bar{\eta}, \bar{\bar{\eta}})=P_b\left(F\,|\,(\Delta+T)\xi, \frac{\bar{T}}{\Delta+T}\bar{\eta}, \frac{\bar{\bar{T}}}{\Delta+T}\bar{\bar{\eta}}\right). \tag{4.3}$$

Then the Theorem 6.3 of [4] states that the distribution of the F whose c.d.f. being given by the expected value of (4.2) with respect to $(\xi, \bar{\eta}, \bar{\bar{\eta}})$, i.e., the distribution of the F after the randomization, is asymptotically equivalent in the sense of type $(M)_d$ to the distribution whose c.d.f. being given by

$$P_b\left(F\,|\,\Delta+T, \frac{\bar{T}}{\Delta+T}, \frac{\bar{\bar{T}}}{\Delta+T}\right), \qquad \text{as } b\to\infty\,, \tag{4.4}$$

if, in addition to the condition (3.28), the following condition is also satisfied.

For any given $\varepsilon>0$, there exist a positive number $\delta=\delta(\varepsilon)$ and a positive integer $b_0=b_0(\varepsilon)$ such that

(C) $|(\xi, \bar{\eta}, \bar{\bar{\eta}})-(1, 1, 1)|<\delta$ implies $\sup_F |Q_b(F\,|\,\xi, \bar{\eta}, \bar{\bar{\eta}})-Q_b(F\,|\,1, 1, 1)|<\varepsilon$ for all $b\geqq b_0$.

We are now going to show the above condition (C) is satisfied in the present case.

For simplicity, let us put

$$u_{\mu,\nu,\gamma}(\bar{\eta}, \bar{\bar{\eta}})=\frac{l!}{\mu!\nu!\gamma!}\,\bar{\eta}^{\mu}\bar{\bar{\eta}}^{\gamma}(1-\bar{\eta}-\bar{\bar{\eta}})^{\nu}\,, \tag{4.5}$$

and

$$H^b_{\mu,\nu}(F)=\int_{0\leq x\leq F}\frac{\Gamma((n-b-\bar{\alpha})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}\left(\frac{\bar{\alpha}}{n-b-v+1}x\right)^{\bar{\alpha}/2+\mu-1} \cdot\left(1+\frac{\bar{\alpha}}{n-b-v+1}x\right)^{-((n-b-\bar{\alpha})/2+\mu+\nu)}d\left(\frac{\bar{\alpha}}{n-b-v+1}x\right). \tag{4.6}$$

Then, it holds that

$$P_b(F\,|\,\xi, \bar{\eta}, \bar{\bar{\eta}})=\exp\left(\frac{-\xi}{2\sigma^2}\right)\sum_{l=0}^{\infty}\frac{(\xi/2\sigma^2)^l}{l!}\sum_{\mu+\nu+\gamma=l}u_{\mu,\nu,\gamma}(\bar{\eta}, \bar{\bar{\eta}})H^b_{\mu,\nu}(F)\,. \tag{4.7}$$

It is not difficult to see that $H^b_{\mu,\nu}(F)$ is a monotone increasing function of b for any fixed values of μ, ν and F, and

$$H^b_{\mu,\nu}(F)\to H^{\infty}_{\mu}(F) \qquad \text{as } b\to\infty\,, \tag{4.8}$$

where

$$H_\mu^\infty(F)=\int_0^F \frac{1}{\Gamma(\bar{\alpha}/2+\mu)}\left(\frac{\bar{\alpha}x}{2}\right)^{\bar{\alpha}/2+\mu-1}\exp\left(\frac{-\bar{\alpha}x}{2}\right)d\left(\frac{\bar{\alpha}x}{2}\right). \tag{4.9}$$

Notice that the convergence (4.8) is of type $(B)_d$. Furthermore, for the Gamma distribution (4.9), it holds that $H_\mu^\infty(F)$ is monotone decreasing with increasing μ for any fixed F, and

$$H_\mu^\infty(F)\to 0 \qquad \text{as } \mu\to\infty . \tag{4.10}$$

Replacing $H_{\mu,\nu}^b(F)$ in the right-hand side of (4.7) by $H_\mu^\infty(F)$, one gets

$$P_\infty(F\mid\xi,\bar{\eta})=\exp\left(\frac{-\xi}{2\sigma^2}\right)\sum_{l=0}^\infty \frac{(\xi/2\sigma^2)^l}{l!}\sum_{\mu+\nu=l} u_{\mu,\nu}(\bar{\eta})H_\mu^\infty(F) , \tag{4.11}$$

where

$$u_{\mu,\nu}(\bar{\eta})=\frac{l!}{\mu!\nu!}\bar{\eta}^\mu(1-\bar{\eta})^\nu . \tag{4.12}$$

Now, in the first step, we show that

$$\sup_{F,\xi,\bar{\eta},\bar{\bar{\eta}}} |P_b(F\mid\xi,\bar{\eta},\bar{\bar{\eta}})-P_\infty(F\mid\xi,\bar{\eta})|\to 0 \qquad \text{as } b\to\infty . \tag{4.13}$$

In the first place, we note that

$$0\leqq P_b(F\mid\xi,\bar{\eta},\bar{\bar{\eta}})\leqq P_\infty(F\mid\xi,\bar{\eta})\leqq 1 , \tag{4.14}$$

for any given values of b, F, ξ, $\bar{\eta}$ and $\bar{\bar{\eta}}$, and these functions are all continuous with respect to F, ξ, $\bar{\eta}$ and $\bar{\bar{\eta}}$. We calculate the difference

$$1-P_b(F\mid\xi,\bar{\eta},\bar{\bar{\eta}})=\exp\left(\frac{-\xi}{2\sigma^2}\right)\sum_{l=0}^\infty \frac{(\xi/2\sigma^2)^l}{l!} \cdot \sum_{\mu+\nu+\tau=l} u_{\mu,\nu,\tau}(\bar{\eta},\bar{\bar{\eta}})(1-H_{\mu,\nu}^b(F)) . \tag{4.15}$$

Since the function $1-H_{\mu,\nu}^b(F)$ is monotone-decreasing with increasing b for any fixed F, μ and ν, it can be seen that

$$1-H_{\mu,\nu}^{b_0}(F)\geqq 1-H_{\mu,\nu}^b(F)\geqq 0 \tag{4.16}$$

for all $b\geqq b_0$. Since

$$\int_0^\infty F dH_{\mu,\nu}^{b_0}(F)=\frac{n_0-b_0-v+1}{\bar{\alpha}}\,\frac{\bar{\alpha}/2+\mu}{(n_0-b_0-\alpha)/2+\mu+\nu} , \qquad n_0=b_0k \tag{4.17}$$

is bounded uniformly for all μ and ν, the Markov inequality assures us that for any given $\varepsilon>0$, there exists a positive number $F_0=F_0(\varepsilon)$ such that

$$1-H_{\mu,\nu}^{b_0}(F_0)<\varepsilon$$

and hence by (4.16)

$$0\leqq 1-H_{\mu,\nu}^{b}(F)<\varepsilon \tag{4.18}$$

uniformly for all $F\geqq F_0$, $b\geqq b_0$, μ and ν.

From (4.14), (4.15) and (4.18), it follows that, for all $b\geqq b_0$,

$$\sup_{\xi,\bar{\eta},\bar{\bar{\eta}},\ F\geqq F_0} |P_b(F|\xi,\bar{\eta},\bar{\bar{\eta}})-P_\infty(F|\xi,\bar{\eta})|<\varepsilon\,. \tag{4.19}$$

In the next place one has to examine the case where $F\geqq F_0$. (4.8) and (4.10) assure us that there exists a positive integer μ_0 such that $H_{\mu_0}^{\infty}(F_0)<\varepsilon$, and

$$|H_{\mu,\nu}^{b}(F)-H_{\mu}^{\infty}(F)|\leqq H_{\mu_0}^{\infty}(F_0)<\varepsilon \tag{4.20}$$

uniformly for all $F\leqq F_0$, $\mu\geqq\mu_0$, b and ν.

For $\mu<\mu_0$, we have by (4.8)

$$|H_{\mu,\nu}^{b}(F)-H_{\mu}^{\infty}(F)|<\varepsilon \tag{4.21}$$

uniformly for all ν and $F\leqq F_0$, provided that $b\geqq b_0'$, where $b_0'=b_0'(\varepsilon)$ is some positive integer.

From (4.20) and (4.21), it now follows that

$$\sup_{\xi,\bar{\eta},\bar{\bar{\eta}},\ F\leqq F_0} |P_b(F|\xi,\bar{\eta},\bar{\bar{\eta}})-P_\infty(F|\xi,\bar{\eta})|<\varepsilon \tag{4.22}$$

for all $b\geqq b_0'$.

Combining (4.19) and (4.22) one gets (4.13), as was to be proved in the first step.

In the second step, we shall show that for any given $\varepsilon>0$, there exists a positive number $\xi_0=\xi_0(\varepsilon)$ such that $\xi\geqq\xi_0$ and $\xi^*\geqq\xi_0$ imply

$$\sup_{\bar{\eta},F} |P_\infty(F|\xi,\bar{\eta})-P_\infty(F|\xi^*,\bar{\eta})|<\varepsilon\,. \tag{4.23}$$

From (4.14), (4.15) and (4.19), it follows that

$$\sup_{\xi,\bar{\eta},\ F>F_0} |1-P_\infty(F|\xi,\bar{\eta})|<\varepsilon \tag{4.24}$$

and hence

$$\sup_{\bar{\eta},\ F>F_0} |P_\infty(F|\xi,\bar{\eta})-P_\infty(F|\xi^*,\bar{\eta})|<\varepsilon \tag{4.25}$$

for any ξ and ξ^*.

Suppose $F\leqq F_0$. First, we have

$$\frac{\partial}{\partial\xi}P_\infty(F|\xi,\bar{\eta})=L_+(F|\xi,\bar{\eta})-L_-(F|\xi,\bar{\eta})\,, \tag{4.26}$$

where

$$(4.27)\quad \begin{aligned} L_+(F|\xi,\bar{\eta}) &= \frac{1}{2\sigma^2}\exp\left(\frac{-\xi}{2\sigma^2}\right)\Big[\sum_{\xi/2\sigma^2\leq l}\left\{\frac{(\xi/2\sigma^2)^{l-1}}{l!}-\frac{(\xi/2\sigma^2)^l}{l!}\right\} \\ &\quad\cdot\sum_{\mu+\nu=l} u_{\mu,\nu}(\bar{\eta})H_\mu^\infty(F)+H_0^\infty(F)\Big] \\ L_-(F|\xi,\bar{\eta}) &= \frac{1}{2\sigma^2}\exp\left(\frac{-\xi}{2\sigma^2}\right)\Big[\sum_{1\leq l<\xi/2\sigma^2}\left\{\frac{(\xi/2\sigma^2)^{l}}{l!}-\frac{(\xi/2\sigma^2)^{l-1}}{(l-1)!}\right\} \\ &\quad\cdot\sum_{\mu+\nu=l} u_{\mu,\nu}(\bar{\eta})H_\mu^\infty(F)\Big] \end{aligned}$$

which are both positive for all values of F, ξ and $\bar{\eta}$. Note that L_+ and $-L_-$ are both non-increasing functions of ξ for any given values of F and $\bar{\eta}$, and hence, for any given F and $\bar{\eta}$, $(\partial/\partial\xi)P_\infty(F|\xi,\bar{\eta})$ is a non-increasing function of ξ. Furthermore $(\partial/\partial\xi)P_\infty(F|\xi,\bar{\eta})\geqq 0$ at $\xi=0$, with equality holding if and only if $F=0$.

By (4.26) and (4.27), it is seen that

$$\frac{\partial}{\partial\xi}P_\infty(F|\xi,\bar{\eta})\leqq|L_+(F|\xi,\bar{\eta})|\leqq\frac{1}{2\sigma^2}\exp(-\zeta)\left(\frac{\zeta^{\zeta-1}}{\Gamma(\zeta)}+1\right).$$

where we have put $\zeta=\xi/2\sigma^2$. Then, since

$$\exp(-\zeta)\frac{\zeta^{\zeta-1}}{\Gamma(\zeta)}\sim c/\sqrt{\zeta}\qquad\text{as }\zeta\to\infty$$

by the Stirling formula, one gets

$$\sup_{F,\bar{\eta}}\left|\frac{\partial}{\partial\xi}P_\infty(F|\xi,\bar{\eta})\right|\to 0\qquad\text{as }\xi\to\infty .$$

Thus, it can be argued that $(\partial/\partial\xi)P_\infty(F|\xi,\bar{\eta})\geqq 0$ for all values of F, ξ and $\bar{\eta}$, and hence $P_\infty(F|\xi,\bar{\eta})$ is a non-decreasing function of ξ. Since $P_\infty(F|\xi,\bar{\eta})$ is bounded uniformly for all F, ξ and $\bar{\eta}$, there exists a limit function, $\bar{P}_\infty(F|\bar{\eta})$ say, such that

$$\lim_{\xi\to\infty}P_\infty(F|\xi,\bar{\eta})=\bar{P}_\infty(F|\bar{\eta}) .$$

By the continuity of $P_\infty(F|\xi,\bar{\eta})$ as a function of $(F,\bar{\eta})$ for every fixed ξ, and the monotonicity of the sequence $\{P_\infty(F|\xi,\bar{\eta})\}(\xi\to\infty)$, the limit function $\bar{P}_\infty(F|\bar{\eta})$ is continuous with respect to $(F,\bar{\eta})$, and hence, uniformly continuous for all F and $\bar{\eta}$ such that $0\leqq F\leqq F_0$ and $0\leqq\bar{\eta}\leqq 1$. Again, from the monotonicity of the sequence, the continuity of the function $P_\infty(F|\xi,\bar{\eta})$ and $\bar{P}_\infty(F|\bar{\eta})$, and the compactness of the domain of $(F,\bar{\eta})$ under consideration, it follows that the convergence

$$P_\infty(F|\xi,\bar{\eta})\to\bar{P}_\infty(F|\bar{\eta})\qquad\text{as }\xi\to\infty$$

is uniform in $(F, \bar{\eta})$, i.e.,

$$\sup_{\bar{\eta}, F \leqq F_0} |P_\infty(F|\xi, \bar{\eta}) - P_\infty(F|\bar{\eta})| \to 0 \qquad \text{as } \xi \to \infty . \tag{4.28}$$

Hence there exists a positive number $\xi_0 = \xi_0(\varepsilon)$ such that

$$\sup_{\bar{\eta}, F \leqq F_0} |P_\infty(F|\xi, \bar{\eta}) - P_\infty(F|\xi^*, \bar{\eta})| < \varepsilon \tag{4.29}$$

for any given values of ξ and ξ^* such that $\xi, \xi^* \geqq \xi_0$. From (4.25) and (4.29), one can conclude (4.23), as was to be proved in the second step.

Let us put, for the sake of simplicity,

$$c_b = \Delta + T , \qquad \bar{d}_b = \bar{T}/(\Delta + T) \quad \text{and} \quad \bar{\bar{d}}_b = \bar{\bar{T}}/(\Delta + T) , \tag{4.30}$$

then it is clear that

$$c_b/b \to \Delta_0 + T_0 , \qquad \bar{d}_b \to \bar{T}_0/(\Delta_0 + T_0) \quad \text{and} \quad \bar{\bar{d}}_b \to \bar{\bar{T}}_0/(\Delta_0 + T_0) \qquad \text{as } b \to \infty . \tag{4.31}$$

Furthermore, let us define

$$Q^b_\infty(F|\xi, \bar{\eta}) = P_\infty(F|c_b\xi, \bar{d}_b\bar{\eta}) . \tag{4.32}$$

Now, in the third step, we will show that, for any given $\varepsilon > 0$, there exist a positive number $\delta = \delta(\varepsilon)$ and a positive integer $b_0 = b_0(\varepsilon)$ such that

$$|(\xi, \bar{\eta}) - (1, 1)| < \delta$$

implies

$$\sup_F |Q^b_\infty(F|\xi, \bar{\eta}) - Q^b_\infty(F|1, 1)| < 2\varepsilon \tag{4.33}$$

for all $b \geqq b_0$.

To show (4.33), we consider the case $F > F_0$ in the first place, where F_0 is the same as in (4.18). From (4.24) we see that

$$\sup_{\xi, \bar{\eta}, F > F_0} |1 - Q^b_\infty(F|\xi, \bar{\eta})| \to 0 \qquad \text{and} \qquad \sup_{F > F_0} |1 - Q^b_\infty(F|1, 1)| \to 0 \qquad \text{as } b \to \infty ,$$

and therefore

$$\sup_{\xi, \bar{\eta}, F > F_0} |Q^b_\infty(F|\xi, \bar{\eta}) - Q^b_\infty(F|1, 1)| \to 0 \qquad \text{as } b \to \infty . \tag{4.34}$$

Suppose that $F \leqq F_0$ in the second place. Since $P_\infty(F|\xi, \bar{\eta})$ is uniformly continuous in $(F, \xi, \bar{\eta})$ over the domain $0 \leqq F \leqq F_0$, $0 \leqq \xi \leqq \xi_0$, $0 \leqq \bar{\eta} \leqq 1$, ξ_0 being the same as in (4.23), there can be found a positive number $\delta_0 = \delta_0(\varepsilon)$ such that

$$|(\xi, \bar{\eta})-(\xi^*, \bar{\eta}^*)|<\delta_0$$

implies

$$\sup_{F\leqq F_0} |P_\infty(F|\xi, \bar{\eta})-P_\infty(F|\xi^*, \bar{\eta}^*)|<\varepsilon \ . \tag{4.35}$$

Let b_* be the minimum integer such that $2\xi_0\leqq c_b$, and let us put

$$\delta=\min(1/2, \delta_0/c_{b_*}) \ ,$$

then it is easy to see that $|(\xi, \bar{\eta})-(1, 1)|<\delta$ implies $|(c_b\xi, \bar{d}_b\bar{\eta})-(c_b, \bar{d}_b)|<\delta_0$ for all $b\leqq b_*$, and if $b>b_*$, then $c_b\xi>\xi_0$ and $c_b>\xi_0$ so long as $|(\xi, \bar{\eta})-(1, 1)|<\delta$. Hence from (4.35) and (4.29), it follows that for $(\xi, \bar{\eta})$ such that $|(\xi, \bar{\eta})-(1, 1)|<\delta$

$$\sup_{F\leqq F_0} |Q^b_\infty(F|\xi, \bar{\eta})-Q^b_\infty(F|1, 1)|<\varepsilon \qquad \text{if } b\leqq b_* \ ,$$

$$\sup_{F\leqq F_0} |Q^b_\infty(F|\xi, \bar{\eta})-Q^b_\infty(F|1, \bar{\eta})|<\varepsilon \qquad \text{if } b>b_* \ ,$$

and from the uniform continuity one can see that

$$\sup_{F\leqq F_0} |Q^b_\infty(F|1, \bar{\eta})-Q^b_\infty(F|1, 1)|<\varepsilon \ .$$

Thus one can show that (4.33) is true.

We are now in a position to be able to show that the condition (C) is satisfied in our present case.

$$\begin{aligned}\sup_F &|Q_b(F|\xi, \bar{\eta}, \bar{\bar{\eta}})-Q_b(F|1, 1, 1)|\\ &\leqq \sup_F |Q_b(F|\xi, \bar{\eta}, \bar{\bar{\eta}})-Q^b_\infty(F|\xi, \bar{\eta})|+\sup_F |Q_b(F|1, 1, 1)\\ &\quad -Q^b_\infty(F|1, 1)|+\sup_F |Q^b_\infty(F|\xi, \bar{\eta})-Q^b_\infty(F|1, 1)| \ .\end{aligned}$$

For a given $\varepsilon>0$, one can choose $b_0=b_0(\varepsilon)$ such that

$$\sup_F |Q_b(F|\xi, \bar{\eta}, \bar{\bar{\eta}})-Q^b_\infty(F|\xi, \bar{\eta})|\leqq \sup_{F,\xi,\bar{\eta},\bar{\bar{\eta}}} |Q_b(F|\xi, \bar{\eta}, \bar{\bar{\eta}})-Q^b_\infty(F|\xi, \bar{\eta})|<\varepsilon/3$$

for all $b\geqq b_0$, and hence

$$\sup_F |Q_b(F|1, 1, 1)-Q^b_\infty(F|1, 1)|<\varepsilon/3 \ ,$$

and choose $\delta=\delta(\varepsilon)$ such that

$$\sup_F |Q^b_\infty(F|\xi, \bar{\eta})-Q^b_\infty(F|1, 1)|<\varepsilon/3$$

so long as $|(\xi, \bar{\eta}, \bar{\bar{\eta}})-(1, 1, 1)|<\delta$ for all b. Hence there exists a positive number $\delta=\delta(\varepsilon)$ such that

$$|(\xi, \bar{\eta}, \bar{\bar{\eta}})-(1, 1, 1)|<\delta(\varepsilon)$$

implies

$$\sup_F |Q_b(F|\xi, \bar{\eta}, \bar{\bar{\eta}})-Q_b(F|1, 1, 1)|<\varepsilon$$

for all $b \geqq b_0$, which is the condition (C).

5. Asymptotic power function of the F after the randomization

As was mentioned in the beginning part of Section 4, the distribution of the F whose c.d.f. should be obtained by taking mathematical expectation of (4.2) with respect to the permutation distribution of $(\xi, \bar{\eta}, \bar{\bar{\eta}})$ due to the randomization, is asymptotically equivalent in the sense of type $(M)_d$ as $b\to\infty$ to a distribution whose c.d.f. is given by (4.4), or equivalently a distribution with the following probability element:

$$(5.1)\quad \exp\left(-\frac{\Delta+T}{2\sigma^2}\right)\sum_{l=0}^{\infty}\frac{((\Delta+T)/2\sigma^2)^l}{l!}\sum_{\mu+\nu+\gamma=l}\frac{l!}{\mu!\nu!\gamma!}\left(\frac{\bar{T}}{\Delta+T}\right)^{\mu}\left(\frac{\bar{\bar{T}}}{\Delta+T}\right)^{\gamma}$$
$$\cdot\left(1-\frac{T}{\Delta+T}\right)^{\nu}\frac{\Gamma((n-b-\bar{\alpha})/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}$$
$$\cdot\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\alpha})/2+\mu+\nu)}$$
$$\cdot d\left(\frac{\bar{\alpha}}{n-b-v+1}F\right).$$

This can be rewritten as

$$(5.2)\qquad \exp\left(-\frac{\Delta+\bar{T}}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\bar{T}/2\sigma^2)^{\mu}}{\mu!}\frac{(\Delta/2\sigma^2)^{\nu}}{\nu!}h^b_{\mu\nu}(F)dF,$$

where

$$(5.3)\quad h^b_{\mu,\nu}(F)=\frac{\bar{\alpha}}{n-b-v+1}\frac{\Gamma((n-b-v+1)/2+\mu+\nu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2+\nu)}$$
$$\cdot\left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1}\left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\alpha})/2+\mu+\nu)}.$$

Let us put

$$(5.4)\quad G_b(F; \Delta, \bar{T})=\exp\left(-\frac{\Delta+\bar{T}}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\bar{T}/2\sigma^2)^{\mu}}{\mu!}\frac{(\Delta/2\sigma^2)^{\nu}}{\nu!}H^b_{\mu,\nu}(F),$$

where $H^b_{\mu,\nu}(F)$ being the same as (4.6). Furthermore, using the function $H^{\infty}_{\mu}(F)$ given by (4.9), we define

$$(5.5)\quad G^{\infty}_b(F; \Delta, \bar{T})=\exp\left(-\frac{\Delta+\bar{T}}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\frac{(\bar{T}/2\sigma^2)^{\mu}}{\mu!}\frac{(\Delta/2\sigma^2)^{\nu}}{\nu!}H^{\infty}_{\mu}(F).$$

It should be noticed that the first step of the proof of the condition (C) given in Section 4 implies that

$$\sup_F |G_b(F; \Delta, \bar{T}) - G_b^\infty(F; \Delta, \bar{T})| \to 0 \quad \text{as } b \to \infty \tag{5.6}$$

for any given values of Δ and $\bar{T}$. Hence in the special case $\Delta=0$, one gets

$$\sup_F |G_b(F; 0, \bar{T}) - G_b^\infty(F; 0, \bar{T})| \to 0 \quad \text{as } b \to \infty . \tag{5.7}$$

Since $H_\mu^\infty(F)$ is independent of ν, we have by (5.5)

$$\begin{aligned} G_b^\infty(F; \Delta, \bar{T}) &= \exp\left(-\frac{\bar{T}}{2\sigma^2}\right) \sum_{\mu=0}^{\infty} \frac{(\bar{T}/2\sigma^2)^\mu}{\mu!} H_\mu^\infty(F) \\ &= G_b^\infty(F; 0, \bar{T}) . \end{aligned} \tag{5.8}$$

Hence from (5.6) and (5.7), it follows that

$$\sup_F |G_b(F; \Delta, \bar{T}) - G_b(F; 0, \bar{T})| \to 0 \quad \text{as } b \to \infty . \tag{5.9}$$

This means that the distribution whose c.d.f. being given by $G_b(F; 0, \bar{T})$ is asymptotically equivalent in the sense of type $(M)_d$ to the power function of the F after the randomization.

Thus one concludes this section by stating the following.

THEOREM. *The power function of the F-statistic given by* (1.3) *after the randomization is asymptotically equivalent in the sense of type* $(M)_d$ *to the non-central F-distribution whose probability element is given by*

$$\begin{aligned} &\exp\left(-\frac{\bar{T}}{2\sigma^2}\right) \sum_{\mu=0}^{\infty} \frac{(\bar{T}/2\sigma^2)^\mu}{\mu!} \frac{\Gamma((n-b-\bar{\alpha})/2+\mu)}{\Gamma(\bar{\alpha}/2+\mu)\Gamma((n-b-v+1)/2)} \\ &\cdot \left(\frac{\bar{\alpha}}{n-b-v+1}F\right)^{\bar{\alpha}/2+\mu-1} \left(1+\frac{\bar{\alpha}}{n-b-v+1}F\right)^{-((n-b-\bar{\alpha})/2+\mu)} \\ &\cdot \left(\frac{\bar{\alpha}}{n-b-v+1}F\right) , \end{aligned} \tag{5.10}$$

under the limiting process (3.13), *provided that the condition* (3.14) *and* (3.15) *are satisfied.*

In the special case $h=m$, this becomes

$$\begin{aligned} &\exp\left(-\frac{T}{2\sigma^2}\right) \sum_{\mu=0}^{\infty} \frac{(T/2\sigma^2)^\mu}{\mu!} \frac{\Gamma((n-b)/2+\mu)}{\Gamma((v-1)/2+\mu)\Gamma((n-b-v+1)/2)} \\ &\cdot \left(\frac{v-1}{n-b-v+1}F\right)^{(v-1)/2+\mu-1} \left(1+\frac{v-1}{n-b-v+1}\right)^{-((n-b)/2+\mu)} \end{aligned} \tag{5.11}$$

$$\cdot d\left(\frac{v-1}{n-b-v+1}F\right) .$$

It should be remarked here, that the distribution having the probability element

$$\text{(5.12)} \qquad \exp\left(-\frac{\bar{T}}{2\sigma^2}\right)\sum_{\mu=0}^{\infty}\frac{(\bar{T}/2\sigma^2)^{\mu}}{\mu!}\frac{1}{\Gamma(\bar{\alpha}/2+\mu)}\left(\frac{\bar{\alpha}}{2}F\right)^{\bar{\alpha}/2+\mu-1} \cdot \exp\left(-\frac{\bar{\alpha}}{2}F\right)d\left(\frac{\bar{\alpha}}{2}F\right)$$

i.e., the non-central chi-square distribution is also asymptotically equivalent in the sense of type $(M)_d$ as $b\to\infty$ to the power function of the F-statistic.

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REFERENCES

[1] Ikeda, S. (1963). Asymptotic equivalence of probability distributions with applications to some problems of asymptotic independence, *Ann. Inst. Statist. Math.*, **15**, 87–116.

[2] Ikeda, S., Ogawa, J. and Ogasawara, M. (1965). On the asymptotic distribution of the F-statistic under the null-hypothesis in a randomized PBIB design with m associate classes under the Neyman model, *UNC Inst. Statist., Mimeo Series*, 454.

[3] Ikeda, S. and Ogawa, J. (1966). On the asymptotic non-null distribution of the F-statistic for testing a partial null-hypothesis in a randomized PBIB design with m associate classes under the Neyman model, *UNC Inst. Statist., Mimeo Series*, 466.

[4] Ikeda, S. (1968). Asymptotic equivalence of real probability distributions, *Ann. Inst. Statist. Math.*, **20**, 339-362.

[5] Ogawa, J. (1961). The effect of randomization on the analysis of randomized block design, *Ann. Inst. Statist. Math.*, **13**, 105–117.

[6] Ogawa, J. (1962). On the randomization in Latin-square design under the Neyman model, *Proc. Inst. Statist. Math.*, **10**, 1-16.

[7] Ogawa, J. (1963). On the null-distribution of the F-statistic in a randomized BIB design under the Neyman model, *Ann. Math. Statist.*, **34**, 1558–1568.

[8] Ogawa, J., Ikeda, S. and Ogasawara, M. (1964). On the null-distribution of the F-statistic in a randomized PBIB design with two associate classes under the Neyman model, Essays in Probability and Statistics, *The Univ. of North Carolina Monograph Series in Probability and Statistics*, No. 3, 517-548.

[9] Ogawa, J. and Ishii, G. (1965). The relationship algebra and the analysis of a PBIB design, *Ann. Math. Statist.*, **36**, 1815-1828.

[10] Ogawa, J., Ikeda, S. and Ogasawara, M. (1967). On the null-distribution of the F-statistic for testing a partial null-hypothesis in a randomized PBIB design with m associate classes under the Neyman model, *Ann. Inst. Statist. Math.*, **19**, 313–330.

[11] Ogawa, J. and Ikeda, S. (1969). On the asymptotic null-distribution of the F-statistic for testing a partial null-hypothesis in a randomized PBIB design with m associate classes under the Neyman model, presented at the 37th Session of the ISI meeting, held in London from September 3rd through 11th, 1969.