NOTE ON THE DISTRIBUTION OF THE MINIMUM LATENT ROOT

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1. Summary

In this paper we shall discuss the distribution problem of the minimum latent root.

2. Notations and preliminary results

The following integrals are useful in the sequel for the derivation of the distribution.

The fundamental property of zonal polynomials is given by

$$(1) \qquad \int_{O(m)} C_\kappa(SHTH')dH = [C_\kappa(S)C_\kappa(T)/C_\kappa(I_m)]$$

where I_m is the $(m\times m)$ identity matrix and dH is an invariant Haar measure on the orthogonal group $O(m)$, normalised to make its volume unity and $C_\kappa(S)$ is the zonal polynomial corresponding to a partition $\kappa=(k_1, k_2, \cdots, k_m)$, $k_1\geqq k_2\geqq \cdots \geqq k_m \geqq 0$, $\sum_{i=1}^m k_i=k$, of an integer k (James [3]).

The Laplace transform of $|S|^{t-(m+1)/2}C_\kappa(ST)$ is given by

$$(2) \qquad \int_{S>0} \mathrm{etr}\,(-RS)|S|^{t-(m+1)/2}C_\kappa(ST)dS = \Gamma_m(t, \kappa)|R|^{-t}C_\kappa(TR^{-1})$$

where $\Gamma_m(t, \kappa)=\pi^{m(m-1)/4}\prod_{i=1}^m \Gamma\left(t+k_i-\frac{1}{2}(i-1)\right)$ and R is an $(m\times m)$ complex symmetric matrix whose real part is positive definite, and T is an arbitrary $(m\times m)$ complex symmetric matrix and integration is over the space of positive definite $(m\times m)$ symmetric matrices, complex number t satisfying $R(t)>(m-1)/2$ (Constantine [1]).

The Beta-type integral is given by

$$(3) \qquad \int_{I_m>S>0} |S|^{t-(m+1)/2}|I_m-S|^{u-(m+1)/2}C_\kappa(TS) = [\Gamma_m(t, \kappa)\Gamma_m(u)/\Gamma_m(t+u, \kappa)]C_\kappa(T)$$

where $\Gamma_m(u)=\pi^{m(m-1)/4}\prod_{i=1}^{m}\Gamma\left(u-\frac{1}{2}(i-1)\right)$ (Constantine [1]).

The generating function for the Laguerre polynomials is

$$(4)\qquad |I_m-S|^{-(r+(m+1)/2)}\int_{O(m)}\operatorname{etr}(-THS(I_m-S)^{-1}H')dH$$
$$=\sum_k\sum_\kappa [C_\kappa(S)L_\kappa^r(T)/k!C_\kappa(I_m)]$$

where $L_\kappa^r(S)$ is the generalised Laguerre polynomial corresponding to κ and $||S||<1$ (Constantine [2]).

The zonal polynomials are used as a basis for the symmetric homogeneous function of the latent roots of a symmetric matrix. The exponential function is

$$(5)\qquad \operatorname{etr}(S)=\sum_k\sum_\kappa [C_\kappa(S)/k!] \qquad \text{(Constantine [1])}.$$

The generalised binomial expansion is

$$(6)\qquad |I_m-S|^{-a}=\sum_k\sum_\kappa (a)_\kappa[C_\kappa(S)/k!]$$

where $(a)_\kappa=\prod_{i=1}^{m}\left(a-\frac{1}{2}(i-1)\right)_{k_i}$, and $(x)_k=x(x+1)(x+2)\cdots(x+k-1)$, and $||S||<1$.

3. The evaluation of the integral

The following integrals are useful for the derivation of the distribution of the minimum latent root.

THEOREM 1. *Let R be a positive definite $(m\times m)$ symmetric matrix and $0<x<1$, then*

$$(7)\qquad \int_{I_m>S>xI_m}|S|^{t-(m+1)/2}C_\kappa(RS)dS$$
$$=\left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\left(t+\frac{1}{2}(m+1)\right)_\kappa\right]$$
$$\times[C_\kappa(R)/C_\kappa(I_m)]\sum_{k_1}\sum_{\kappa_1}\left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1}\Big/k_1!(m+1)_{\kappa_1}\right]$$
$$\times\sum_\nu g_{\kappa\kappa_1}^\nu\left(t+\frac{1}{2}(m+1)\right)_\nu C_\nu(I_m)x^{mt+k}(1-x)^{m(m+1)/2+k_1}$$

where

$$(8)\qquad C_\kappa(S)C_{\kappa_1}(S)=\sum_\nu g_{\kappa\kappa_1}^\nu C_\nu(S)\,,$$

in which $g_{\kappa\kappa_1}^\nu$ is a constant (Khatri and Pillai [5]).

PROOF. The left-hand side of (7) is a symmetric function $f(R)$ of R, so that transforming $R \to HRH'$ and integrating with respect to H, we have

$$f(R)=[f(I_m)/C_\kappa(I_m)]C_\kappa(R) . \tag{9}$$

On the other hand, transforming $S=xR^{-1/2}TR^{-1/2}$, then $J(S\to T)=x^{m(m+1)/2}|R|^{-(m+1)/2}$, gives

$$f(R)=x^{mt+k}|R|^{-t}\int_{R/x>T>R}|T|^{t-(m+1)/2}C_\kappa(T)dT .$$

Hence, multiplying both sides by $|R|^t$, we get

$$f(R)|R|^t=x^{mt+k}\int_{R/x>T>R}|T|^{t-(m+1)/2}C_\kappa(T)dT . \tag{10}$$

Substituting (9) in (10), and taking the Laplace transforms of the each side, we get

$$\begin{aligned}[f(I_m)/C_\kappa(I_m)]&\int_{R>0}\operatorname{etr}(-R)|R|^tC_\kappa(R)dR \\ &=x^{mt+k}\int_{R>0}\int_{R/x>T>R}\operatorname{etr}(-R)|T|^{t-(m+1)/2}C_\kappa(T)dTdR .\end{aligned} \tag{11}$$

Using (2), the left-hand side of (11) becomes

$$[f(I_m)/C_\kappa(I_m)]\Gamma_m\left(t+\frac{1}{2}(m+1),\,\kappa\right)C_\kappa(I_m) . \tag{12}$$

Performing the transformation, $U=T-R$, $T=T$, the right-hand side of (11) becomes

$$x^{mt+k}\int_{T>0}\operatorname{etr}(-T)|T|^{t-(m+1)/2}C_\kappa(T)\int_{(1-x)T>U>0}\operatorname{etr}(U)dUdT . \tag{13}$$

By using

$$\begin{aligned}\int_{(1-x)T>U>0}\operatorname{etr}(U)dU=&\left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\right]|T|^{(m+1)/2}(1-x)^{m(m+1)/2} \\ &\times{}_1F_1\left(\frac{1}{2}(m+1),\,m+1,\,(1-x)T\right)\end{aligned}$$

(Constantine [1]),

(13) becomes

$$\begin{aligned}x^{mt+k}&\left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\right]\sum_{k_1}\sum_{\kappa_1}(1-x)^{m(m+1)/2+k_1} \\ &\times\left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1}\Big/(m+1)_{\kappa_1}k_1!\right]\int_{T>0}\operatorname{etr}(-T)|T|^tC_\kappa(T)C_{\kappa_1}(T)dT .\end{aligned} \tag{14}$$

Using (8) and (2), (14) becomes

$$\text{(15)}\qquad \left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Gamma_m\left(t+\frac{1}{2}(m+1)\right)\Big/\Gamma_m(m+1)\right]$$
$$\times\sum_{k_1}\sum_{\kappa_1}\left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1}\Big/(m+1)_{\kappa_1}k_1!\right]$$
$$\times\sum_{\nu} g^{\nu}_{\kappa\kappa_1}\left(t+\frac{1}{2}(m+1)\right)_{\nu} C_{\nu}(I_m)x^{mt+k}(1-x)^{m(m+1)/2+k_1}\,.$$

Equaling (12) to (15) and solving for $f(I_m)/C_\kappa(I_m)$ and substituting in (9), we have the required result.

From Theorem 1, we have the following corollary immediately.

COROLLARY 1.

$$\int_{I_m>\Lambda>xI_m}|\Lambda|^{t-(m+1)/2}C_\kappa(\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i=1}^{m}d\lambda_i$$
$$=\left[\Gamma_m\left(\frac{1}{2}m\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\pi^{m^2/2}\left(t+\frac{1}{2}(m+1)\right)_\kappa\right]$$
$$\times\sum_{k_1}\sum_{\kappa_1}\left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1}\Big/(m+1)_{\kappa_1}k_1!\right]\sum_{\nu} g^{\nu}_{\kappa\kappa_1}\left(t+\frac{1}{2}(m+1)\right)_{\nu}$$
$$\times C_\nu(I_m)x^{mt+k}(1-x)^{m(m+1)/2+k_1}\,,$$

where $\Lambda=\text{diag}\,(\lambda_1,\lambda_2,\cdots,\lambda_m)$.

Using (6), (8) and Theorem 1, we have the following theorem.

THEOREM 2.

$$\text{(17)}\qquad \int_{I_m>S>xI_m}|S|^{t-(m+1)/2}|I_m-S|^{u-(m+1)/2}C_\kappa(RS)dS$$
$$=\left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\right][C_\kappa(R)/C_\kappa(I_m)]$$
$$\times\sum_{k_1,k_2}\sum_{\kappa_1,\kappa_2}\left[\left(-u+\frac{1}{2}(m+1)\right)_{\kappa_1}\left(\frac{1}{2}(m+1)\right)_{\kappa_2}\Big/k_1!k_2!(m+1)_{\kappa_2}\right]$$
$$\times\sum_{\nu}\left[g^{\nu}_{\kappa\kappa_1}\Big/\left(t+\frac{1}{2}(m+1)\right)_\nu\right]\sum_{\tau} g^{\tau}_{\nu\kappa_2}\left(t+\frac{1}{2}(m+1)\right)_\tau$$
$$\times C_\tau(I_m)x^{mt+k+k_1}(1-x)^{m(m+1)/2+k_2}\,.$$

COROLLARY 2.

$$\text{(18)}\qquad \int_{I_m>S>xI_m}|\Lambda|^{t-(m+1)/2}|I_m-\Lambda|^{u-(m+1)/2}C_\kappa(\Lambda)\prod_{i<j}(\lambda_i-\lambda_j)\prod_{i=1}^{m}d\lambda_i$$
$$=\left[\Gamma_m\left(\frac{1}{2}m\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\pi^{m^2/2}\right]$$

$$\times\sum_{k_1,k_2}\sum_{\kappa_1,\kappa_2}\left[\left(-u+\frac{1}{2}(m+1)\right)_{\kappa_1}\left(\frac{1}{2}(m+1)\right)_{\kappa_2}\Big/k_1!k_2!(m+1)_{\kappa_2}\right]$$
$$\times\sum_{\nu}\left[g^{\nu}_{\kappa\kappa_1}\Big/\left(t+\frac{1}{2}(m+1)\right)_{\nu}\right]\sum_{\tau}g^{\tau}_{\nu\kappa_2}\left(t+\frac{1}{2}(m+1)\right)_{\tau}$$
$$\times C_{\tau}(I_m)x^{mt+k+k_1}(1-x)^{m(m+1)/2+k_2}$$

where $\Lambda=\operatorname{diag}(\lambda_1,\lambda_2,\cdots,\lambda_m)$.

4. The distribution of the minimum latent root

(i) *The case of the Wishart matrix*

Let A be an $(m\times m)$ Wishart matrix with n degrees of freedom and covariance matrix $\sum$, distributed as

$$(19)\qquad \left[1\Big/\Gamma_m\left(\frac{1}{2}n\right)|2\Sigma|^{n/2}\right]\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}A\right)|A|^{(n-m-1)/2}.$$

Then the joint density function of the latent roots a_i's of A is

$$\left[\pi^{m^2/2}\Big/\Gamma_m\left(\frac{1}{2}n\right)|2\Sigma|^{n/2}\Gamma_m\left(\frac{1}{2}m\right)\right]$$
$$\times\int_{O(m)}\operatorname{etr}\left(-\frac{1}{2}\Sigma^{-1}H\Lambda H'\right)dH|\Lambda|^{(n-m-1)/2}\prod_{i<j}(a_i-a_j),$$

where $\Lambda=\operatorname{diag}(a_1,a_2,\cdots,a_m)$. Now, let $a_i=b_i/(1-b_i)$, then the joint density function of b_i's is, using (4)

$$\left[\pi^{m^2/2}\Big/\Gamma_m\left(\frac{1}{2}m\right)\Gamma_m\left(\frac{1}{2}n\right)|2\Sigma|^{n/2}\right]\sum_k\sum_{\kappa}\left[L^{n/2}_{\kappa}\left(\frac{1}{2}\Sigma^{-1}\right)\Big/k!C_{\kappa}(I_m)\right]$$
$$\times|\Lambda_b|^{(n-m-1)/2}C_{\kappa}(\Lambda_b)\prod_{i<j}(b_i-b_j)$$

where $\Lambda_b=\operatorname{diag}(b_1,b_2,\cdots,b_m)$.

Noting that $P_r(a_m\leqq x)=P_r(b_m\leqq x/(1+x))=1-P_r(b_m>x/(1+x))$, and using Corollary 1, we get

$$(20)\qquad P_r(a_m\leqq x)=1-\left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m(m+1)\Gamma_m\left(\frac{1}{2}n\right)|2\Sigma|^{n/2}\right]$$
$$\times\sum_k\sum_{\kappa}\left[L^{n/2}_{\kappa}\left(\frac{1}{2}\Sigma^{-1}\right)\Big/k!\left(\frac{1}{2}(n+m+1)\right)_{\kappa}C_{\kappa}(I_m)\right]$$
$$\times\sum_{k_1}\sum_{\kappa_1}\left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1}\Big/k_1!(m+1)_{\kappa_1}\right]$$
$$\times\sum_{\nu}g^{\nu}_{\kappa\kappa_1}\left(\frac{1}{2}(n+m+1)\right)_{\nu}$$
$$\times C_{\nu}(I_m)x^{mn/2+k}(1/(1+x))^{m(n+m+1)/2+k+k_1}.$$

Differentiating (20) with respect to x, we get the density function of the minimum latent a_m as follows:

THEOREM 3. *Let A be distributed as* (19), *then the density function of the minimum latent root a_m of A is given by*

$$
\begin{aligned}
f(a_m) = & \left[\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2 \Big/ \Gamma_m\left(\frac{1}{2}n\right)\Gamma_m(m+1)\,|2\Sigma|^{n/2}\right] \\
& \times \sum_k \sum_\kappa \left[L_\kappa^{n/2}\left(\frac{1}{2}\Sigma^{-1}\right) \Big/ k!\left(\frac{1}{2}(n+m+1)\right)_\kappa C_\kappa(I_m)\right] \\
& \times \sum_{k_1} \sum_{\kappa_1} \left[\left(\frac{1}{2}(m+1)\right)_{\kappa_1} \Big/ k_1!\,(m+1)_{\kappa_1}\right] \sum_\nu g_{\kappa\kappa_1}^\nu \left(\frac{1}{2}(n+m+1)\right)_\nu \\
& \times C_\nu(I_m) a_m^{mn/2+k-1} (1/(1+a_m))^{m(n+m+1)/2+k+k_1+1} \\
& \times \left[\left(\frac{1}{2}m(m+1)+k_1\right)a_m - \left(\frac{1}{2}mn+k\right)\right] .
\end{aligned}
$$

(ii) *The case of MANOVA MODEL*

Let A and B be independent $(m\times m)$ matrices distributed as the noncentral Wishart with s degrees of freedom and noncentrality parameters Ω, and the central Wishart with t degrees of freedom and the same covariance martix Σ respectively, $s, t \geqq m$, then the latent roots b_i's of $A(A+B)^{-1}$ are distributed in the form as

$$
\begin{aligned}
(21) \qquad & \left[\pi^{m^2/2}\Gamma_m\left(\frac{1}{2}(s+t)\right) \Big/ \Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m\left(\frac{1}{2}m\right)\right] \\
& \times \operatorname{etr}(-\Omega)\,|\Lambda|^{(s-m-1)/2}\,|I_m-\Lambda|^{(t-m-1)/2} \prod_{i<j}(b_i-b_j) \\
& \times \sum_k \sum_\kappa \left[\left(\frac{1}{2}(s+t)\right)_\kappa C_\kappa(\Omega)C_\kappa(\Lambda) \Big/ \left(\frac{1}{2}s\right)_\kappa k!\,C_\kappa(I_m)\right]
\end{aligned}
$$

(Constantine [1])

where $\Lambda=\operatorname{diag}(b_1, b_2, \cdots, b_m)$ and the latent roots f_i's of AB^{-1} are distributed with the joint density function such that

$$
\begin{aligned}
(22) \qquad & \left[\pi^{m^2/2}\Gamma_m\left(\frac{1}{2}(s+t)\right) \Big/ \Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m\left(\frac{1}{2}m\right)\right] \\
& \times \operatorname{etr}(-\Omega)\,|F|^{(s-m-1)/2}\,|I_m+F|^{-(s+t)/2} \prod_{i<j}(f_i-f_j) \\
& \times \sum_k \sum_\kappa \left[\left(\frac{1}{2}(s+t)\right)_\kappa C_\kappa(\Omega)C_\kappa(F(1+F)^{-1}) \Big/ \left(\frac{1}{2}s\right)_\kappa k!\,C_\kappa(I_m)\right]
\end{aligned}
$$

where $F=\operatorname{diag}(f_1, f_2, \cdots, f_m)$. If $s<m$, the joint density functions of b_i's and f_i's are obtained from (21) and (22) respectively making the substitutions

$$
s \to m\,, \qquad m \to s\,, \qquad t \to s+t-m\,.
$$

Using (21) and (22) and Corollary 2, the cdf's of the minimum latent roots, b_m, f_m, of Λ and F as

$$
\begin{aligned}
(23)\qquad & 1-\left[\Gamma_m\left(\frac{1}{2}(s+t)\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m(m+1)\right] \\
& \times \operatorname{etr}(-\Omega)\sum_k\sum_\kappa\left[\left(\frac{1}{2}(s+t)\right)_\kappa\Big/\left(\frac{1}{2}s\right)_\kappa k!\right][C_\kappa(\Omega)/C_\kappa(I_m)] \\
& \times \sum_{k_1,k_2}\sum_{\kappa_1,\kappa_2}\left[\left(\frac{1}{2}(-t+m+1)\right)_{\kappa_1}\left(\frac{1}{2}(m+1)\right)_{\kappa_2}\Big/k_1!k_2!(m+1)_{\kappa_2}\right] \\
& \times \sum_\nu\left[g^\nu_{\kappa\kappa_1}\Big/\left(\frac{1}{2}(s+m+1)\right)_\nu\right]\sum_\tau g^\tau_{\kappa_2\nu}\left(\frac{1}{2}(s+m+1)\right)_\tau \\
& \times C_\tau(I_m)x^{ms/2+k+k_1}(1-x)^{m(m+1)/2+k_2}\,,
\end{aligned}
$$

$$
\begin{aligned}
(24)\qquad & 1-\left[\Gamma_m\left(\frac{1}{2}(s+t)\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m(m-1)\right] \\
& \times \operatorname{etr}(-\Omega)\sum_k\sum_\kappa\left[\left(\frac{1}{2}(s+t)\right)_\kappa\Big/\left(\frac{1}{2}s\right)_\kappa k!\right][C_\kappa(\Omega)/C_\kappa(I_m)] \\
& \times \sum_{k_1,k_2}\sum_{\kappa_1,\kappa_2}\left[\left(\frac{1}{2}(-t+m+1)\right)_{\kappa_1}\left(\frac{1}{2}(m+1)\right)_{\kappa_2}\Big/k_1!k_2!(m+1)_{\kappa_2}\right] \\
& \times \sum_\nu\left[g^\nu_{\kappa\kappa_1}\Big/\left(\frac{1}{2}(s+m+1)\right)_\nu\right]\sum_\tau g^\tau_{\kappa_2\nu}\left(\frac{1}{2}(s+m+1)\right)_\tau \\
& \times C_\tau(I_m)x^{ms/2+k+k_1}(1+x)^{-(m(s+m+1)/2+k+k_1+k_2)}\,.
\end{aligned}
$$

From (23) and (24) we have the following theorem immediately.

THEOREM 4. *Let b_i's and f_i's be distributed with* (21) *and* (22) *respectively, then the density functions of the minimum latent roots, b_m, f_m, are given by*

$$
\begin{aligned}
f(b_m)= & \left[\Gamma_m\left(\frac{1}{2}(s+t)\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m(m+1)\right] \\
& \times \operatorname{etr}(-\Omega)\sum_k\sum_\kappa\left[\left(\frac{1}{2}(s+t)\right)_\kappa\Big/\left(\frac{1}{2}s\right)_\kappa k!\right][C_\kappa(\Omega)/C_\kappa(I_m)] \\
& \times \sum_{k_1,k_2}\sum_{\kappa_1,\kappa_2}\left[\left(\frac{1}{2}(-t+m+1)\right)_{\kappa_1}\left(\frac{1}{2}(m+1)\right)_{\kappa_2}\Big/k_1!k_2!(m+1)_{\kappa_2}\right] \\
& \times \sum_\nu\left[g^\nu_{\kappa\kappa_1}\Big/\left(\frac{1}{2}(s+m+1)\right)_\nu\right]\sum_\tau g^\tau_{\kappa_2\nu}\left(\frac{1}{2}(s+m+1)\right)_\tau \\
& \times C_\tau(I_m)b^{ms/2+k+k_1-1}(1-b_m)^{m(m+1)/2+k_2-1} \\
& \times\left[\left(\frac{1}{2}m(s+m+1)+k+k_1+k_2\right)b_m-\left(\frac{1}{2}ms+k+k_1\right)\right],
\end{aligned}
$$

$$
f(f_m)=\left[\Gamma_m\left(\frac{1}{2}(s+t)\right)\left(\Gamma_m\left(\frac{1}{2}(m+1)\right)\right)^2\Big/\Gamma_m\left(\frac{1}{2}s\right)\Gamma_m\left(\frac{1}{2}t\right)\Gamma_m(m+1)\right]
$$

$$\times \operatorname{etr}(-\Omega) \sum_{k} \sum_{\kappa} \left[\left(\frac{1}{2}(s+t)\right)_{\kappa} \Big/ \left(\frac{1}{2}s\right)_{\kappa} k! \right] [C_{\kappa}(\Omega)/C_{\kappa}(I_m)]$$

$$\times \sum_{k_1, k_2} \sum_{\kappa_1, \kappa_2} \left[\left(\frac{1}{2}(-t+m+1)\right)_{\kappa_1} \left(\frac{1}{2}(m+1)\right)_{\kappa_2} \Big/ k_1! k_2! (m+1)_{\kappa_2} \right]$$

$$\times \sum_{\nu} g^{\nu}_{\kappa \kappa_1} \Big/ \left(\frac{1}{2}(s+m+1)\right)_{\nu} \sum_{\tau} g^{\tau}_{\kappa_2 \nu} \left(\frac{1}{2}(s+m+1)\right)_{\tau}$$

$$\times C_{\tau}(I_m) f_m^{ms/2+k+k_1-1} (1+f_m)^{-(m(s+m+1)/2+k+k_1+k_2-1)}$$

$$\times \left[\left(\frac{1}{2} m(m+1) + k_2\right) f_m - \left(\frac{1}{2} ms + k + k_1\right) \right] .$$

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