FURTHER ASYMPTOTIC FORMULAS FOR THE NON-NULL DISTRIBUTIONS OF THREE STATISTICS FOR MULTIVARIATE LINEAR HYPOTHESIS

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Summary

New asymptotic expansions of the non-null distributions of the likelihood ratio, Hotelling's and Pillai's statistics for multivariate linear hypothesis are given in terms of normal distribution function and its derivatives, assuming the matrix of noncentrality parameters is of the same order as the sample size.

1. Introduction

Starting from the canonical form of multivariate linear hypothesis with the same notation as in Sugiura and Fujikoshi [12], we can express the likelihood ratio (=LR) statistic by $-2\rho\log\lambda=-\rho N\log|S_e(S_e+S_h)^{-1}|$, Hotelling's statistic by $T_0^2=(N-s)\operatorname{tr} S_hS_e^{-1}$ and Pillai's statistic by $V=(N-s+b)\operatorname{tr} S_h(S_e+S_h)^{-1}$, where S_h has the noncentral Wishart distribution $W_p(\Sigma, b; \Omega)$ and S_e has $W_p(\Sigma, N-s)$, independently. The parameters b and $n=N-s$ $(b\leqq s)$ are the degrees of freedom for the hypothesis and for the error respectively, with the sample size N from p-variate normal population and ρ is the correction factor given by $\rho N=N-s+(b-p-1)/2$.

In this paper new asymptotic formulas for the distributions of the three statistics mentioned above, are derived in terms of normal distribution and its derivatives under the assumption of $\Omega=O(n)$. The naturality of this assumption can be illustrated by considering the typical example of testing the equality of several means $\mu_1=\mu_2=\cdots=\mu_k$ $(p\times 1)$, based on random samples of size $N_1, N_2, \cdots, N_k$ from normal populations with common covariance matrix Σ (unknown), in which $b=k-1$, $s=k$ and the matrix of noncentrality parameters takes $\Omega=\Sigma^{-1}(N_1\mu_1\mu_1'+\cdots+N_k\mu_k\mu_k')/2$. If $\mu_i=O(1)$ and N_i tends to infinity for fixed N_i/N with $N=\sum_{\alpha=1}^{k} N_\alpha$, we have $\Omega=O(N)$.

All the asymptotic expansions for the multivariate linear hypothesis have been developed, hitherto, under $\Omega=O(1)$, in terms of noncentral χ^2-distributions, namely, Hotelling's T_0^2 by Siotani [8], [9], Ito [4], Fujikoshi [3] and Lee [5]; the LR statistic by Sugiura and Fujikoshi [12] and Pillai's V by Fujikoshi [3], which are valid only under alternatives near to the null hypothesis. Whereas our present asymptotic expansions are useful for alternatives far from the null hypothesis. We have already obtained the asymptotic expansions in terms of normal distribution and its derivatives for the non-null distributions of test statistics on covariance matrix in Sugiura and Fujikoshi [12], Sugiura [10], Nagao [7] and Sugiura and Nagao [14], from which we can foresee that normal distribution will appear as a limiting distribution for our present problem. In fact the same technique as in Sugiura and Nagao [14] is used in this paper. Expansions in terms of noncentral χ^2-distributions for test statistics on covariance matrix were obtained in Sugiura [11] and Fujikoshi [3].

2. Preliminary lemmas

Since only the invariant tests for the multivariate linear hypothesis are considered in this paper, we may assume $\Sigma=I$ and put $\Omega=m\Theta=m\,\mathrm{diag}(\theta_1,\cdots,\theta_p)$ for $n=m+2\Delta$, where $\Delta=O(1)$ is a correction factor corresponding to each statistic. By Anderson [1], the characteristic function of S_h having $W_p(I,b;\Omega)$, is expressed by

$$E[\mathrm{etr}\,(iTS_h)]=|I-2iT|^{-b/2}\,\mathrm{etr}\,\{2i\Omega T(I-2iT)^{-1}\}\,, \tag{2.1}$$

for $T=((1+\delta_{ij})t_{ij}/2)$, from which we can see that the statistics

$$T_e=\sqrt{m}(S_e/m-I) \qquad \text{and} \qquad T_h=\sqrt{m}(S_h/m-2\theta) \tag{2.2}$$

converge in law to $p(p+1)/2$ variate normal distributions with mean 0 as m tends to infinity. The following lemma is useful for our asymptotic expansions of the three distributions.

LEMMA 2.1. *Let S_e have $W_p(I,n)$ and S_h have $W_p(I,b;\Omega)$. For an analytic function $f(\Lambda_e,\Lambda_h)$ of two positive definite matrices $\Lambda_e=(\lambda_{ij}^{(e)})$ and $\Lambda_h=(\lambda_{ij}^{(h)})$ of order p, put*

$$\partial_e=\left(\frac{1}{2}(1+\delta_{ij})\partial/\partial\lambda_{ij}^{(e)}\right) \qquad \textit{and} \qquad \partial_h=\left(\frac{1}{2}(1+\delta_{ij})\partial/\partial\lambda_{ij}^{(h)}\right)\,. \tag{2.3}$$

Then for any given diagonal matrices A and B of order p, the following asymptotic formula holds:

$$\begin{aligned}(2.4)\qquad & E[f(S_e/m,S_h/m)\,\mathrm{etr}\,\{it(AT_e+BT_h)\}]\\ &=\mathrm{etr}\,\{-(A^2+4B^2\theta)t^2\}\Big[1+m^{-1/2}\sum_{\alpha=1}^{2}d_{2\alpha-1}(\partial)(it)^{2\alpha-1}\end{aligned}$$

$$+m^{-1}\sum_{\alpha=0}^{3} g_{2\alpha}(\partial)(it)^{2\alpha}+m^{-3/2}\sum_{\alpha=1}^{5} h_{2\alpha-1}(\partial)(it)^{2\alpha-1}$$

$$+m^{-2}\left\{\sum_{\alpha=0}^{2} k_{2\alpha}(\partial)(it)^{2\alpha}+\begin{pmatrix}\text{the lower order}\\ \text{derivatives}\end{pmatrix}\right\}$$

$$+O(m^{-5/2})\Big]f(\Lambda_e,\Lambda_h)|_{\Lambda_e=I,\Lambda_h=2\theta} ,$$

where

$$d_1(\partial)=\mathrm{tr}\,(2A\partial_e+8\theta B\partial_h+bB+2\Delta A) ,$$

$$d_3(\partial)=\mathrm{tr}\left(\frac{4}{3}A^3+8\theta B^3\right) ,$$

$$g_0(\partial)=\mathrm{tr}\,(\partial_e^2+4\theta\partial_h^2+b\partial_h+2\Delta\partial_e) ,$$

$$g_2(\partial)=\mathrm{tr}\,(4A^2\partial_e+24\theta B^2\partial_h+2\Delta A^2+bB^2+d_1(\partial)^2/2 ,$$

$$g_4(\partial)=\mathrm{tr}\,(2A^4+16\theta B^4)+d_1(\partial)d_3(\partial) ,$$

$$g_6(\partial)=d_3(\partial)^2/2 ,$$

$$h_1(\partial)=\mathrm{tr}\,(4A\partial_e^2+16\theta B\partial_h^2+8\theta\partial_h B\partial_h+4\Delta A\partial_e+2bB\partial_h)+d_1(\partial)g_0(\partial) ,$$

(2.5) $$h_3(\partial)=\mathrm{tr}\left(8A^3\partial_e+64\theta B^3\partial_h+\frac{8}{3}\Delta A^3+\frac{4}{3}bB^3\right)$$

$$+d_1(\partial)\left\{g_2(\partial)-\frac{1}{2}d_1(\partial)^2\right\}+d_3(\partial)g_0(\partial)+\frac{1}{6}d_1(\partial)^3 ,$$

$$h_5(\partial)=\mathrm{tr}\left(32\theta B^5+\frac{16}{5}A^5\right)+d_1(\partial)\,\mathrm{tr}\,(2A^4+16\theta B^4)$$

$$+d_3(\partial)\{g_2(\partial)-d_1(\partial)^2/2\}+d_1(\partial)^2d_3(\partial)/2 ,$$

$$h_7(\partial)=d_3(\partial)\,\mathrm{tr}\,(2A^4+16\theta B^4)+d_1(\partial)d_3(\partial)^2/2 ,$$

$$h_9(\partial)=d_3(\partial)^3/6 ,$$

$$k_0(\partial)=\{\mathrm{tr}\,(\partial_e^2+4\theta\partial_h^2)\}^2/2 ,$$

$$k_2(\partial)=\mathrm{tr}\,(\partial_e^2+4\theta\partial_h^2)\{\mathrm{tr}\,(2A\partial_e+8\theta B\partial_h)\}^2/2 ,$$

$$k_4(\partial)=2\{\mathrm{tr}\,(A\partial_e+4\theta B\partial_h)\}^4/3 .$$

PROOF. Since S_e/m and S_h/m converge in probability to I and 2θ respectively, Taylor expansion of f at $\Lambda_e=I$, $\Lambda_h=2\theta$ gives

(2.6) $$\mathrm{etr}\,\{(S_e/m-I)\partial_e+(S_h/m-2\theta)\partial_h\}f(\Lambda_e,\Lambda_h)|_{\Lambda_e=I,\Lambda_h=2\theta} .$$

From (2.1) we can write the left-hand side of (2.4) as

(2.7) $$\mathrm{etr}\,\{(-\partial_e-2\theta\partial_h)-\sqrt{m}\,it(A+2\theta B)\}|I-2m^{-1/2}(itA+m^{-1/2}\partial_e)|^{-n/2}$$
$$\times|I-2m^{-1/2}(itB+m^{-1/2}\partial_h)|^{-b/2}\,\mathrm{etr}\,[2\Omega m^{-1/2}(itB+m^{-1/2}\partial_h)$$
$$\times\{I-2m^{-1/2}(itB+m^{-1/2}\partial_h)\}^{-1}] .$$

Applying the asymptotic formula, $-\log|I-m^{-1/2}A|=\sum_{\alpha=1}^{l} m^{-\alpha/2}\operatorname{tr} A^\alpha/\alpha+O(m^{-(l+1)/2})$ for symmetric matrix A having the absolute value of the maximum latent root$=||A||<\sqrt{m}$, to (2.7), we can get the result after some straight forward computation.

Although the following lemma was used in Sugiura and Nagao [13], [14], which is still useful in this paper in the derivation of the asymptotic expansions in the next section, we shall state it here for completeness. Let E_{ii} be a $p\times p$ matrix having 1 at ith diagonal and 0 at other places. Also let E_{ij} $(i\neq j)$ be a symmetric matrix having 1/2 at (i, j) and (j, i) places and 0 at other places. The matrix E_{ij} is generated by operating (i, j) element of ∂_e or ∂_h to Λ_e or Λ_h respectively.

LEMMA 2.2 *Let $\theta=\operatorname{diag}(\theta_1,\cdots,\theta_p)$ and $\Gamma=\operatorname{diag}(\gamma_1,\cdots,\gamma_p)$. For any diagonal matrices A and B, the following identities hold. Further the second identity holds for any matrices A and B not necessarily diagonal.*

$$\sum_{i,j=1}^{p}\theta_i\gamma_j \operatorname{tr} AE_{ij}BE_{ij}=\frac{1}{4}(2\operatorname{tr} AB\theta\Gamma+\operatorname{tr} A\theta\operatorname{tr} B\Gamma+\operatorname{tr} A\Gamma\operatorname{tr} B\theta)\ ,$$

$$\sum_{i,j=1}^{p}\theta_i\gamma_j \operatorname{tr} AE_{ij}\operatorname{tr} BE_{ij}=\frac{1}{4}\operatorname{tr}\theta(A+A')\Gamma(B+B')\ ,$$

$$\begin{aligned}(2.8)\qquad &\sum_{i,j,k,l=1}^{p}\theta_i\gamma_k(\operatorname{tr} AE_{ij}BE_{kl})^2\\ &=\frac{1}{2}\operatorname{tr} A^2B^2\theta\Gamma+\frac{1}{8}(\operatorname{tr} AB\theta\Gamma\operatorname{tr} AB+\operatorname{tr} AB\theta\operatorname{tr} AB\Gamma)\\ &\quad+\frac{1}{16}(\operatorname{tr} A^2\theta\Gamma\operatorname{tr} B^2+\operatorname{tr} A^2\operatorname{tr} B^2\theta\Gamma+\operatorname{tr} A^2\theta\operatorname{tr} B^2\Gamma\\ &\quad+\operatorname{tr} B^2\theta\operatorname{tr} A^2\Gamma)\ .\end{aligned}$$

3. Asymptotic expansions of the LR statistic

The likelihood ratio statistic $-2\rho\log\lambda=-\rho N\log|S_e(S_e+S_h)^{-1}|$ can be expressed by T_e and T_h given by (2.2) for $m=\rho N$, which are $O_p(1)$, as

$$(3.1)\quad m\{\log|I+2\theta|-\log|I+m^{-1/2}T_e|+\log|I+m^{-1/2}(I+2\theta)^{-1}(T_e+T_h)|\}\ .$$

Expanding the second and the third term within the region $R_N=\{(T_e, T_h)|\,||T_e||<\sqrt{m}$ and $||(I+2\theta)^{-1}(T_e+T_h)||<\sqrt{m}\}$, and using the Theorem 1 in Mann and Wald [6], the characteristic function of $\{-2\rho\log\lambda-m\log|I+2\theta|\}/\sqrt{m}$ can be expressed by

$$\begin{aligned}(3.2)\qquad &E\Big[\operatorname{etr}\{(AT_e+BT_h)it\}\Big\{1+m^{-1/2}itl_1(T)\\ &\quad+m^{-1}\Big[itl_2(T)+\frac{1}{2}(it)^2l_1(T)^2\Big]\Big\}\Big]+O(m^{-3/2})\ ,\end{aligned}$$

where $A=B-I$, $B=(I+2\theta)^{-1}$ and

$$l_1(T)=\frac{1}{2}[\operatorname{tr} T_e^2-\operatorname{tr}\{B(T_e+T_h)\}^2] ,$$
(3.3)
$$l_2(T)=-\frac{1}{3}[\operatorname{tr} T_e^3-\operatorname{tr}\{B(T_e+T_h)\}^3] .$$

We shall compute each expectation in (3.2). Note that $\Delta=(-b+p+1)/4$ in this case. Taking $f(\Lambda_e, \Lambda_h)\equiv 1$ in Lemma 2.1 and arranging each term with respect to $s_j=\operatorname{tr} A^j$, we immediately obtain

(3.4)
$$E[\operatorname{etr}\{it(AT_e+BT_h)\}] \\ =\exp(-\tau^2t^2/2)[1+m^{-1/2}\{itd_1+(it)^3d_3\} \\ +m^{-1}\{(it)^2g_2+(it)^4g_4+(it)^6g_6\}+O(m^{-3/2})] ,$$

where $\tau^2=-2\operatorname{tr} A^2-4\operatorname{tr} A$ and

(3.5)
$$d_1=\frac{1}{2}(b+p+1)s_1+bp ,$$
$$d_3=-\frac{8}{3}s_3-8s_2-4s_1 ,$$
$$g_2=\frac{1}{2}(b+p+1)s_2+2bs_1+bp+\frac{1}{2}d_1^2 ,$$
$$g_4=(-6s_4-24s_3-24s_2-8s_1)+d_1d_3 ,$$
$$g_6=\frac{1}{2}d_3^2 .$$

Putting $f=\operatorname{tr}(\Lambda_e-I)^2$ or $\operatorname{tr}\{B(\Lambda_e+\Lambda_h-I-2\theta)\}^2$ in Lemma 2.1 and noting that

(3.6)
$$\partial_{ij}^{(e)}\partial_{kl}^{(e)}\operatorname{tr}(\Lambda_e-I)^2=2\operatorname{tr} E_{ij}E_{kl} ,$$
$$\partial_{ij}^{(e)}\partial_{kl}^{(e)}\operatorname{tr}\{B(\Lambda_e+\Lambda_h-I-2\theta)\}^2=2\operatorname{tr}(BE_{ij}BE_{kl}) ,$$

for $\partial_e=(\partial_{ij}^{(e)})$ and $E_{ij}=\partial_{ij}^{(e)}\Lambda_e=((1/2)(1+\delta_{ij})(\partial/\partial\lambda_{ij})\lambda_{\alpha\beta})_{\alpha,\beta=1,\cdots,p}$, we can get from Lemma 2.2,

(3.7)
$$E[l_1(T)\operatorname{etr}\{it(AT_e+BT_h)\}] \\ =\exp(-\tau^2t^2/2)[g_0+(it)^2g_2+m^{-1/2}\{ith_1+(it)^3h_3+(it)^5h_5\}] \\ +O(m^{-1}) ,$$

where

$$g_0=\frac{1}{2}\{s_2+s_1^2\}\ ,$$

$$g_2=-2s_4-4s_3\ ,$$

(3.8) $$h_1=(b+p+5)s_3+\frac{1}{4}(b+p+1)s_1^3+\frac{1}{4}(b+p+17)s_2s_1 +\frac{1}{2}(bp+4b+8p+20)s_2+\frac{1}{2}(bp+12)s_1^2+8(p+1)s_1\ ,$$

$$h_3=-16s_5-(b+p+1)s_4s_1-\frac{4}{3}s_3s_2-\frac{4}{3}s_3s_1^2-2(bp+28)s_4 -2(b+p+1)s_1s_3-4s_2^2-4s_2s_1^2-4(bp+14)s_3-2s_2s_1-2s_1^3-24s_2\ ,$$

$$h_5=\frac{16}{3}s_4s_3+\frac{32}{3}s_3^2+16s_2s_4+32s_2s_3+8s_4s_1+16s_3s_1\ .$$

Similarly we have

(3.9) $$E[l_2(T)\ \mathrm{etr}\{it(AT_e+BT_h)\}] =\exp(-\tau^2t^2/2)\Big[it\{2s_4+2s_3s_1+2s_3+2s_2s_1-2(p+1)s_2-4(p+1)s_1\} -\frac{8}{3}(it)^3(s_6+3s_5+3s_4+2s_3)\Big]+O(m^{-1/2})\ .$$

Finally in the computation of the fourth expectation in (3.2), the leading term appears only from the operation $k_0(\partial)$, $k_2(\partial)$ and $k_4(\partial)$. Noting that

$$\partial_{ii'}^{(e)}\partial_{jj'}^{(e)}\partial_{kk'}^{(e)}\partial_{ll'}^{(e)}\ \mathrm{tr}\,(A\Lambda_e)^2\ \mathrm{tr}\,(B\Lambda_e)^2 =4\ \mathrm{tr}\,AE_{ii'}AE_{jj'}\ \mathrm{tr}\,BE_{kk'}BE_{ll'}+4\ \mathrm{tr}\,AE_{kk'}AE_{ll'}\ \mathrm{tr}\,BE_{ii'}BE_{jj'} +4\ \mathrm{tr}\,AE_{ii'}AE_{kk'}\ \mathrm{tr}\,BE_{jj'}BE_{ll'}+4\ \mathrm{tr}\,AE_{jj'}AE_{ll'}\ \mathrm{tr}\,BE_{ii'}BE_{kk'} +4\ \mathrm{tr}\,AE_{ii'}AE_{ll'}\ \mathrm{tr}\,BE_{jj'}BE_{kk'}+4\ \mathrm{tr}\,AE_{jj'}AE_{kk'}\ \mathrm{tr}\,BE_{ii'}BE_{ll'}\ ,$$

(3.10) $$\partial_{ii'}^{(e)}\partial_{jj'}^{(e)}\partial_{kk'}^{(e)}\partial_{ll'}^{(h)}\ \mathrm{tr}\,\Lambda_e^2\ \mathrm{tr}\,\{B(\Lambda_e+\Lambda_h)\}^2 =4\ \mathrm{tr}\,E_{ii'}E_{jj'}\ \mathrm{tr}\,BE_{kk'}BE_{ll'}+4\ \mathrm{tr}\,E_{ii'}E_{kk'}\ \mathrm{tr}BE_{jj'}BE_{ll'} +4\ \mathrm{tr}\,E_{jj'}E_{kk'}\ \mathrm{tr}\,BE_{ii'}BE_{ll'}\ ,$$

$$\partial_{ii'}^{(e)2}\partial_{jj'}^{(h)}\partial_{kk'}^{(h)}\ \mathrm{tr}\,\Lambda_e^2\ \mathrm{tr}\,\{B(\Lambda_e+\Lambda_h)\}^2 =4\ \mathrm{tr}\,E_{ii'}^2\ \mathrm{tr}\,BE_{jj'}BE_{kk'}+8\ \mathrm{tr}\,E_{ii'}E_{jj'}\ \mathrm{tr}\,BE_{ii'}BE_{kk'} +8\ \mathrm{tr}\,E_{ii'}E_{kk'}\ \mathrm{tr}\,BE_{ii'}BE_{jj'}+4\ \mathrm{tr}\,E_{jj'}E_{kk'}\ \mathrm{tr}\,(BE_{ii'})^2\ ,$$

we can get

(3.11) $$E\Big[\frac{1}{2}l_1(T)^2\ \mathrm{etr}\,(AT_e+BT_h)\Big]=\exp(-\tau^2t^2/2)[k_0+(it)^2k_2+it^4k_4] +O(m^{-1/2})\ ,$$

where

$$k_0=\frac{1}{2}s_4+\frac{1}{8}s_1^4+\frac{5}{8}s_2^2+\frac{1}{4}s_2s_1^2-2s_1^2-2s_2-2(p+1)s_1\ ,$$

(3.12)
$$k_2=-4s_6-s_4s_2-s_4s_1^2-8s_5-2s_3s_1^2-2s_3s_2+8s_4+24s_3+16s_2\ ,$$

$$k_4=2s_4^2+8s_4s_3+8s_3^2\ .$$

It follows that the characteristic function (3.2) can be expressed by

(3.13)
$$\exp(-\tau^2t^2/2)\Big[1+m^{-1/2}\{(it)v_1+(it)^3v_3\}+m^{-1}\sum_{\alpha=1}^{3}(it)^{2\alpha}w_{2\alpha}+O(m^{-3/2})\Big]\ ,$$

where

$$v_1=\frac{1}{2}s_2+\frac{1}{2}s_1^2+\frac{1}{2}(b+p+1)s_1+bp\ ,$$

$$v_3=-2s_4-\frac{20}{3}s_3-8s_2-4s_1\ ,$$

(3.14)
$$\begin{aligned} w_2=&\frac{5}{2}s_4+\frac{1}{8}s_1^4+\frac{5}{8}s_2^2+\frac{1}{4}s_2s_1^2+2s_1s_3+(b+p+7)s_3+\frac{1}{4}(b+p+1)s_1^3\\ &+\frac{1}{4}(b+p+25)s_1s_2+\frac{1}{2}(bp+5b+5p+13)s_2\\ &+\left\{\frac{1}{8}(b+p+1)^2+\frac{1}{2}bp+4\right\}s_1^2\\ &+(b+p+1)\left(\frac{bp}{2}+2\right)s_1+bp\left(\frac{bp}{2}+1\right)\ ,\end{aligned}$$

$$\begin{aligned} w_4=&-\frac{20}{3}s_6-s_4s_2-s_4s_1^2-32s_5-\frac{10}{3}s_3s_1^2-\frac{10}{3}s_2s_3-(b+p+1)s_1s_4\\ &-4s_2^2-4s_2s_1^2-2(bp+31)s_4-\frac{10}{3}(b+p+1)s_1s_3-\frac{4}{3}(5bp+46)s_3\\ &-2s_1^3-2(2b+2p+3)s_1s_2-8(bp+4)s_2-2(b+p+1)s_1^2-4(bp+2)s_1\ ,\end{aligned}$$

$$\begin{aligned} w_6=&2s_4^2+\frac{40}{3}s_3s_4+\frac{200}{9}s_3^2+16s_2s_4+\frac{160}{3}s_3s_2+8s_1s_4+32s_2^2\\ &+\frac{80}{3}s_1s_3+32s_1s_2+8s_1^2\ .\end{aligned}$$

Inversion of the characteristic function (3.13) yields:

THEOREM 3.1. *Let* $-2\rho\log\lambda=-\rho N\log|S_e(S_e+S_h)^{-1}|$ *be the likelihood ratio statistic for testing the multivariate linear hypothesis where* S_e *has* $W_p(\Sigma, N-s)$ *and* S_h *has* $W_p(\Sigma, b;\Omega)$. *Correction factor* ρ *is given by* $m=\rho N=N-s+(1/2)(b-p-1)$. *Under* $\Omega=m\theta$, *we have*

$$P(\{-2\rho\log\lambda-m\log|I+2\theta|\}/(\tau\sqrt{m})<x)$$
$$=\Phi(x)-m^{-1/2}\{v_1\Phi^{(1)}(x)/\tau+v_3\Phi^{(3)}(x)/\tau^3\}$$
$$+m^{-1}\sum_{\alpha=1}^{3} w_{2\alpha}\Phi^{(2\alpha)}(x)\tau^{-2\alpha}+O(m^{-3/2}) ,$$

where $\tau^2=-2s_2-4s_1$ *for* $s_j=\mathrm{tr}\{(I+2\theta)^{-1}-I\}^j$ *and* $\Phi^{(j)}(x)$ *means the jth order derivative of the standard normal distribution function* $\Phi(x)$. *The coefficients* v_j *and* w_j *are given by* (3.14).

4. Asymptotic expansions of Hotelling's and Pillai's statistics

The characteristic function of Hotelling's statistic, $\sqrt{n}\,\mathrm{tr}\,S_hS_e^{-1}-\sqrt{n}\,\mathrm{tr}\,2\theta$ with $n=N-s$ can be expressed by (3.2) for $A=-2\theta$, $B=I$ and

$$l_1(T)=\mathrm{tr}\,2\theta T_e^2-\mathrm{tr}\,T_hT_e ,\qquad (4.1)$$
$$l_2(T)=-\mathrm{tr}\,2\theta T_e^3+\mathrm{tr}\,T_hT_e^2 ,$$

Based on Lemma 2.1 and Lemma 2.2, the similar computation as in Section 3 yields:

THEOREM 4.1. *Using the same notation as in Theorem* 3.1, *we have, for Hotelling's* $T_0^2=n\,\mathrm{tr}\,S_hS_e^{-1}$,

$$P(\sqrt{n}\{\mathrm{tr}\,S_hS_e^{-1}-\mathrm{tr}\,2\theta\}/\tau<x)\qquad (4.2)$$
$$=\Phi(x)-n^{-1/2}\{v_1\Phi^{(1)}(x)/\tau+v_3\Phi^{(3)}(x)/\tau^3\}$$
$$+n^{-1}\sum_{\alpha=1}^{3} w_{2\alpha}\Phi^{(2\alpha)}(x)\tau^{-2\alpha}+O(n^{-3/2}) ,$$

where $\tau^2=8\,\mathrm{tr}\,\theta(I+\theta)$ *and for* $s_j=\mathrm{tr}\,\theta^j$,

$$v_1=bp+2(p+1)s_1 ,$$
$$v_3=\frac{64}{3}s_3+32s_2+8s_1 ,$$
$$w_2=4(3p+4)s_2+2(p^2+2p+3)s_1^2+\{2b(p^2+p+2)+12p+12\}s_1+\frac{1}{2}bp(bp+2) ,\qquad (4.3)$$
$$w_4=160s_4+\frac{128}{3}(p+1)s_1s_3+\left(\frac{64}{3}bp+320\right)s_3+64(p+1)s_1s_2+16(p+1)s_1^2+32(bp+5)s_2+8(bp+2)s_1 ,$$
$$w_6=\frac{2048}{9}s_3^2+\frac{2048}{3}s_3s_2+\frac{512}{3}s_1s_3+512s_2^2+256s_1s_2+32s_1^2 .$$

By Constantine [2], it is known that under the null hypothesis $\Omega=0$,

$$E[\mathrm{tr}\,(S_h S_e^{-1})]=bp/(n-p-1)\ . \tag{4.4}$$

Consequently if we take $m=n-p-1$ ($\Delta=(p+1)/2$) and put $\tilde{T}_0^2=m\,\mathrm{tr}\,S_h S_e^{-1}$, then the expectation of $\tilde{T}_0^2$ is the same as the expectation of the limiting distribution of $\tilde{T}_0^2$, namely, the χ^2-distribution with bp degrees of freedom. This will suggest the use of m in the asymptotic expansion instead of n. In fact it is easy to see that Theorem 4.1 holds, when we replace n by m in (4.2) and v_j, w_α by $\tilde{v}_j$, $\tilde{w}_\alpha$ given below:

$$\begin{aligned}
&\tilde{v}_1=bp\ , \qquad \tilde{v}_3=v_3\ ,\\
&\tilde{w}_2=4s_2+4s_1^2+4(b+p+1)s_1+\frac{1}{2}bp(bp+2)\ ,\\
&\tilde{w}_4=160s_4+\left(\frac{64}{3}bp+320\right)s_3+32(bp+5)s_2+8(bp+2)s_1\ ,\\
&\tilde{w}_6=w_6\ .
\end{aligned} \tag{4.5}$$

It may be interesting to note that each coefficients in the asymptotic expansion by m in (4.5) and (3.14) is symmetric with respect to b and p, whereas v_i and w_j in (4.3) are not.

For Pillai's statistic $V=(n+b)\,\mathrm{tr}\,S_h(S_e+S_h)^{-1}$, we easily have $E[V]=bp$ under the null hypothesis. Noting that V has asymptotically χ^2-distribution with bp degrees of freedom by Fujikoshi [3] under $\Omega=0$, $m=n+b$ ($\Delta=-b/2$) may be recommended for the asymptotic expansion as is traditionally used.

We can write the characteristic function of $m^{-1/2}V-\sqrt{m}\,\mathrm{tr}\,2\theta(I+2\theta)^{-1}$ as (3.2) for $A=-(I+2\theta)^{-1}+(I+2\theta)^{-2}$, $B=(I+2\theta)^{-2}$ and

$$\begin{aligned}
&l_1(T)=\mathrm{tr}\,CT_eC(T_e+T_h)-\mathrm{tr}\,C\{C(T_e+T_h)\}^2\ ,\\
&l_2(T)=-\mathrm{tr}\,CT_e\{C(T_e+T_h)\}^2+\mathrm{tr}\,C\{C(T_e+T_h)\}^3\ ,
\end{aligned} \tag{4.6}$$

where $C=(I+2\theta)^{-1}$. A similar computation as in Hotelling's T_0^2 statistic, though it is more tedious, yields the following result:

THEOREM 4.2. *Under $\Omega=m\theta$ for $m=n+b$, the non-null distribution of Pillai's statistic $V=m\,\mathrm{tr}\,S_h(S_e+S_h)^{-1}$ can be written by*

$$\begin{aligned}
&P(\{m^{-1/2}V-\sqrt{m}\,\mathrm{tr}\,2\theta C\}/\tau<x)\\
&\quad=\Phi(x)+m^{-1/2}\{v_1\Phi^{(1)}(x)/\tau+v_3\Phi^{(3)}(x)/\tau^3\}\\
&\qquad+m^{-1}\sum_{\alpha=1}^{3}w_{2\alpha}\Phi^{(2\alpha)}(x)\tau^{-2\alpha}+O(m^{-3/2})\ ,
\end{aligned}$$

where $C=(I+2\theta)^{-1}$, $\tau^2=2\,\mathrm{tr}\,C^2(I-C^2)$ and for $t_j=\mathrm{tr}\,L^j=\mathrm{tr}\,(I-C)^j$, each coefficient is given by

$$v_1 = -bp + (b+p+1)t_1 - 2t_2 + t_3 - 2t_1^2 + t_1 t_2 ,$$

$$v_3 = 4 \operatorname{tr} L(I-L)^3\left(-I + 4L - \frac{11}{3}L^2 + L^3\right) ,$$

$$w_2 = \frac{1}{2}v_1^2 + bp - 6(b+p+1)t_1 + (11b+11p+34)t_2 - (8b+8p+74)t_3$$
$$+ (2b+2p+78)t_4 - 40t_5 + 8t_6 + 23t_1^2 - 66t_1t_2 + 56t_1t_3 + 20t_2^2$$
$$- 24t_1t_4 - 16t_2t_3 + 3t_2t_4 + 4t_1t_5 + t_3^2 ,$$

$$w_4 = 2 \operatorname{tr} L(I-L)^4(4I - 40L + 112L^2 - 127L^3 + 64L^4 - 12L^5) + v_1 v_3 ,$$

$$w_6 = \frac{1}{2}v_3^2 .$$

Here we can see again that each coefficient before t_j is symmetric with respect to b and p.

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