ON THE COMPLEX ANALOGUE OF BAYESIAN ESTIMATION OF A MULTIVARIATE REGRESSION MODEL

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(Received Feb. 20, 1970; revised Feb. 1, 1972)

1. Introduction

Classical statistical analysis based on the complex Gaussian distribution has been investigated recently in Goodman [9], James [10], Khatri [12], Giri [8], and Saxena [15] in the sampling theory framework. In this paper attempts are made to discuss some Bayesian estimation procedure for the complex multivariate regression model and hence to derive the complex analogues of the analyses given in Geisser [6], Geisser and Confield [7], and Tiao and Zellner [16].

In Section 3, we discuss the prior and the posterior distributions of the parameters θ and Σ for the model (2.1). Properties of the posterior distributions of these parameters are then derived in Sections 4–6. For the notations of this paper we follow in essence Khatri [12] unless otherwise stated.

2. Specification of the model

Consider the following complex multivariate regression model

$$Y_j=\theta X+U_j\ , \qquad j=1, 2,\cdots, n \tag{2.1}$$

where $Y_j=Y_{j1}+iY_{j2} : p\times k$ is the jth observation matrix, $\theta=\theta_1+i\theta_2 :$ $p\times q$ the unknown parameter matrix in the pq-dimensional complex plane, $X=X_1+iX_2 : q\times k$ a given matrix of design variables with rank q, and $U_j=U_{j1}+iU_{j2} : p\times k$ the complex matrix of random disturbances. In this paper it will be assumed that the U_j's are independently distributed as complex Gaussian with zero expectation (that is, $EU_{j1}=EU_{j2}=0$) and covariance matrix $V_U=E\{\boldsymbol{u}_j^*\bar{\boldsymbol{u}}_j^{*\prime}\}=K\otimes\Sigma$, where $\boldsymbol{u}_j^*$ is the $pk\times 1$ column vector of columns of U_j, and further that $\Sigma=\Sigma_1+i\Sigma_2 : p\times p$ is a hermitian positive definite matrix (hpd) and $K : k\times k$ a known real symmetric positive definite matrix (spd). Then, following Goodman [9], we have for the likelihood function of (θ, Σ):

$$(2.2)\qquad L(\theta, \Sigma | Y) \propto |\Sigma|^{-kn} \exp\left\{-\sum_{j=1}^{n} \operatorname{tr} \Sigma^{-1}(Y_j-\theta X)K^{-1}(\overline{Y_j-\theta X})'\right\} .$$

Now, if we put

$$\hat{\theta}=Y.K^{-1}\bar{X}'(XK^{-1}\bar{X}')^{-1} \qquad \text{and}$$

$$S=\sum_{j=1}^{n}(Y_j-\hat{\theta}X)K^{-1}(\overline{Y_j-\hat{\theta}X})' , \qquad \text{where } Y.=\frac{1}{n}\sum_{j=1}^{n} Y_j ,$$

then one can easily observe that $\hat{\theta}$ and S form a minimal set of sufficient statistics for θ and Σ, and whence,

$$(2.3)\qquad L(\theta, \Sigma | Y) \propto |\Sigma|^{-kn} \exp\{-\operatorname{tr} \Sigma^{-1}[S+n(\theta-\hat{\theta})XK^{-1}\bar{X}'(\overline{\theta-\hat{\theta}})']\} .$$

In Section 3 we shall make use of (2.3) to derive the posterior distributions of θ and Σ.

3. Prior and posterior distribution of θ and Σ

For the prior distribution of θ and Σ, we follow Jeffrey ([11], p. 182) to assume that *a priori* θ and Σ are statistically independent and further that

$$(3.1)\qquad P(\theta) \propto \text{const.}$$

This prior purports to reflect to a large degree prior ignorance or relative diffuseness of θ and can be justified by the principle of stable estimation or heuristic arguments or others. For the prior of Σ we shall follow the invariance principle due to Jeffrey ([11], p. 180) by taking

$$P(\Sigma) \propto \left|\begin{pmatrix} -E\dfrac{\partial^2 \log L}{\partial\sigma_{ij(1)}\partial\sigma_{i'j'(1)}} , & -E\dfrac{\partial^2 \log L}{\partial\sigma_{ij(1)}\partial\sigma_{h's'(2)}} \\ -E\dfrac{\partial^2 \log L}{\partial\sigma_{hs(2)}\partial\sigma_{i'j'(1)}} , & -E\dfrac{\partial^2 \log L}{\partial\sigma_{hs(2)}\partial\sigma_{h's'(2)}} \end{pmatrix}\right|^{\frac{1}{2}}$$

$$i \leqq j,\; i' \leqq j',\; h<s,\; h'<s' ,$$

where $\Sigma=\Sigma_1+i\Sigma_2=(\sigma_{ij(1)})+i(\sigma_{hs(2)})$.

Now, if we put $\Sigma^{-1}=W=W_1+iW_2=(w_{ij(1)})+i(w_{hs(2)})$, then W_1 is spd and W_2 skew symmetric so that $w_{hh(2)}=0$, $h=1, 2, \cdots, p$, and it can readily be seen that,

$$\frac{\partial \log L}{\partial w_{ij(1)}}=kn \operatorname{tr} \Sigma\left(\frac{\partial \Sigma^{-1}}{\partial w_{ij(1)}}\right)-\operatorname{tr}\left(\frac{\partial \Sigma^{-1}}{\partial w_{ij(1)}}\right)[S+n(\theta-\hat{\theta})XK^{-1}\bar{X}'(\overline{\theta-\hat{\theta}})']$$

$$i \leqq j$$

$$\begin{cases} = kn\sigma_{ii(1)} - C_{ii(1)} \\ = 2kn\sigma_{ij(1)} - C_{ij(1)}\,, & i \neq j\,, \end{cases}$$

and

$$\frac{\partial \log L}{\partial w_{hs(2)}} = 2kn\sigma_{hs(2)} - C_{hs(2)}\,, \qquad h < s\,,$$

where $C_{ii(1)}$, $C_{ij(1)}$ and $C_{hs(2)}$ are numbers independent of elements of Σ (and hence of $W = \Sigma^{-1}$).

Thus, the adoption of the Jeffrey's invariance principle leads immediately to take as prior of Σ^{-1}:

$$\begin{aligned} P(\Sigma^{-1}) &\propto \left| \begin{pmatrix} -E\dfrac{\partial^2 \log L}{\partial w_{ij(1)}\partial w_{i'j'(1)}}\,, & -E\dfrac{\partial^2 \log L}{\partial w_{ij(1)}\partial w_{h's'(2)}} \\ -E\dfrac{\partial^2 \log L}{\partial w_{hs(2)}\partial w_{i'j'(1)}}\,, & -E\dfrac{\partial^2 \log L}{\partial w_{hs(2)}\partial w_{h's'(2)}} \end{pmatrix} \right|^{\frac{1}{2}} \\ &\propto \left| \frac{\partial(\Sigma_1, \Sigma_2)}{\partial(W_1, W_2)} \right|^{1/2} = |\Sigma|^p\,, \qquad i \leqq j,\ i' \leqq j',\ h < s,\ h' < s'\,, \end{aligned}$$

the last equality of which follows from Khatri [11] and Deemer and Olkin [4] and whence,

$$P(\Sigma) \propto |\Sigma|^{-p}\,. \tag{3.2}$$

Thus, if we combine (3.1) and (3.2) with (2.3) we obtain the joint posterior distribution of θ and Σ as

$$P(\theta, \Sigma | Y) = C|\Sigma|^{-(kn+p)} \exp\{-\mathrm{tr}\,\Sigma^{-1}[S + n(\theta - \hat{\theta})XK^{-1}\bar{X}'\overline{(\theta - \hat{\theta})'}]\} \tag{3.3}$$

where

$$C = \pi^{-pq}\{\Gamma_p(nk-q)\}^{-1}|S|^{kn-q}|nXK^{-1}\bar{X}'|^p\,, \qquad \Gamma_p(\lambda) = \pi^{p(p-1)/2}\prod_{j=1}^{p}\Gamma(\lambda - j + 1)\,.$$

The normalizing constants C in (3.3) is obtained by first transforming Σ into Σ^{-1} and then making use of the complex Gaussian and complex Wishart densities given in Goodman [9].

From (3.3), it follows immediately that the posterior distribution of Σ is

$$P(\Sigma | Y) = \{\Gamma_p(nk-q)\}^{-1}|S|^{kn-q}|\Sigma|^{-(kn-q+p)} \exp\{-\mathrm{tr}\,\Sigma^{-1}S\}\,. \tag{3.4}$$

This is the complex analogue of the inverted Wishart density (Tiao and Zellner [16]) and will be denoted in this paper by $\Sigma_{p\times p} \sim W_{I,c}(p, kn-q; S)$.

Further, one can integrate out Σ easily from (3.3) by using the density (3.4) to obtain the posterior distribution of θ as

$$P(\theta|Y)=\pi^{-pq}\Gamma_p(nk)\{\Gamma_p(nk-q)\}^{-1}|S|^{kn-q}|nXK^{-1}\bar{X}'|^{p} \cdot |S+n(\theta-\hat{\theta})XK^{-1}\bar{X}'\overline{(\theta-\hat{\theta})}'|^{-kn}\,. \tag{3.5}$$

This is the complex analogue of the matric *t*-distribution as defined in Dickey [5] and will be denoted in this paper by

$$\theta_{p\times q}\sim GMt_c(p,\, q,\, kn;\, \hat{\theta},\, S,\, (nXK^{-1}\bar{X}')^{-1})\,.$$

Properties of the above two densities are to be discussed in Sections 4 and 6 and then posterior distributions of some functions of the parameters and their H.P.D. (Highest posterior density) regions will be derived in Sections 5 and 7.

4. The complex matric *t*-distribution

In this section we derive some properties of the complex matric *t*-distribution, $X_{p\times q}\sim GMt_c(p,\, q,\, n;\, \theta,\, \Sigma,\, V)$, as defined in the previous section. We shall first show that the marginal and the conditional distributions of X are also the complex matric *t*-distributions. Making use of these results we then proceed to the derivation of the distribution of linear functions of X.

THEOREM 4.1. *Let* $X_{p\times q}\sim GMt_c(p,\, q,\, n;\, \theta,\, \Sigma,\, V)$ *and*

$$X=(\overset{q_1}{X_{c1}},\, \overset{q_2}{X_{c2}})_p\,, \qquad \theta=(\overset{q_1}{\theta_{c1}},\, \overset{q_2}{\theta_{c2}})_p\,, \qquad V=\begin{array}{cc} q_1 & q_2 \\ \end{array}\!\!\begin{pmatrix} V_{11} & V_{12} \\ \bar{V}_{12}' & V_{22} \end{pmatrix}\!\begin{array}{c} q_1 \\ q_2 \end{array},$$

where $q_1+q_2=q$. *Then* $X_{c1}\sim GMt_c(p,\, q_1,\, n-q_2;\, \theta_{c1},\, \Sigma,\, V_{11})$ *and* $X_{c2}|X_{c1}\sim GMt_c(p,\, q_2,\, n;\, \theta_{c2}+(X_{c1}-\theta_{c1})V_{11}^{-1}V_{12},\, \Sigma+(X_{c1}-\theta_{c1})V_{11}^{-1}\overline{(X_{c1}-\theta_{c1})}',\, V_{22\cdot 1})$, *where* $V_{22\cdot 1}=V_{22}-\bar{V}_{12}'V_{11}^{-1}V_{12}$.

PROOF. We have $|V|=|V_{11}||V_{22\cdot 1}|$. Further, if we put $V^{-1}=\begin{pmatrix} A_{11} & A_{12} \\ \bar{A}_{12}' & A_{22} \end{pmatrix}$, then $V_{11}^{-1}=A_{11}-A_{12}A_{22}^{-1}\bar{A}_{12}'$, $A_{22}=V_{22\cdot 1}^{-1}$ and $\bar{A}_{12}'=-V_{22\cdot 1}^{-1}\bar{V}_{12}'V_{11}^{-1}$, and whence, by completing the square,

$$\begin{aligned}(X-\theta)V^{-1}\overline{(X-\theta)}'=&(X_{c1}-\theta_{c1})V_{11}^{-1}\overline{(X_{c1}-\theta_{c1})}'\\ &+(X_{c2}-\theta_{c2}-(X_{c1}-\theta_{c1})V_{11}^{-1}V_{12})V_{22\cdot 1}^{-1}\\ &\cdot\overline{(X_{c2}-\theta_{c2}-(X_{c1}-\theta_{c1})V_{11}^{-1}V_{12})}'\,.\end{aligned}$$

Thus

$$\begin{aligned}P(X)&=\pi^{-pq}\{\Gamma_p(n)/\Gamma_p(n-q)\}|\Sigma|^{n-q}|V|^{-p}|\Sigma+(X-\theta)V^{-1}\overline{(X-\theta)}'|^{-n}\\ &=\pi^{-pq_1}\{\Gamma_p(n-q_2)/\Gamma_p(n-q_2-q_1)\}|\Sigma|^{n-q_2-q_1}|V_{11}|^{-p}\\ &\quad\cdot|\Sigma+(X_{c1}-\theta_{c1})V_{11}^{-1}\overline{(X_{c1}-\theta_{c1})}'|^{-(n-q_2)}\pi^{-pq_2}\{\Gamma_p(n)/\Gamma_p(n-q_2)\}\end{aligned}$$

$$\cdot\, |V_{22\cdot 1}|^{-p}|\Sigma+(X_{c1}-\theta_{c1})V_{11}^{-1}(\overline{X_{c1}-\theta_{c1}})'|^{n-q_2}$$
$$\cdot\, |\Sigma+(X_{c1}-\theta_{c1})V_{11}^{-1}(\overline{X_{c1}-\theta_{c1}})'+(X_{c2}-\theta_{c2}-(X_{c1}-\theta_{c1})V_{11}^{-1}V_{12})V_{22\cdot 1}^{-1}$$
$$\cdot\, (\overline{X_{c2}-\theta_{c2}-(X_{c1}-\theta_{c1})V_{11}^{-1}V_{12}})'|^{-n}\,.$$

This completes the proof of Theorem 4.1.

Now, one can readily show that

$$|\Sigma+(X-\theta)V^{-1}(\overline{X-\theta})'|=|\Sigma||V+(\overline{X-\theta})'\Sigma^{-1}(X-\theta)||V^{-1}|\,.$$

Thus, if $X\sim GMt_c(p, q, n; \theta, \Sigma, V)$, then $\bar{X}'\sim GMt_c(q, p, n; \bar{\theta}', V, \Sigma)$ and conversely.

Utilizing this result, Theorem 4.1 leads immediately to the following results.

THEOREM 4.2. *Let* $X_{p\times q}\sim GMt_c(p, q, n; \theta, \Sigma, V)$ *and* $X=\begin{array}{cc} q & \\ \begin{pmatrix} X_{r1} \\ X_{r2} \end{pmatrix} & \begin{matrix} p_1 \\ p_2 \end{matrix} \end{array}$, $\theta=\begin{array}{cc} q & \\ \begin{pmatrix} \theta_{r1} \\ \theta_{r2} \end{pmatrix} & \begin{matrix} p_1 \\ p_2 \end{matrix} \end{array}$, *and* $\Sigma=\begin{array}{cc} \begin{matrix} p_1 & \quad p_2 \end{matrix} & \\ \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \bar{\Sigma}_{12}' & \Sigma_{22} \end{pmatrix} & \begin{matrix} p_1 \\ p_2 \end{matrix} \end{array}$. *Then* $X_{r1}\sim GMt_c(p_1, q, n-p_2; \theta_{r1}, \Sigma_{11}, V)$ *and* $X_{r2}|X_{r1}\sim GMt_c(p_2, q, n; \theta_{r2}+\bar{\Sigma}_{12}'\Sigma_{11}^{-1}(X_{r1}-\theta_{r1}), \Sigma_{22\cdot 1}, V+(\overline{X_{r1}-\theta_{r1}})'\Sigma_{11}^{-1}(X_{r1}-\theta_{r1}))$ *where* $\Sigma_{22\cdot 1}=\Sigma_{22}-\bar{\Sigma}_{12}'\Sigma_{11}^{-1}\Sigma_{12}$.

By making use of the above two theorems we now derive the distribution of linear functions of X.

THEOREM 4.3. *Let* $X_{p\times q}\sim GMt_c(p, q, n; \theta, \Sigma, V)$. *Let* $D: q\times s$ *and* $R: r\times p$ *be matrices* (*complex in general*) *of ranks* s *and* r *respectively. Then* $XD\sim GMt_c(p, s, n-q+s; \theta D, \Sigma, \bar{D}'VD)$ *and*

$$RX\sim GMt_c(r, q, n-p+r; R\theta, R\Sigma\bar{R}', V)\,.$$

PROOF. Consider the transformation $T: Z=XV^{-1/2}$. Following Khatri [12], the Jacobian of the transformation is $J=\partial X/\partial Z=\partial(X_1, X_2)/\partial(Z_1, Z_2)=|V|^p$, where $X=X_1+iX_2$ and $Z=Z_1+iZ_2$, so that $Z_{p\times q}\sim GMt_c(p, q, n; \theta V^{-1/2}, \Sigma, I_q)$. If we put $H_1=V^{1/2}D$, then $\operatorname{rank} H_1=\operatorname{rank} D=s$ and there exists a matrix $H_2: q\times(q-s)$ of rank $q-s$ such that $H=(H_1, H_2)$ is nonsingular. Thus, if we make the transformation $W=ZH=(ZH_1, ZH_2)$, we obtain the Jacobian as $\partial Z/\partial W=|\bar{H}'H|^{-p}$ and hence,

$$W_{p\times q}\sim GMt_c(p, q, n; \theta V^{-1/2}H, \Sigma, \bar{H}'H)\,.$$

From Theorem 4.1, it follows that

$$W_1=ZH_1=XD\sim GMt_c(p, s, n-q+s; \theta D, \Sigma, \bar{H}_1'H_1=\bar{D}'VD)\,.$$

Now, $\bar{X}'\sim GMt_c(q, p, n; \bar{\theta}', V, \Sigma)$ so that

$$\bar{X}'\bar{R}'\sim GMt_c(q, r, n-p+r; \bar{\theta}'\bar{R}', V, R\Sigma\bar{R}')\,.$$

Thus, $RX \sim GMt_c(r, q, n-p+r; R\theta, R\Sigma\bar{R}', V)$. Q.E.D.

If $s=1$, then $\boldsymbol{XD}$ is a complex linear combination of columns of X and $\bar{\boldsymbol{D}}'V\boldsymbol{D}$ is real. We have for the density of $(\overline{X\boldsymbol{D}})'$:

$$P(\bar{\boldsymbol{D}}'\bar{X}')=\pi^{-p}\{\Gamma(n-q+1)/\Gamma(n-q+1-p)\}(\bar{\boldsymbol{D}}'V\boldsymbol{D})^{n+1-p-q}|\Sigma|^{-1}$$
$$\cdot\ \{\bar{\boldsymbol{D}}'V\boldsymbol{D}+\bar{\boldsymbol{D}}'(\bar{X}'-\bar{\theta}')\Sigma^{-1}(X-\theta)\boldsymbol{D}\}^{-(n-q+1)}\,.$$

Or the density of $\boldsymbol{Y}=(1/\sqrt{\bar{\boldsymbol{D}}'V\boldsymbol{D}})\bar{\boldsymbol{D}}'\bar{X}'$ is

$$P(\boldsymbol{Y})=\pi^{-p}\{\Gamma(n+1-q)/\Gamma(n+1-q-p)\}$$
$$\cdot\ |\Sigma|^{-1}(1+(\boldsymbol{Y}-\boldsymbol{\mu})\Sigma^{-1}(\overline{\boldsymbol{Y}-\boldsymbol{\mu}})')^{-(n-q+1)}\,,$$

where $\boldsymbol{\mu}=(1/\sqrt{\bar{\boldsymbol{D}}'V\boldsymbol{D}})\bar{\boldsymbol{D}}'\bar{\theta}'$. This is the complex analogue of the multivariate t-distribution as given in Cornish [3]. We shall use the notation $\boldsymbol{Y} \sim Mt_c(p, n-q+1; \boldsymbol{\mu}, \Sigma)$.

Similarly, if $r=1$, then RX is the complex linear combination of rows of X and we have for the density of $\boldsymbol{Z}=(1/\sqrt{R\Sigma\bar{R}'})\boldsymbol{R}X$:

$$P(\boldsymbol{Z})=\pi^{-q}\{\Gamma(n-p+1)/\Gamma(n-p+1-q)\}$$
$$\cdot\ |V|^{-1}(1+(Z-W)V^{-1}(\overline{Z-W})')^{-(n-p+1)}\,,$$

where $\boldsymbol{W}=(1/\sqrt{\boldsymbol{R}\Sigma\bar{\boldsymbol{R}}'})\boldsymbol{R}\theta$.

Or $\boldsymbol{Z}=(1/\sqrt{\boldsymbol{R}\Sigma\bar{\boldsymbol{R}}'})\boldsymbol{R}X \sim Mt_c(q, n-p+1; \boldsymbol{W}, V)$ where $\boldsymbol{R}\Sigma\bar{\boldsymbol{R}}'$ is real.

The following theorem gives a connection between the complex matric t-distribution and the complex multivariate Beta II distribution.

THEOREM 4.4. *Let $X_{p\times q} \sim GMt_c(p, q, n; \theta, \Sigma, V)$. Then, putting $W=(X-\theta)V^{-1}(\overline{X-\theta})'$, the density of W is*

$$P_w(W)=\{\Gamma_p(n)/[\Gamma_p(n-q)\Gamma_p(q)]\}|\Sigma|^{n-q}|W|^{q-p}|\Sigma+W|^{-n}\,, \qquad n>q>p\,.$$

PROOF. Utilizing the complex Wishart density established by Goodman [9], it is easy to establish that

$$\int_{\substack{YV^{-1}Y'=W\\ Y:\,p\times q}} dY=|V|^{p}\int_{\substack{Y_1\bar{Y}_1'=W\\ Y_1:\,p\times q}} dY_1=\pi^{pq}\{\Gamma_p(q)\}^{-1}|W|^{q-p}|V|^{p}\,.$$

Thus,

$$P_w(W)=\pi^{-pq}\Gamma_p(n)\{\Gamma_p(n-q)\}^{-1}|V|^{-p}|\Sigma|^{n-p}\int_{\left\{\begin{array}{l}(X-\theta)V^{-1}(\overline{X-\theta})'=W\\ X:\,p\times q\end{array}\right.}|\Sigma+(X-\theta)V^{-1}(\overline{X-\theta})'|^{-n}dX$$
$$=\{\Gamma_p(n)/[\Gamma_p(n-q)\Gamma_p(q)]\}|\Sigma|^{n-q}|W|^{q-p}|\Sigma+W|^{-n}\,, \qquad n>q>p\,.$$

This is the complex analogue of the nonstandardized multivariate Beta II distribution and will be denoted in this paper by $W_{p\times p}\sim B_{\mathrm{II},c}(q, n-q; \Sigma)$.

5. The posterior distributions of θ and of constants of θ

In Section 3 we obtained the posterior distribution of θ as $\theta\sim GMt_c(p, q, kn; \hat{\theta}, S, (nXK^{-1}\bar{X}')^{-1})$. Thus, from Theorem 4.4 it follows that the linear combinations of columns or rows of θ are distributed as complex multivariate-*t* *a posteriori*. Specifically, if $\theta\boldsymbol{d}$ is a column linear combination of θ, then a *posteriori*,

$$\begin{aligned}\boldsymbol{\mu}&=\frac{1}{\sqrt{\bar{\boldsymbol{d}}'(nXK^{-1}\bar{X}')^{-1}\boldsymbol{d}}}(\overline{\theta\boldsymbol{d}})'\\&\sim Mt_c\left(p, nk-q+1; \frac{1}{\sqrt{\bar{\boldsymbol{d}}'(nXK^{-1}\bar{X}')^{-1}\boldsymbol{d}}}(\overline{\hat{\theta}\boldsymbol{d}})', S\right)\end{aligned}$$

Similarly, if $\boldsymbol{c}'\theta$ is a row contrast of θ, then *a posteriori*,

$$\boldsymbol{w}=\frac{1}{\sqrt{\bar{\boldsymbol{c}}'S\boldsymbol{c}}}\bar{\boldsymbol{c}}'\theta\sim Mt_c\left(q, nk-p+1; \frac{1}{\sqrt{\bar{\boldsymbol{c}}'S\boldsymbol{c}}}\bar{\boldsymbol{c}}'\hat{\theta}, (nXK^{-1}\bar{X}')^{-1}\right).$$

Utilizing these results we now proceed to obtain the H.P.D. regions for θ, $\theta\boldsymbol{d}$ and $\bar{\boldsymbol{c}}'\theta$ as defined in Box and Tiao [2], which then provide tests of Hypotheses for $H_{01}:\theta=\theta_0$, $H_{02}:\theta\boldsymbol{d}=\boldsymbol{\mu}_0$ and $H_{03}:\bar{\boldsymbol{c}}'\theta=\boldsymbol{w}_0'$ respectively.

Following Box and Tiao [2], we have for the H.P.D. region R of θ with content $1-\alpha$:

$$R: U(\theta)=|S|\,|S+(\theta-\hat{\theta})(nXK^{-1}\bar{X}')(\overline{\theta-\hat{\theta}})'|^{-1}\geqq C_\alpha\,,$$

where C_α is determined such that $Pr\{U(\theta)\geqq C_\alpha|Y\}=1-\alpha$.

But, from the calculation of moments of $U(\theta)$, it follows that $U(\theta)\sim U_{p,q,kn-q}$, where $U_{p,q,kn-q}$ is the product of p independent Beta I variables, $B_{\mathrm{I}}(q, kn-q-j+1)$, $j=1, 2,\cdots, p$, as defined in Anderson [1]. Thus, we have $C_\alpha=U_{\alpha,p,q,kn-q}$, where $U_{\alpha,p,q,kn-q}$ is the lower α point of $U_{p,q,kn-q}$. Putting $\alpha=0.05$ (or some small nonnegative number), one would then tend to reject the Hypothesis $H_0:\theta=\theta_0$ if $U(\theta_0)<C_{0.05}$, for *a posteriori* the probability that the data will fit the hypothesis is at most 0.05. This argument parallels the classical hypothesis testing approach as given in Lehmann [13]. In fact, it can readily be shown that the likelihood ratio test for $H_0:\theta=\theta_0$ in the Pearson-Neyman sense with level of significance $\alpha=0.05$ is given by $U(\theta_0)<C_{0.05}$, as given above.

For the derivation of the H.P.D. region for $\theta\boldsymbol{d}$ we notice that, by putting $\boldsymbol{\mu}=(1/\sqrt{\bar{\boldsymbol{d}}'(nXK^{-1}\bar{X}')^{-1}\boldsymbol{d}})\bar{\boldsymbol{d}}'(\overline{\theta-\hat{\theta}})'=\boldsymbol{\mu}_1'+i\boldsymbol{\mu}_2'$ and $S^{-1}=R_1+iR_2$, we have

$$P(\boldsymbol{\mu})=\{\Gamma(nk-q+1)/[\pi^p\Gamma(nk-q-p+1)]\}$$
$$\cdot\begin{vmatrix}R_1 & -R_2\\ R_2 & R_1\end{vmatrix}^{1/2}\left(1+(\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')\begin{pmatrix}R_1 & -R_2\\ R_2 & R_1\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix}\right)^{-(nk-q+1)}.$$

Thus, by utilizing a property of the multivariate t-distribution as given in Cornish [3],

$$\frac{nk-p-q+1}{2}(\boldsymbol{\mu}_1', \boldsymbol{\mu}_2')\begin{pmatrix}R_1 & -R_2\\ R_2 & R_1\end{pmatrix}\begin{pmatrix}\boldsymbol{\mu}_1\\ \boldsymbol{\mu}_2\end{pmatrix}\sim F_{2p,2(nk-q-p+1)}\,, \qquad \textit{a posteriori.}$$

Therefore, by adopting the approach given in Box and Tiao [2], we obtain the H.P.D. region of content $1-\alpha$ for $\theta\boldsymbol{d}$ as

$$[\overline{(\theta-\hat{\theta})\underset{\sim}{d}}]'S^{-1}[(\theta-\hat{\theta})\underset{\sim}{d}]\leqq p[\bar{\boldsymbol{d}}'(nXK^{-1}\bar{X}')^{-1}\boldsymbol{d}]F_{\alpha,2p,2(nk-p-q+1)}/(nk-p-q+1)$$

where $F_{\alpha,2p,2(nk-q-p+1)}$ is the upper α point of $F_{2p,2(nk-q-p+1)}$.

Similarly, we obtain the H.P.D. region of content $1-\alpha$ for $\bar{\boldsymbol{c}}'\theta$ as

$$n\bar{\boldsymbol{c}}'(\theta-\hat{\theta})(XK^{-1}\bar{X}')\overline{(\theta-\hat{\theta})}'\boldsymbol{c}\leqq q[\bar{\boldsymbol{c}}'S\boldsymbol{c}]F_{\alpha,2q,2(nk-p-q+1)}/(nk-p-q+1)$$

where $F_{\alpha,2q,2(nk-p-q+1)}$ is the upper α point of $F_{2q,2(nk-p-q+1)}$.

Thus, letting α be a small positive number (0.05 for instance), one would then tend to reject $H_{01}:\theta\boldsymbol{d}=\boldsymbol{\mu}_0$ if $\overline{(\underset{\sim}{\mu}_0-\hat{\theta}\boldsymbol{d})}'S^{-1}(\underset{\sim}{\mu}_0-\hat{\theta}\underset{\sim}{d})>p[\bar{\boldsymbol{d}}'(nXK^{-1}\cdot\bar{X}')^{-1}\boldsymbol{d}]F_{\alpha,2p,2(nk-q-p+1)}/(nk-p-q+1)$ and reject $H_{02}:\bar{\boldsymbol{c}}'\theta=\boldsymbol{w}_0$ if $n(\underset{\sim}{w}_0-\bar{\boldsymbol{c}}'\hat{\theta})\cdot(XK^{-1}\bar{X}')\overline{(\underset{\sim}{w}_0-\bar{\boldsymbol{c}}'\hat{\theta})}'>q[\bar{\boldsymbol{c}}'S\boldsymbol{c}]F_{\alpha,2q,2(nk-p-q+1)}/(nk-p-q+1)$.

6. The complex inverted Wishart distribution

Let $X:p\times p$ be a hermitian random matrix distributed as $X_{p\times p}\sim W_{\mathrm{I},c}(p, n; V)$, where $V:p\times p$ is hpd. In this section we shall derive the distributions of some interesting functions of X.

THEOREM 6.1. *If* $X_{p\times p}\sim W_{\mathrm{I},c}(p, n; V)$, *then* $Y=X^{-1}\sim W_c(p, n; V^{-1})$ *and conversely.*

PROOF. This theorem is trivial by (3.4), since, following Khatri [12], one can easily show that the Jacobian of the transformation $T:Y=X^{-1}$ is $\partial Y/\partial X=|X^{-1}|^{2p}$.

THEOREM 6.2. *Let* $X_{p\times p}\sim W_{\mathrm{I},c}(p, n; V)$ *and let* X *and* V *be partitioned as* $X=\begin{array}{cc} p_1 & p_2 \\ \end{array}\!\!\begin{pmatrix}X_{11} & X_{12}\\ \bar{X}_{12}' & X_{22}\end{pmatrix}\begin{matrix}p_1\\ p_2\end{matrix}$ and $V=\begin{array}{cc} p_1 & p_2 \\ \end{array}\!\!\begin{pmatrix}V_{11} & V_{12}\\ \bar{V}_{12}' & V_{22}\end{pmatrix}\begin{matrix}p_1\\ p_2\end{matrix}$. *Then* $X_{11}\sim W_{\mathrm{I},c}(p_1, n-p_2; V_{11})$, $X_{22\cdot1}=X_{22}-\bar{X}_{12}'X_{11}^{-1}X_{12}\sim W_{\mathrm{I},c}(p_2, n; V_{22\cdot1})$ *independently of* X_{11}, *where* $V_{22\cdot1}=V_{22}-\bar{V}_{12}'V_{11}^{-1}V_{12}$, *and*

$$Z=X_{11}^{-1}X_{12}\sim GMt_c(p_1, p_2, n+p_1; V_{11}^{-1}V_{12}, V_{11}^{-1}, V_{22\cdot1})\ .$$

Proof. Put $Y=X^{-1}=\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, where $Y_{21}=\bar{Y}_{12}'$, then, by completing the square,

$$\begin{aligned} \operatorname{tr} VX^{-1} = & \operatorname{tr} V_{11}X_{11}^{-1}+\operatorname{tr} V_{11}(X_{11}^{-1}X_{12}-V_{11}^{-1}V_{12})X_{22\cdot 1}^{-1} \\ & \cdot (\overline{X_{11}^{-1}X_{12}-V_{11}^{-1}V_{12}})'+\operatorname{tr} V_{22\cdot 1}X_{22\cdot 1}^{-1} \ . \end{aligned}$$

For fixed X_{11}, if we make the transformation

$$T: \left\{ \begin{aligned} X_{22\cdot 1} &= X_{22}-\bar{X}_{12}'X_{11}^{-1}X_{12} \\ Z &= X_{11}^{-1}X_{12} \end{aligned} \right.$$

then we obtain the Jacobian of the transformation T as

$$\frac{\partial(X_{22}, X_{12})}{\partial(X_{22\cdot 1}, Z)}=\frac{\partial X_{12}}{\partial Z}=|X_{11}\bar{X}_{11}'|^{p_2}=|X_{11}|^{2p_2} \ .$$

Thus, noting that $|V|=|V_{11}||V_{22\cdot 1}|$, $|X|=|X_{11}||X_{22\cdot 1}|$ and $\Gamma_p(n)=\pi^{p_1p_2}\Gamma_{p_2}(n) \cdot\Gamma_{p_1}(n-p_2)$, we have

$$\begin{aligned} P(X_{11}, X_{22\cdot 1}, Z) = & \{\pi^{p_1p_2}\Gamma_{p_2}(n)\Gamma_{p_1}(n-p_2)\}^{-1}|V_{11}|^n|V_{22\cdot 1}|^n \\ & \cdot |X_{11}|^{-(n-p_2+p_1)}|X_{22\cdot 1}|^{-(n+p)} \\ & \cdot \exp\{-\operatorname{tr} V_{11}X_{11}^{-1}-\operatorname{tr} V_{22\cdot 1}X_{22\cdot 1}^{-1} \\ & \quad -\operatorname{tr} V_{11}(Z-V_{11}^{-1}V_{12})X_{22\cdot 1}^{-1}(\overline{Z-V_{11}^{-1}V_{12}})'\} \\ = & W_{\mathrm{I},c}(X_{11}; p_1, n-p_2; V_{11})W_{\mathrm{I},c}(X_{22\cdot 1}; p_2, n; V_{22\cdot 1}) \\ & \cdot \pi^{-p_1p_2}|V_{11}|^{p_2}|X_{22\cdot 1}|^{-p_1} \\ & \cdot \exp\{-\operatorname{tr} V_{11}(Z-V_{11}^{-1}V_{12})X_{22\cdot 1}^{-1}(\overline{Z-V_{11}^{-1}V_{12}})'\} \end{aligned}$$

which implies the first and the second statements of Theorem 6.1.

If we integrate out $X_{22\cdot 1}$ from the joint density of $X_{22\cdot 1}$ and Z by making use of the complex inverted Wishart density, we obtain the distribution of Z as

$$\bar{Z}'\sim GMt_c(p_2, p_1, n+p_1; \bar{V}_{12}'V_{11}^{-1}, V_{22\cdot 1}V_{11}^{-1}) \ .$$

Thus

$$Z\sim GMt_c(p_1, p_2, n+p_1; V_{11}^{-1}V_{12}, V_{11}^{-1}, V_{22\cdot 1}) \ . \qquad \text{Q.E.D.}$$

If $p_1=1$, then $X_{11}=x_{11}$ and $V_{11}=\sigma_{11}$ are real and $P(x_{11})=\{\Gamma(n-p+1)\}^{-1} \cdot\sigma_{11}^{n-p+1}x_{11}^{-(n-p+2)}\exp\{-\sigma_{11}/x_{11}\}$. Thus $2(\sigma_{11}/x_{11})\sim\chi^2_{2(n-p+1)}$, a Chi-square distribution with d.f.$=2(n-p+1)$. If $p_2=1$, then $X_{22\cdot 1}=x_{22\cdot 1}$ and $V_{22\cdot 1}=\sigma_{22\cdot 1}$ are real and so, $2\sigma_{22\cdot 1}/x_{22\cdot 1}\sim\chi^2_{2n}$.

The next theorem derives the distribution of $x_{ij}/\sqrt{x_{ii}x_{jj}}=r_{ij}$, $i\neq j$, which will be called the complex correlation coefficient.

Theorem 6.3. *Let $X_{p\times p}\sim W_{\mathrm{I},c}(p, n; \Sigma)$ and let x_{ij} be the (i, j)th ele-*

ment of X. *Then* $r_{ij}=r_{ij(1)}+ir_{ij(2)}=x_{ij}/\sqrt{x_{ii}x_{jj}}=(x_{ij(1)}+ix_{ij(2)})/\sqrt{x_{ii}x_{jj}}$ *has density*

$$P(r_{ij})=\frac{\Gamma(n-p+2+1/2)}{\pi^{3/2}\Gamma(n-p+1)}(1-\rho_{ij}\bar{\rho}_{ij})^{n-p+2}(1-r_{ij}\bar{r}_{ij})^{(n-p+2)-2}$$
$$\cdot\, I_{2(n-p+2)}(\text{R.P.}\,(\bar{\rho}_{ij}r_{ij}))\,,$$

where $\rho_{ij}=\sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$ *with* σ_{ij} *being the* $(i,\ j)$*th element of* Σ *and*

$$I_{2(n-p+2)}(\text{R.P.}\,(\bar{\rho}_{ij}r_{ij}))=\int_{-\infty}^{\infty}[\cosh\beta-\text{R.P.}\,(\bar{\rho}_{ij}r_{ij})]^{-2(n-p+2)}d\beta\ .$$

In this paper we shall use the notation $r_{ij}\sim h_c(\rho_{ij},\ n-p+2)$.

PROOF. From Theorem 6.2,

$$\begin{pmatrix} x_{ii} & x_{ij} \\ \bar{x}_{ij} & x_{jj}\end{pmatrix}\sim W_{\text{I},c}\left(2,\ n-p+2;\ \begin{pmatrix}\sigma_{ii} & \sigma_{ij}\\ \bar{\sigma}_{ij} & \sigma_{jj}\end{pmatrix}\right)\,,$$

where x_{ii}, x_{jj}, σ_{ii} and σ_{jj} are real quantities, that is,

$$P(x_{ii},\ x_{jj},\ x_{ij})=\{\pi\Gamma(n-p+2)\Gamma(n-p+1)\}^{-1}$$
$$\cdot\,(\sigma_{ii}\sigma_{jj}(1-\rho_{ij}\bar{\rho}_{ij}))^{n-p+2}(x_{ii}x_{jj}(1-r_{ij}\bar{r}_{ij}))^{-(n-p+4)}$$
$$\cdot\,\exp\left\{-\frac{1}{(1-r_{ij}\bar{r}_{ij})}\left[\frac{\sigma_{ii}}{x_{ii}}+\frac{\sigma_{jj}}{x_{jj}}-2\sqrt{\frac{\sigma_{ii}\sigma_{jj}}{x_{ii}x_{jj}}}\,\text{R.P.}\,(\bar{\rho}_{ij}r_{ij})\right]\right\}\,.$$

If we make the transformation

$$T:\left\{\begin{aligned} &\alpha=\sqrt{\frac{\sigma_{ii}\sigma_{jj}}{x_{ii}x_{jj}}}\\ &e^{\beta}=\sqrt{\frac{\sigma_{jj}x_{ii}}{\sigma_{ii}x_{jj}}}\\ &r_{ij}=\frac{x_{ij}}{\sqrt{x_{ii}x_{jj}}}=\alpha\frac{x_{ij}}{\sqrt{\sigma_{ii}\sigma_{jj}}}\end{aligned}\right.$$

then $0\leqq\alpha<\infty$, $-\infty<\beta<\infty$, $|r_{ij}|\leqq 1$ and

$$\left|\frac{\partial(x_{ii},\ x_{jj},\ x_{ij})}{\partial(\alpha,\ \beta,\ r_{ij})}\right|=\left|\frac{\partial(x_{ii},\ x_{jj},\ x_{ij(1)},\ x_{ij(2)})}{\partial(\alpha,\ \beta,\ r_{ij(1)},\ r_{ij(2)})}\right|$$
$$=(\sigma_{ii}\sigma_{jj})^2\frac{2}{\alpha^5}\ .$$

Thus,

$$P(\alpha,\ \beta,\ r_{ij})=2\{\pi\Gamma(n-p+2)\Gamma(n-p+1)\}^{-1}(1-\rho_{ij}\bar{\rho}_{ij})^{n-p+2}\alpha^{2(n-p+4)-5}$$

$$\cdot (1-r_{ij}\bar{r}_{ij})^{-(n-p+4)} \exp\left\{-\frac{2\alpha}{1-r_{ij}\bar{r}_{ij}}(\cosh\beta-\text{R.P.}\ (\bar{\rho}_{ij}r_{ij}))\right\}$$

$$0\leqq\alpha<\infty,\ -\infty<\beta<\infty \text{ and } |r_{ij}|\leqq 1\,.$$

Integrating out α and β and noting that $\Gamma(1/2)\Gamma(2r)=2^{2r-1}\Gamma(r)\Gamma(r+1/2)$ for any positive integer r, we obtain the desired conclusion. Q.E.D.

If $\rho_{ij}=0$, then $I_{2(n-p+2)}(\text{R.P.}\ (\bar{\rho}_{ij}r_{ij}))=B(1/2, n-p+2)$. Hence,

$$P(r_{ij})=\frac{\Gamma(n-p+2)}{(\Gamma(1/2))^2\Gamma(n-p+1)}(1-r_{ij}\bar{r}_{ij})^{n-p}\,.$$

Thus, $r_{ij}\bar{r}_{ij}\sim B_1(1, n-p+1)$ so that, using the relationship between the Beta and F variables, $(n-p+1)r_{ij}\bar{r}_{ij}/(1-r_{ij}\bar{r}_{ij})\sim F_{2,2(n-p+1)}$. Before closing this section we shall derive the distribution of the characteristic roots $\delta_1^2, \delta_2^2, \cdots, \delta_{p_1}^2$ (if $p_1\leqq p_2$) of $X_{12}X_{22}^{-1}\bar{X}_{12}'X_{11}^{-1}$ by virture of which one has immediately the distribution of complex multiple correlation coefficient. Putting $Y=X^{-1}=\begin{array}{cc|c} p_1 & p_2 & \\ Y_{11} & Y_{12} & p_1 \\ \bar{Y}_{12}' & Y_{22} & p_2 \end{array}$, then it is easy to see that $Y_{11}^{-1}Y_{12}\cdot Y_{22}^{-1}\bar{Y}_{12}'=X_{12}X_{22}^{-1}\bar{X}_{12}'X_{11}^{-1}$. From Theorem 6.1, $Y\sim W_c(p, n; V^{-1})$, and hence, following James [10], we have for the distribution of the characteristic roots $\delta_1^2, \delta_2^2, \cdots, \delta_{p_1}^2$ (real numbers):

$$P(\delta_1^2, \delta_2^2, \cdots, \delta_{p_1}^2)=\frac{\Gamma_{p_1}(n)}{\Gamma_{p_1}(n-p_2)\Gamma_{p_1}(p_2)\Gamma_{p_1}(p_1)}\pi^{p_1(p_1-1)}|I_{p_1}-P^2|^n$$
$$\cdot |R^2|^{p_2-p_1}|I-R^2|^{n-p_2-p_1}\prod_{i<j}(\delta_i^2-\delta_j^2)^2$$
$$\cdot {}_2\tilde{F}_1(n, n; p_2, P^2, R^2)\,,$$

where $R=\text{diag}\,(\delta_1, \delta_2, \cdots, \delta_{p_1})$, $P=\text{diag}\,(\rho_1, \rho_2, \cdots, \rho_{p_1})$ with $\rho_1^2, \rho_2^2, \cdots, \rho_{p_1}^2$ being the characteristic root of $V_{11}^{-1}V_{12}V_{22}^{-1}\bar{V}_{12}'$, and ${}_2\tilde{F}_1(n, n; p_2; P^2, R^2)$ is the hypergeometric functions (in terms of Zonal polynomials) as defined in James ([10], p. 488). In this paper we shall use the notation $(\delta_1^2, \delta_2^2, \cdots, \delta_{p_1}^2)\sim h(\rho_1^2, \rho_2^2, \cdots, \rho_{p_1}^2;\ p_1, p_2, n)$.

If $p_1=1$, then ${}_2\tilde{F}_1(n, n; p_2, P^2, R^2)=F(n, n; p-1, \delta_1^2\rho_1^2)$ so that

$$P(\delta_1^2)=\frac{\Gamma(n)}{\Gamma(n-p+1)\Gamma(p-1)}(1-\rho_1^2)^n(\delta_1^2)^{p-2}(1-\delta_1^2)^{n-p}F(n, n; p-1, \delta_1^2\rho_1^2)\,,$$

where $F(n, n; p-1, \delta_1^2\rho_1^2)$ is the ordinary hypergeometric series.

7. Posterior distributions of Σ and some functions of elements of Σ

In Section 3 we obtained the posterior distribution of Σ as $\Sigma\sim W_{\text{I},c}(p, kn-q; S)$. Utilizing this result together with those given in the previous section we obtain in this section the posterior distributions of some interesting functions of Σ. The H.P.D. regions for these functions are then derived and utilized to make inferences about them.

7.1. Let σ_{ij} be the (i, j)th element of Σ. Then σ_{jj}, $j=1, 2, \cdots, p$, are real, and, from Theorem 6.2, $(2s_{jj}/\sigma_{jj}) \sim \chi^2_{2(kn-q-p+1)}$, where s_{ij} is the (i, j)th element of S.

Similarly, $\sigma_{11\cdot 23\cdot\cdot p}$ and $s_{11\cdot 23\cdot\cdot p}$ are real and, from Theorem 6.2, $(2s_{11\cdot 23\cdot\cdot p}/\sigma_{11\cdot 23\cdot\cdot p}) \sim \chi^2_{2(kn-q)}$.

In the sampling theory framework, it can readily be shown that $S \sim W_c(p, kn-q; \Sigma)$ so that $(2s_{jj}/\sigma_{jj}) \sim \chi^2_{2(kn-q)}$, $j=1, 2, \cdots, p$ and $(2s_{11\cdot 23\cdot\cdot p}/\sigma_{11\cdot 23\cdot\cdot p}) \sim \chi^2_{2(kn-q-p+1)}$. This shows that the sampling theory results are in the reverse order of the Bayesian results. We have for the respective H.P.D. regions of content $1-\alpha$ for σ_{jj} and $\sigma_{11\cdot 23\cdot\cdot p}$:

$$I_{\sigma_{jj}} = \left(\frac{2s_{jj}}{\chi^2_{1, 2(kn-q-p+1)}}, \frac{2s_{jj}}{\chi^2_{2, 2(kn-q-p+1)}} \right) \quad \text{and}$$

$$I_{\sigma_{11\cdot 23\cdot\cdot p}} = \left(\frac{2s_{11\cdot 23\cdot\cdot p}}{\chi^2_{1, 2(kn-q)}}, \frac{2s_{11\cdot 23\cdot\cdot p}}{\chi^2_{2, 2(kn-q)}} \right),$$

where $\chi^2_{1,m}$ and $\chi^2_{2,m}$ are constants satisfying

(a) $Pr\{\chi^2_m \leqq \chi^2_{2,m}\} + Pr\{\chi^2_m \geqq \chi^2_{1,m}\} = \alpha$ and

(b) $(\chi^2_{1,m})^{m/2-1} \exp(-\chi^2_{1,m}/2) = (\chi^2_{2,m})^{m/2-1} \exp(-\chi^2_{2,m}/2)$.

7.2. From Theorem 6.2 and Theorem 6.3 we have for the respective posterior distributions of the complex correlation coefficient $\rho_{ij} = \sigma_{ij}/\sqrt{\sigma_{ii}\sigma_{jj}}$, $(i \neq j,\ i, j=1, 2, \cdots, p)$ and the complex partial correlation coefficient $\rho_{ht\cdot 12\cdot\cdot p_1} = \sigma_{ht\cdot 12\cdot\cdot p_1}/\sqrt{\sigma_{hh\cdot 12\cdot\cdot p_1}\sigma_{tt\cdot 12\cdot\cdot p_1}}$, $(h \neq t,\ h, t=p_1+1, \cdots, p)$: $\rho_{ij} | Y \sim h_c(r_{ij}; kn-q-p+2)$ and $\rho_{ht\cdot 12\cdot\cdot p_1} | Y \sim h_c(r_{ht\cdot 12\cdot\cdot p_1}; kn-q-p_2+2)$, where $r_{ij} = s_{ij}/\sqrt{s_{ii}s_{jj}}$ with s_{ij} being the (i, j)th element of S and $r_{ht\cdot 12\cdot\cdot p_1} = s_{ht\cdot 12\cdot\cdot p_1}/\sqrt{s_{hh\cdot 12\cdot\cdot p_1}s_{tt\cdot 12\cdot\cdot p_1}}$ with $s_{ht\cdot 12\cdot\cdot p_1}$ being the (h, t)th element of $S_{22\cdot 1} = S_{22} - \bar{S}'_{12}S_{11}^{-1}S_{12}$ with $S = \begin{pmatrix} S_{11} & S_{12} \\ \bar{S}'_{12} & S_{22} \end{pmatrix}$.

These posterior distributions are too complicated to be utilized to make inferences about ρ_{ij} and $\rho_{ht\cdot 12\cdot\cdot q_1}$ but asymptotic results can readily be obtained. We now proceed to derive the respective asymptotic H.P.D. regions for ρ_{ij} and $\rho_{ht\cdot 12\cdot\cdot p_1}$ and for such a purpose we shall need the following well-known lemma.

LEMMA. *Let* $\boldsymbol{x}_n : p \times 1$ *converge to* $\boldsymbol{\mu}$ *in probability and* $\boldsymbol{x}_n \sim N(\boldsymbol{\mu}, \Sigma)$ *asymptotically. Let* $f_i(\boldsymbol{x})$, $i=1, 2, \cdots, k$, *be* k *functionally independent functions in the p-dimensional Euclidean space which possess continuous first partial derivative* $\partial f_i/\partial x_j = g_{ij}$ *at the neighborhood of* $\boldsymbol{\mu}$ *and*$(g_{ij})_{\boldsymbol{\mu}} = g_{ij}(\boldsymbol{\mu}) \neq 0$ *for at least one* j *and all* i. *Then, by writing* $(\boldsymbol{f}(\boldsymbol{x}))' = (f_1(\boldsymbol{x}), \cdots, f_k(\boldsymbol{x}))$, $\boldsymbol{f}(\boldsymbol{x}_n) \sim N\{\boldsymbol{f}(\boldsymbol{\mu}), (g_{ij}(\boldsymbol{\mu}))\Sigma(g_{ij}(\boldsymbol{\mu}))'\}$ *asymptotically.*

Let now $\begin{pmatrix} X_{11} & X_{12} \\ \bar{X}_{12} & X_{22} \end{pmatrix} \sim W_c\left(2, N; \begin{pmatrix} 1 & \rho \\ \bar{\rho} & 1 \end{pmatrix}\right)$, where $X_{12}=X_{12(1)}+iX_{12(2)}$ and $\rho=\rho_1+i\rho_2$. Then, by adopting the same approach as in proving Theorem 6.3, it is easy to establish that

$$r_p=r_1+ir_2=\frac{X_{12}}{\sqrt{X_{11}X_{12}}}\sim h(\rho; N) .$$

Now, by making use of the characteristic function, one can readily show that asymptotically, $(1/\sqrt{N})(X_{11}, X_{22}, X_{12(1)}, X_{12(2)})$ is distributed as normal with expectation $(1, 1, \rho_1, \rho_2)$ and covariance matrix

$$\frac{1}{N}\begin{pmatrix} 1 & \rho_1^2+\rho_2^2 & \rho_1 & \rho_2 \\ \rho_1^2+\rho_2^2 & 1 & \rho_1 & \rho_2 \\ \rho_1 & \rho_1 & (1+\rho_1^2-\rho_2^2)/2 & \rho_1\rho_2 \\ \rho_2 & \rho_2 & \rho_1\rho_2 & (1-\rho_1^2+\rho_2^2)/2 \end{pmatrix} .$$

Thus, we obtain, by utilizing the lemma, $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \sim N\left\{\begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}, \frac{1}{2N}(1-\rho_1^2-\rho_2^2)\begin{pmatrix} 1-\rho_1^2 & -\rho_1\rho_2 \\ -\rho_1\rho_2 & 1-\rho_2^2 \end{pmatrix}\right\}$, asymptotically. Or, $\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} \rightarrow \begin{pmatrix} \rho_1 \\ \rho_2 \end{pmatrix}$ in probability, by using the bivariate Tchebychev's Inequality as given in Olkin and Pratt [14].

From this result it follows that the respective asymptotic H.P.D. regions of content $1-\alpha$ for $\rho_{ij}=\rho_{ij(1)}+i\rho_{ij(2)}$ and $\rho_{ht\cdot 12\cdot\cdot p_1}=\rho_{ht\cdot 1(1)}+i\rho_{ht\cdot 1(2)}$ are given by:

$$(\rho_{ij(1)}-r_{ij(1)}, \quad \rho_{ij(2)}-r_{ij(2)})\begin{pmatrix} 1-r_{ij(1)}^2, & -r_{ij(1)}r_{ij(2)} \\ -r_{ij(1)}r_{ij(2)}, & 1-r_{ij(2)}^2 \end{pmatrix}^{-1}\begin{pmatrix} \rho_{ij(1)}-r_{ij(1)} \\ \rho_{ij(2)}-r_{ij(2)} \end{pmatrix}$$
$$\leqq [\chi_{\alpha,2}^2(1-r_{ij(1)}^2-r_{ij(2)}^2)]/2(kn-q-p+2)$$

and

$$(\rho_{ht\cdot 1(1)}-r_{ht\cdot 1(1)}, \quad \rho_{ht\cdot 1(2)}-r_{ht\cdot 1(2)})$$
$$\cdot\begin{pmatrix} 1-r_{ht\cdot 1(1)}^2, & -r_{ht\cdot 1(1)}r_{ht\cdot 1(2)} \\ -r_{ht\cdot 1(1)}r_{ht\cdot 1(2)}, & 1-r_{ht\cdot 1(2)}^2 \end{pmatrix}^{-1}\begin{pmatrix} \rho_{ht\cdot 1(1)}-r_{ht\cdot 1(1)} \\ \rho_{ht\cdot 1(2)}-r_{ht\cdot 1(2)} \end{pmatrix}$$
$$\leqq [\chi_{\alpha,2}^2(1-r_{ht\cdot 1(1)}^2-r_{ht\cdot 1(2)}^2)]/(2(kn-q-p_2+2))$$

where $r_{ij}=r_{ij(1)}+ir_{ij(2)}$, $r_{ht\cdot 12\cdot\cdot p_1}=r_{ht\cdot 1(1)}+ir_{ht\cdot 1(2)}$, and $\chi_{\alpha,2}^2$ is the upper α point of χ_2^2.

In the sampling theory framework, if one adopts the approach as given for proving Theorem 6.3, one can readily show that

$$r_{ij}=\frac{s_{ij}}{\sqrt{s_{ii}s_{jj}}}\sim h(\rho_{ij}; kn-q) \qquad \text{and}$$

$$r_{ht\cdot 12\cdot\cdot p_1}=\frac{s_{ht\cdot 12\cdot\cdot p_1}}{\sqrt{s_{hh\cdot 12\cdot\cdot p_1}s_{tt\cdot 12\cdot\cdot p_1}}}\sim h(\rho_{ht\cdot 12\cdot\cdot p_1};\, kn-q-p_1)\ ,$$

where r_{ij} is the sample complex correlation coefficient between the ith and the jth variables and $r_{ht\cdot 12\cdot\cdot p_1}$ the sample complex partial correlation coefficient between the hth and the tth variables given the first p_1 variables.

Notice that r_{ij} is independent of p and $r_{ht\cdot 12\cdot\cdot p_1}$ independent of p_2, in contrast to the respective posterior distributions of ρ_{ij} and $\rho_{ht\cdot 12\cdot\cdot p_1}$.

7.3. Let Σ and S be partitioned as

$$\Sigma=\begin{array}{cc} p_1 & p_2 \\ \end{array}\!\!\begin{pmatrix}\Sigma_{11} & \Sigma_{12}\\ \bar{\Sigma}'_{12} & \Sigma_{22}\end{pmatrix}\!\begin{array}{c} p_1 \\ p_2 \end{array} \quad\text{and}\quad S=\begin{array}{cc} p_1 & p_2 \\ \end{array}\!\!\begin{pmatrix}S_{11} & S_{12}\\ \bar{S}'_{12} & S_{22}\end{pmatrix}\!\begin{array}{c} p_1 \\ p_2 \end{array} .$$

Then from Theorem 6.2, we have for the posterior distribution of the complex regression matrix $Z=\Sigma_{11}^{-1}\Sigma_{12}$:

$$Z\,|\,Y\sim GMt_c(p_1,\, p_2,\, kn-q+p_1;\, S_{11}^{-1}S_{12},\, S_{11}^{-1},\, S_{22\cdot 1})\ .$$

Thus, utilizing Theorem 4.1, we obtain the posterior distribution of the complex regression coefficient $\beta_{12}=\sigma_{12}/\sigma_{11}$ as

$$P(\beta_{12}\,|\,Y)=\frac{\Gamma(kn-q-p+3)}{\pi\Gamma(kn-q-p+2)}(s_{11}/[s_{22}(1-r_{12}\bar{r}_{12})]) \cdot\left(1+\frac{s_{11}}{s_{22}(1-r_{12}\bar{r}_{12})}(\beta_{12}-b_{12})(\overline{\beta_{12}-b_{12}})\right)^{-(kn-q-p+3)}\ ,$$

where s_{ij} is the $(i,\, j)$th element of S,

$$b_{12}=\frac{s_{12}}{s_{11}} \quad\text{and}\quad r_{12}=\frac{s_{22}}{\sqrt{s_{11}s_{22}}}=b_{12}\sqrt{\frac{s_{22}}{s_{11}}}\ .$$

Or, equivalently,

$$(kn-q-p+2)\,|\,(\beta_{12}-b_{12})\,|^2\frac{s_{11}}{s_{22}(1-r_{12}\bar{r}_{12})}\Bigg|\,Y\sim F_{2,2(kn-q-p+2)}\ .$$

Making use of the result given in Section 5, we obtain the respective H.P.D. regions of content $1-\alpha$ for Z and β_{12} as

$$|\,I_{p_1}+S_{11}(Z-S_{11}^{-1}S_{12})S_{22\cdot 1}^{-1}(\overline{Z-S_{11}^{-1}S_{12}})'\,|^{-1}\geqq U_{\alpha,p_1,p_2,kn-q+p_1-p_2}\ ,$$

and

$$|\,(\beta_{12}-b_{12})\,|^2\leqq\left\{\frac{s_{11}(kn-q-p+2)}{s_{22}(1-r_{12}\bar{r}_{12})}\right\}^{-1}F_{\alpha,2,2(kn-q-p+2)}\ ,$$

where $U_{\alpha,p_1,p_2,kn-q+p_1-p_2}$ is the lower α point of $U_{p_1,p_2,kn-q+p_1-p_2}$ and $F_{\alpha,2,2(kn-q-p+2)}$ the upper α point of $F_{2,2(kn-q-p+2)}$.

Consider now the partitions

$$\Sigma=\begin{array}{cccl} m_1 & m_2 & p-m_1-m_2 & \\ \Sigma^*_{11} & \Sigma^*_{12} & \Sigma^*_{13} & m_1 \\ \bar{\Sigma}^{*\prime}_{12} & \Sigma^*_{22} & \Sigma^*_{23} & m_2 \\ \bar{\Sigma}^{*\prime}_{13} & \bar{\Sigma}^{*\prime}_{23} & \Sigma^*_{33} & p-m_1-m_2 \end{array} \quad \text{and} \quad S=\begin{array}{cccl} m_1 & m_2 & p-m_1-m_2 & \\ S^*_{11} & S^*_{12} & S^*_{13} & m_1 \\ \bar{S}^{*\prime}_{12} & S^*_{22} & S^*_{23} & m_2 \\ \bar{S}^{*\prime}_{13} & \bar{S}^{*\prime}_{23} & S^*_{33} & p-m_1-m_2 \end{array} .$$

From Theorem 6.2, if we put

$$\begin{pmatrix} \Sigma_{11\cdot3} & \Sigma_{12\cdot3} \\ \bar{\Sigma}'_{12\cdot3} & \Sigma_{22\cdot3} \end{pmatrix}=\begin{pmatrix} \Sigma^*_{11} & \Sigma^*_{12} \\ \bar{\Sigma}^{*\prime}_{12} & \Sigma^*_{22} \end{pmatrix}-\begin{pmatrix} \Sigma^*_{13} \\ \Sigma^*_{23} \end{pmatrix}\Sigma^{*-1}_{33}(\bar{\Sigma}^{*\prime}_{13},\ \bar{\Sigma}^{*\prime}_{23}) ,$$

then

$$\begin{pmatrix} \Sigma_{11\cdot3} & \Sigma_{12\cdot3} \\ \bar{\Sigma}'_{12\cdot3} & \Sigma_{22\cdot3} \end{pmatrix} \Bigg| Y\sim W_{\mathrm{I},c}\left(m_1+m_2,\ kn-q;\ \begin{pmatrix} S_{11\cdot3} & S_{12\cdot3} \\ \bar{S}'_{12\cdot3} & S_{22\cdot3} \end{pmatrix}\right) ,$$

where

$$\begin{pmatrix} S_{11\cdot3} & S_{12\cdot3} \\ \bar{S}'_{12\cdot3} & S_{22\cdot3} \end{pmatrix}=\begin{pmatrix} S^*_{11} & S^*_{12} \\ \bar{S}^{*\prime}_{12} & S^*_{22} \end{pmatrix}-\begin{pmatrix} S^*_{13} \\ S^*_{23} \end{pmatrix}S^{*-1}_{33}(\bar{S}^{*\prime}_{13},\ \bar{S}^{*\prime}_{23}) .$$

Thus, we have for the posterior distributions of the complex partial regression matrix $Z_{\cdot3}=\Sigma^{-1}_{11\cdot3}\Sigma_{12\cdot3}$ and the complex partial regression coefficient $\beta_{12\cdot3}=\sigma_{12\cdot3}/\sigma_{11\cdot3}$:

$$Z_{\cdot3}\,|\,Y\sim GMt_c(m_1,\ m_2,\ kn-q+m_1;\ S^{-1}_{11\cdot3}S_{12\cdot3},\ S^{-1}_{11\cdot3},\ S_{22\cdot13})$$

and

$$P(\beta_{12\cdot3}\,|\,Y)=\frac{\Gamma(kn-q-m_1-m_2+3)}{\pi\Gamma(kn-q-m_1-m_2+2)}\left(\frac{s_{11\cdot3}}{s_{22\cdot3}(1-r_{12\cdot3}\bar{r}_{12\cdot3})}\right)$$
$$\cdot\left(1+\frac{s_{11\cdot3}}{s_{22\cdot3}(1-r_{12\cdot3}\bar{r}_{12\cdot3})}(\beta_{12\cdot3}-b_{12\cdot3})(\overline{\beta_{12\cdot3}-b_{12\cdot3}})\right)^{-(kn-q-m_1-m_2+3)} ,$$

where, $S_{22\cdot13}=S_{22\cdot3}-\bar{S}'_{12\cdot3}S^{-1}_{11\cdot3}S_{12\cdot3}$, $s_{ij\cdot3}$ is the $(i,\ j)$th element of $\begin{pmatrix} S_{11\cdot3} & S_{12\cdot3} \\ \bar{S}'_{12\cdot3} & S_{22\cdot3} \end{pmatrix}$, $b_{12\cdot3}=s_{12\cdot3}/s_{11\cdot3}$ and $r_{12\cdot3}=s_{12\cdot3}/\sqrt{s_{11\cdot3}s_{22\cdot3}}$ with $m_1=1$.

The H.P.D. regions of content $1-\alpha$ for $Z_{\cdot3}$ and $\beta_{12\cdot3}$ are given respectively by:

$$|I+S_{11\cdot3}(Z_{\cdot3}-S^{-1}_{11\cdot3}S_{12\cdot3})S^{-1}_{22\cdot13}(\overline{Z_{\cdot3}-S^{-1}_{11\cdot3}S_{12\cdot3}})'|^{-1}\geqq U_{\alpha,m_1,m_2,kn-q+m_1-m_2} ,$$

and

$$|\beta_{12\cdot3}-b_{12\cdot3}|^2\leqq\left\{\frac{s_{11\cdot3}(kn-q-m_1-m_2+2)}{s_{22\cdot3}(1-r_{12\cdot3}\bar{r}_{12\cdot3})}\right\}^{-1}F_{\alpha,2,kn-q-m_1-m_2+2} .$$

We now note that the above results differ from those derived by the sampling theory approach. In fact, by adopting the same procedures as in proving Theorem 6.2 we obtain the distributions of the sample complex regression matrix $S_{11}^{-1}S_{12}=W_{12}$ and the sample complex partial regression matrix $S_{11\cdot3}^{-1}S_{12\cdot3}=W_{12\cdot3}$ as

$$W_{12}=S_{11}^{-1}S_{12}\sim GMt_c(p_1,\ p_2,\ kn-q+p_2;\ \Sigma_{11}^{-1}\Sigma_{12},\ \Sigma_{11}^{-1},\ \Sigma_{22\cdot1})$$

and

$$W_{12\cdot3}=S_{11\cdot3}^{-1}S_{12\cdot3}\sim GMt_c(m_1,\ m_2,\ kn-q-p+m_1+m_2+m_2;\ \Sigma_{11\cdot3}^{-1}\Sigma_{12\cdot3},\ \Sigma_{11\cdot3}^{-1},\ \Sigma_{22\cdot13})\ .$$

Thus we have for the distributions of the sample complex regression coefficient $b_{12}=s_{12}/s_{11}$ and the sample complex partial regression coefficient $b_{12\cdot3}=s_{12\cdot3}/s_{11\cdot3}$:

$$P(b_{12})=\frac{\Gamma(kn-q+1)}{\pi\Gamma(kn-q)}\left(\frac{\sigma_{11}}{\sigma_{22}(1-\rho_{12}\bar{\rho}_{12})}\right)\cdot\left(1+\frac{\sigma_{11}}{\sigma_{22}(1-\rho_{12}\bar{\rho}_{12})}(b_{12}-\beta_{12})\overline{(b_{12}-\beta_{12})}\right)^{-(kn-q+1)}$$

and

$$P(b_{12\cdot3})=\frac{\Gamma(kn-q-p+m_1+m_2+1)}{\pi\Gamma(kn-q-p+m_1+m_2)}\left(\frac{\sigma_{11\cdot3}}{\sigma_{22\cdot3}(1-\rho_{12\cdot3}\bar{\rho}_{12\cdot3})}\right)\cdot\left(1+\frac{\sigma_{11\cdot3}}{\sigma_{22\cdot3}(1-\rho_{12\cdot3}\bar{\rho}_{12\cdot3})}(b_{12\cdot3}-\beta_{12\cdot3})\overline{(b_{12\cdot3}-\beta_{12\cdot3})}\right)^{-(kn-q-p+m_1+m_2+1)},\quad m_1=1\ .$$

The differences between the results from the Bayesian approach and from the sampling theory approach arise from the fact that the former uses the complex inverted Wishart density while the latter the complex Wishart density.

7.4. In closing this section we note that the posterior distribution of the characteristic roots $\rho_1^2,\ \rho_2^2,\cdots,\ \rho_{p_1}^2$ $(p_1\leqq p_2)$ of $\Sigma_{12}\Sigma_{22}^{-1}\bar{\Sigma}_{12}'\Sigma_{11}^{-1}$ is

$$(\rho_1^2,\ \rho_2^2,\cdots,\ \rho_{p_1}^2)\,|\,Y\sim h(d_1^2,\ d_2^2,\cdots,\ d_{p_1}^2;\ p_1,\ p_2,\ nk-q)\ ,$$

where $d_1^2,\ d_2^2,\cdots,\ d_{p_1}^2$ are the characteristic roots of $S_{12}S_{22}^{-1}\bar{S}_{12}'S_{11}^{-1}$. For the particular case $p_1=1$, the posterior distribution of the square of the multiple correlation coefficient ρ^2 between the first variable and the rest (which is also defined as the multiple coherence by Goodman [9]) is:

$$P(\rho^2\,|\,Y)=\frac{\Gamma(kn-q)}{\Gamma(p-1)\Gamma(kn-q-p+1)}(1-d^2)^{kn-q}(\rho^2)^{p-2}(1-\rho^2)^{kn-q-p}\cdot F(kn-q,\ nk-q;\ p-1,\ d^2\rho^2)\ ,$$

where d^2 is the square of the sample multiple correlation coefficient between the first variable and the rest.

Similarly, we have for the posterior distributions of the characteristic roots $\rho^2_{1\cdot 3}, \rho^2_{2\cdot 3}, \cdots, \rho^2_{m_1\cdot 3}$ $(m_1 \leqq m_2)$ of $\Sigma_{12\cdot 3}\Sigma^{-1}_{22\cdot 3}\bar{\Sigma}'_{12\cdot 3}\Sigma^{-1}_{11\cdot 3}$ and the square of the partial multiple correlation coefficient $\rho^2_{\cdot 3}$ between the first and the $(2, 3, \cdots, m_1+m_2)$ variables given the rest (the partial multiple coherence in Goodman's terminology):

$$(\rho^2_{1\cdot 3}, \rho^2_{2\cdot 3}, \cdots, \rho^2_{m_1\cdot 3}) \,|\, Y \sim h(d^2_{1\cdot 3}, d^2_{2\cdot 3}, \cdots, d^2_{m_1\cdot 3}; m_1, m_2, kn-q) ,$$

where $d^2_{1\cdot 3}, d^2_{2\cdot 3}, \cdots, d^2_{m_1\cdot 3}$ are the characteristic roots $(m_1 \leqq m_2)$ of $S_{12\cdot 3}S^{-1}_{22\cdot 3}\cdot \bar{S}'_{12\cdot 3}S^{-1}_{11\cdot 3}$, and

$$\begin{aligned} P(\rho^2_{\cdot 3} \,|\, Y) = & \frac{\Gamma(kn-q)}{\Gamma(m_1+m_2-1)\Gamma(kn-q-m_1-m_2+1)}(1-d^2_{\cdot 3})^{kn-q}(\rho^2_{\cdot 3})^{m_1+m_2-2} \\ & \cdot (1-\rho^2_{\cdot 3})^{kn-q-m_1-m_2} F(kn-q, kn-q, m_1+m_2-1, d^2_{\cdot 3}\rho^2_{\cdot 3}) , \end{aligned}$$

where $d^2_{\cdot 3}$ is the square of the sample partial multiple correlation coefficient between the first and the $(2, 3, \cdots, m_1)$ variables given the rest.

In the sampling theory framework, since $S \sim W_c(p, kn-q; \Sigma)$, we have:

$$(d^2_1, d^2_2, \cdots, d^2_{p_1}) \sim h(\rho^2_1, \rho^2_2, \cdots, \rho^2_{p_1}; p_1, p_2, kn-q)$$

and

$$\begin{aligned} P(d^2) = & \frac{\Gamma(kn-q)}{\Gamma(p-1)\Gamma(kn-q-p+1)}(1-\rho^2)^{kn-q}(d^2)^{p-2}(1-d^2)^{kn-q-p} \\ & \cdot F(kn-q, kn-q; p-1, d^2\rho^2) . \end{aligned}$$

These are in exactly the same form as the respective posterior distributions of $(\rho^2_1, \rho^2_2, \cdots, \rho^2_{p_1})$ and ρ^2, interchanging $(d^2_1, d^2_2, \cdots, d^2_{p_1})$ with $(\rho^2_1, \rho^2_2, \cdots, \rho^2_{p_1})$ and d^2 with ρ^2. These results do not hold between $(d^2_{1\cdot 3}, d^2_{2\cdot 3}, \cdots, d^2_{m_1\cdot 3})$ and $(\rho^2_{1\cdot 3}, \rho^2_{2\cdot 3}, \cdots, \rho^2_{m_1\cdot 3})$ and between $d^2_{\cdot 3}$ and $\rho^2_{\cdot 3}$, however. In fact we have

$$(d^2_{1\cdot 3}, d^2_{2\cdot 3}, \cdots, d^2_{m_1\cdot 3}) \sim h(\rho^2_{1\cdot 3}, \rho^2_{2\cdot 3}, \cdots, \rho^2_{m_1\cdot 3}; m_1, m_2, kn-q-p+m_1+m_2)$$

and

$$\begin{aligned} P(d^2_{\cdot 3}) = & \frac{\Gamma(kn-q-p+m_1+m_2)}{\Gamma(m_1+m_2-1)\Gamma(kn-q-p+1)}(1-\rho^2_{\cdot 3})^{kn-q-p+m_1+m_2}(d^2_{\cdot 3})^{m_1+m_2-2} \\ & \cdot (1-d^2_{\cdot 3})^{kn-q-p} F(kn-q-p+m_1+m_2, kn-q-p+m_1+m_2, \\ & \quad m_1+m_2-1, d^2_{\cdot 3}\rho^2_{\cdot 3}) . \end{aligned}$$

For deriving the H.P.D. regions of ρ^2 and $\rho^2_{\cdot 3}$ of any given content, we notice that if ρ^2 has density

$$\begin{aligned} P(\rho^2) = & \frac{\Gamma(N)}{\Gamma(p-1)\Gamma(N-p+1)}(1-d^2)^N(\rho^2)^{p-2}(1-\rho^2)^{N-p} \\ & \cdot F(N, N; p-1, d^2\rho^2) \end{aligned}$$

and if $d^2 \neq 0$, then, ρ^2 is asymptotically normal with

$$E(\rho^2) = d^2 + [(p-1)/(N-1)](1-d^2) - 2(N-p)d^2(1-d^2)/(N^2-1) + O(N^{-2})$$

and

$$\operatorname{Var}(\rho^2) = 4d^2(1-d^2)(N-p)^2/[(N^2-1)(N+3)] + O(N^{-2}) ,$$

(for derivation, see Kendall and Stuart, Vol. II, p. 341).

These facts may be used to approximate the respective H.P.D. regions for ρ^2 and $\rho^2_{\cdot 3}$ of any given content.

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REFERENCES

[1] Anderson, T. W. (1958). *An Introduction to Multivariate Statistical Analysis*, Wiley, New York.

[2] Box, G. E. P. and Tiao, George C. (1965). Multiparameter problems from a Bayesian point of view, *Ann. Math. Statist.*, **36**, 1468-1482.

[3] Cornish, E. A. (1954). The multivariate *t*-distribution associated with a set of normal sample deviates, *Australian Journal of Physics*, **7**, 531-542.

[4] Deemer, W. L. and Olkin, I (1951). The Jacobians of certain matrix transformations useful in multivariate analysis, *Biometrika*, **38**, 345-367. Based on lectures of P. L. Hsu at the University of North Carolina.

[5] Dickey, J. M. (1967). Matricvariate generalizations of the multivariate *t*-distribution and the inverted multivariate *t*-distribution, *Ann. Math. Statist.*, **38**, 511-518.

[6] Geisser, S. (1965). Bayesian estimation in multivariate analysis, *Ann. Math. Statist.*, **36**, 150-159.

[7] Geisser, S. and Cornfield, J. (1963). Posterior distributions for multivariate normal parameters, *J. Roy. Statist. Soc. Ser. B*, **25**, 368-376.

[8] Giri, N. (1965). On the complex analogues of T^2- and R^2-tests, *Ann. Math. Statist.*, **36**, 664-670.

[9] Goodman, N. R. (1963). Statistical analysis based on a certain multivariate complex Gaussian distribution (an introduction), *Ann. Math. Statist.*, **34**, 152-177.

[10] James, A. T. (1964). Distributions of matrix variates and latent roots derived from normal samples, *Ann. Math. Statist.*, **35**, 475-501.

[11] Jeffreys, H. (1961). *Theory of Probability* (3rd edition), Oxford, Clarendon Press.

[12] Khatri, C. G. (1965). Classical statistical analysis based on certain multivariate complex Gaussian distribution, *Ann. Math. Statist.*, **36**, 98-114.

[13] Lehmann, E. L. (1959). *Testing Statistical Hypotheses*, Wiley, New York.

[14] Olkin, I. and Pratt, J. W. (1958). A multivariate Tchebysheff inequality, *Ann. Math. Statist.*, **29**, 226-234.

[15] Saxena, A. K. (1966). On the complex analogue of T^2 for two populations, *J. Indian Statist. Assoc.*, **4**, 99-102.

[16] Tiao, George C. and Zellner, A. (1964). On the Bayesian estimation of multivariate regression, *J. Roy. Statist. Soc. Ser. B*, **26**, 277-285.