ON SOME SEQUENTIAL SIMULTANEOUS CONFIDENCE INTERVALS PROCEDURES*

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Summary

The purpose of this paper is two-fold: (i) to extend the simultaneous confidence intervals procedures (SCIP) of Healy [7] along the lines of Chow and Robbins [3] and (ii) to develop certain robust non-parametric SCIP based on the results of Sen [10] and Sen and Ghosh [11]; the allied efficiency results are also presented.

1. Introduction

The proof of the non-existence of a fixed-sample size procedure for obtaining a fixed-width confidence interval for the mean of a normal population with unknown variance and minimum confidence coefficient $1-\alpha$ $(0<\alpha<1)$ is due to Dantzig [4]. Stein [12] developed a two-stage procedure which meets the requirements. Healy [7] extended the Stein procedure to the k $(\geqq 2)$ sample problem dealing with simultaneous confidence intervals for (i) all the k means, (ii) all possible $\binom{k}{2}$ differences of the means, and (iii) linear functions of the means. These procedures suffer from the principal drawback of usually needing a larger sample size than one might require due to their failure of updating the estimate of σ^2. Later, Chow and Robbins [3] have modified the Stein procedure by a sequential one for which as the prescribed width of the confidence interval is made to converge to zero, the confidence coefficient approaches $1-\alpha$ ("asymptotic consistency"), and the ratio of the ASN to the corresponding sample size assuming σ to be known approaches to unity ("asymptotic efficiency"). Besides, normality of the underlying distribution, assumed by Stein [12] (also by Healy [7]), is not needed; it is enough that $\sigma^2<\infty$ for the unknown population. Our first objective is to extend the Healy procedures along the lines of [3].

Some extensions of the Chow-Robbins procedure based on a general

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class of robust nonparametric rank statistics (and without requiring the existence and finiteness of σ^2) have been considered, very recently, by the precent authors ([11]); we also refer to [5] dealing with the sign and the signed rank statistics. Our second objective is to make use of these results along with the fixed-sample size SCIP results of Sen [10] to provide robust competitors to the procedures referred to earlier. Finally, the allied asymptotic relative efficiency (A.R.E.) results are considered in the concluding section, and these provide good reasons to advocate the use of the robust procedures in many non-standard situations.

2. SCIP based on sample means and variances

Consider the mutually independent random variables $\{X_{ij},\ i=1, 2, \cdots, k;\ j=1, 2, \cdots\}$, where the X_{ij}, $j\geqq 1$, are iidrv (independent and identically distributed random variables) having a cdf (cumulative distribution function) $F_i(x)$ with mean θ_i (unknown), $i=1,\cdots,k$. We assume that these cdf's are homoscedastic i.e.,

$$\int_{-\infty}^{\infty}(x-\theta_i)^2 dF_i(x)=\sigma^2 \text{ (unknown)}, \qquad \text{for all } i=1,\cdots,k, \tag{2.1}$$

where $0<\sigma^2<\infty$.

Problem I. Given a positive d, based on a sample of size n (from each distribution), we want to find intervals $I_n^{(i)}$, $i=1,\cdots,k$, such that (i) the width of each $I_n^{(i)}$ is $\leqq 2d$ and (ii) $P\{\theta_i \in I_n^{(i)},\ i=1,\cdots,k\}\geqq 1-\alpha$ $(0<\alpha<1)$. Since $F_1,\cdots,F_k$ and σ^2 are unknown, no fixed-sample size procedure sounds feasible.

The following sequential procedure, framed along the lines of Chow and Robbins [3], is considered. For every $n\geqq 2$, define

$$\bar{X}_n^{(i)}=n^{-1}\sum_{j=1}^{n}X_{ij}, \quad i=1,\cdots,k; \qquad s_n^2=\frac{1}{k(n-1)}\sum_{i=1}^{k}\sum_{j=1}^{n}(X_{ij}-\bar{X}_n^{(i)})^2. \tag{2.2}$$

Thus, s_n^2 is the so called "pooled within sample mean square" at the nth stage of experiment. Define the stopping time $N=N(d)$ to be the first positive integer m ($\geqq n_0$, an initial sample size, $\geqq 2$) for which $m\geqq d^{-2}s_m^2 M_{k,\alpha}^2$, where $M_{k,\alpha}$ is the upper $100\alpha\%$ point of the distribution of the maximum modulus of k random normal deviates. Our proposed SCI for $\theta_1,\cdots,\theta_k$ are then

$$I_{N(d)}^{(i)}=\{\theta_i : \bar{X}_{N(d)}^{(i)}-d\leqq\theta_i\leqq\bar{X}_{N(d)}^{(i)}+d\}, \qquad i=1, 2,\cdots,k. \tag{2.3}$$

The following theorem depicts the properties of the proposed SCIP for small d.

THEOREM 2.1. *Under* (2.1), $N(d)$ *is a non-increasing function of* d, $N(d)$ *is finite almost surely* (*a.s.*), $\lim_{d\to 0} N(d)=\infty$ *a.s., and* $\lim_{d\to 0} EN(d) =\infty$;

$$\lim_{d\to 0} N(d)/\nu(d)=1 \ a.s. ; \tag{2.4}$$

$$\lim_{d\to 0} P\{\theta_i \in I^{(i)}_{N(d)},\ i=1,\cdots,k\}=1-\alpha ; \tag{2.5}$$

$$\lim_{d\to 0} \{EN(d)\}/\nu(d)=1 , \tag{2.6}$$

where

$$\nu(d)=d^{-2}\sigma^2 M^2_{k,\alpha} . \tag{2.7}$$

PROOF. It follows from definition that $N(d)$ is a non-increasing function of d (>0). Also,

$$P\{N(d)=\infty\}=\lim_{n\to\infty} P\{N(d)>n\}\leqq \lim_{n\to\infty} P\{n<s_n^2 M^2_{k,\alpha} d^{-2}\}=0 , \tag{2.8}$$

since $s_n^2\to\sigma^2$ ($<\infty$) a.s., as $n\to\infty$. Again, from definition, $\lim_{d\to 0} N(d)= \infty$, and using the Monotone Convergence Theorem, one gets now $\lim_{d\to 0} EN(d)=\infty$. Since $\lim_{n\to\infty} s_n^2/\sigma^2=1$ a.s., and,

$$M^2_{k,\alpha} s^2_{N(d)}\sigma^{-2}\leqq N(d)d^2\sigma^{-2}<M^2_{k,\alpha} s^2_{N(d)-1}\sigma^{-2}+d^2\sigma^{-2} , \tag{2.9}$$

the proof of (2.4) follows along the lines of Lemma 1 of Chow and Robbins [3], and hence, the details are omitted. To prove (2.5), let $n(d)=[d^{-2}\sigma^2 M^2_{k,\alpha}]+1$, where $[s]$ denotes the largest integer contained s. Then, $\lim_{d\to 0} n(d)=\infty$, and by the classical central limit theorem, $Y_{ni}= n^{1/2}(\bar{X}^{(i)}_n-\theta_i)/\sigma$, $i=1,\cdots,k$ are independent and asymptotically distributed according to the standard normal distribution. Hence, by the definition of $M_{k,\alpha}$ and $n(d)$ one gets

$$\lim_{d\to 0} P\{\bar{X}^{(i)}_{n(d)}-d\leqq\theta_i\leqq\bar{X}^{(i)}_{n(d)}+d,\ \text{for all } i=1,\cdots,k\}=1-\alpha . \tag{2.10}$$

Now, by (2.4), $N(d)/n(d)\to 1$ a.e. as $d\to 0$. Hence, to prove (2.5), it is sufficient to establish the "uniform continuity in probability" of the sequences $\{\bar{X}^{(i)}_n\}$, $i=1,2,\cdots,k$ with respect to $n^{-1/2}$. The above follows from the results of Anscombe [1]. Finally, (2.6) follows along the lines of Chow and Robbins [3], and, hence, the proof is omitted.

PROBLEM II. We want to determine $\binom{k}{2}$ intervals $I_{nii'}$, $1\leqq i\leqq i'\leqq k$, such that the width of each one is $\leqq 2d$, and $P\{\theta_i-\theta_{i'}\in I_{nii'}$, for all $1\leqq i<i'\leqq k\}\geqq 1-\alpha$. Here also no fixed sample size procedure is available, and the goal is achieved here sequentially when d is small.

Let $R_{k,\alpha}$ be the upper $100\alpha\%$ point of the distribution of the range of k standard normal deviates. Define $\bar{X}_n^{(i)}$, $i=1,\cdots,k$ and s_n^2 as in (2.2), and let

$$Z_{nii'}=\bar{X}_n^{(i)}-\bar{X}_n^{(i')}, \qquad 1\leqq i<i'\leqq k . \tag{2.11}$$

Define then a stopping variable $N=N(d)$ to be first positive integer m $(\geqq n_0)$ for which $m\geqq d^{-2}s_m^2R_{k,\alpha}^2$. Then the proposed SCI for $\theta_i-\theta_{i'}$, $1\leqq i<i'\leqq k$ are

$$I_{N(d)ii'}=\{\theta_i-\theta_{i'}:Z_{N(d)ii'}-d\leqq\theta_i-\theta_{i'}\leqq Z_{N(d)ii'}+d\} , \qquad 1\leqq i<i'\leqq k . \tag{2.12}$$

Define now

$$Z_n^{**}=\max_{1\leqq i<i'\leqq k}|Z_{nii'}-(\theta_i-\theta_{i'})|=\operatorname*{Range}_{1\leqq i\leqq k}(\bar{X}_n^{(i)}-\theta_i) . \tag{2.13}$$

Use the inequality [consequent on (2.13)]

$$|Z_n^{**}-Z_{n'}^{**}|\leqq 2\max_{1\leqq i\leqq k}|\bar{X}_n^{(i)}-\bar{X}_{n'}^{(i)}| , \tag{2.14}$$

and the uniform continuity in probability of $\{Z_n^{**}\}$ with respect to $n^{-1/2}$ follows quite readily from results of Anscombe [1]. Hence, a theorem quite analogous to Theorem 2.1 (with $M_{k,\alpha}$ being replaced by $R_{k,\alpha}$ in the definition of $\nu(d)$) can be proved quite easily. Thus (2.4) and (2.6) remain the same, while analogous to (2.5), we have,

$$\lim_{d\to 0}P\{\theta_i-\theta_{i'}\in I_{N(d)ii'},\ \text{for all } 1\leqq i<i'\leqq k\}=1-\alpha . \tag{2.15}$$

In fact, if $\boldsymbol{l}=(l_1,\cdots,l_k)'$, $\boldsymbol{\theta}=(\theta_1,\cdots,\theta_k)$, $\bar{\boldsymbol{X}}_n=(\bar{X}_n^{(1)},\cdots,\bar{X}_n^{(k)})'$, $\boldsymbol{1}_k=(1,\cdots,1)'$ is a k-component vector with all elements 1's, then one gets from (2.15),

$$\lim_{d\to 0}P\Big\{\boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)}-\frac{1}{2}d\sum_1^k|l_i|\leqq\boldsymbol{l}'\boldsymbol{\theta} \leqq\boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)}+\frac{1}{2}d\sum_1^k|l_i|,\ \text{for all } \boldsymbol{l}\perp\boldsymbol{1}_k\Big\}=1-\alpha . \tag{2.16}$$

This extension covers the entire class of contrasts among $\theta_1,\cdots,\theta_k$.

PROBLEM III. Here we are interested in the entire class of linear combinations of $\boldsymbol{\theta}$ i.e., in $\Phi=\{\phi=\boldsymbol{l}'\boldsymbol{\theta}:\boldsymbol{l}\neq\boldsymbol{0}\}$. We standardize $\boldsymbol{l}$ by $\boldsymbol{l}'\boldsymbol{l}=1$, and our problem is to determine n such that

$$P\{\boldsymbol{l}'\bar{\boldsymbol{X}}_n-d\leqq\boldsymbol{l}'\boldsymbol{\theta}\leqq\boldsymbol{l}'\bar{\boldsymbol{X}}_n+d,\ \text{for all } \boldsymbol{l};\ \boldsymbol{l}'\boldsymbol{l}=1\}\geqq 1-\alpha . \tag{2.17}$$

Here also a fixed-sample size procedure is not feasible, and we propose the following sequential procedure. Let $\chi_{k,\alpha}^2$ be the upper $100\alpha\%$ point of the chi-square distribution with k degrees of freedom. Then our stopping variate $N(d)$=first positive integer m ($\geqq n_0$) for which $m \geqq d^{-2}\chi^2_{k,\alpha}s^2_m$, where s^2_m, is defined by (2.2). The proposed SCI is then $\boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)} - d \leqq \boldsymbol{l}'\boldsymbol{\theta} \leqq \boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)} + d$, for all $\boldsymbol{l}$: $\boldsymbol{l}'\boldsymbol{l}=1$. Note that

$$(2.18) \qquad \sup_{\boldsymbol{l}:\, \boldsymbol{l}'\boldsymbol{l}=1} n[\boldsymbol{l}'(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})]^2/\sigma^2 = n[(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})'(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})]/\sigma^2 = \frac{n}{\sigma^2}\sum_{i=1}^{k}(\bar{X}_n^{(i)}-\theta_i)^2 \sim \chi^2_k\,, \qquad \text{as } n \to \infty\,.$$

Further, $\{\bar{\boldsymbol{X}}_n-\boldsymbol{\theta},\ n \geqq 1\}$, forms a reversed martingale sequence, and, hence, $(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})'(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})$ forms a reversed submartingale sequence. Thus, $\{(\bar{\boldsymbol{X}}_{n+[\delta n]}-\boldsymbol{\theta})'(\bar{\boldsymbol{X}}_{n+[\delta n]}-\boldsymbol{\theta}), \cdots, (\bar{\boldsymbol{X}}_{n-[\delta n]}-\boldsymbol{\theta})'(\bar{\boldsymbol{X}}_{n-[\delta n]}-\boldsymbol{\theta})\}$ forms a forward submartingale sequence on which the Hájek-Rènyi-Chow (cf. [2]) inequality leads to

$$(2.19) \qquad P\left\{\sup_{|n'-n|<\delta n}\left|\frac{n}{\sigma^2}\sum_{i=1}^{k}[(\bar{X}_{n'}^{(i)}-\theta_i)^2-(\bar{X}_n^{(i)}-\theta_i)^2]\right|>\varepsilon\right\} \leqq (2k\delta n^2)/\{(n+[\delta n])(n-[\delta n])\varepsilon\} < \delta'/\varepsilon < \eta\,,$$

by proper choice of δ' (>0). Hence, a theorem similar to Theorem 2.1 can be formulated with $M^2_{k,\alpha}$ replaced by $\chi^2_{k,\alpha}$ (in the definition of $\nu(d)$) and (2.5) being replaced by

$$(2.20) \qquad \lim_{d\to 0} P\{\boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)}-d \leqq \boldsymbol{l}'\boldsymbol{\theta} \leqq \boldsymbol{l}'\bar{\boldsymbol{X}}_{N(d)}+d, \text{ for all } \boldsymbol{l}: \boldsymbol{l}'\boldsymbol{l}=1\} = 1-\alpha\,.$$

If the following section we shall consider some robust competitors to the procedures considered here. Those procedures, being based on suitable rank statistics are less vulnerable to gross errors or outliers and are comparatively efficient for distributions with "heavy tails".

3. SCIP based on rank order statistics

In the development of the three procedures for the three problems considered in the earlier section, the almost sure convergence of s^2_n to σ^2 and the central limit theorem on $\sqrt{n}(\bar{\boldsymbol{X}}_n-\boldsymbol{\theta})$, along with its uniform continuity in probability, play the fundamental role. In the parallel procedures (to be considered below) its role is played by the nonparametric confidence bands and their strong convergence properties, as have been studied in detail in Sen and Ghosh [11].

Let $c(u)=1$ or 0 according as $u \geqq$ or <0, and let,

$$(3.1) \qquad R_{ni}^{(j)} = \sum_{i'=1}^{n} c(|X_{ij}|-|X_{i'j}|)\,, \qquad 1 \leqq i \leqq n,\ j=1,\cdots,k\,;$$

$$(3.2) \qquad T_{nj} = T_{nj}(\boldsymbol{X}_n^{(j)}) = n^{-1}\sum_{i=1}^{n} c(X_{ij}) J_n((n+1)^{-1}R_{ni}^{(j)})\,,$$

where $X_n^{(j)}=(X_{1j},\cdots,X_{nj})$, $j=1,\cdots,k$, and $\{J_n(u): 0<u<1\}$ is generated by a score function $J(u)$ $(0<u<1)$ in either of the following two ways:
(a) $J_n(u)=J((n+1)^{-1}i)$, $(i-1)/n<u\leqq i/n$, $1\leqq i\leqq n$;
(b) $J_n(u)=EJ(U_{ni})$, $(i-1)/n<u\leqq i/n$, $1\leqq i\leqq n$,
where $U_{n1}\leqq\cdots\leqq U_{nn}$ are the ordered random variables from a rectangular $(0,1)$ distribution. Also, we assume that $J(u)=\Psi^{-1}((1+u)/2)$, $0<u<1$, where $\Psi(x)$ is an absolutely continuous cdf symmetric about 0 (i.e., $\Psi(x)+\Psi(-x)=1$, for all x), and the tail of $\Psi(x)$ has an increasing failure rate (i.e., $-\log[1-\Psi(x)]$ is convex for all $x\geqq x_0\geqq 0$ where x_0 is some real value). Note that by definition, $J(u)$ is $\uparrow$ in u and $J(0)=0$; we refer to [11] for other properties of $J(u)$. We may remark that when Ψ is the standard normal cdf (or the uniform cdf over $(-1,1)$), the corresponding T_{nj} is the classical one-sample normal scores statistics (or the Wilcoxon signed rank statistic). Whereas we do not need the assumption that σ^2 exists and is finite, the following conditions on $F_j(x)$, $j=1,\cdots,k$, are imposed for Problems I and III:

(3.3) $$F_j(x)=F(x-\theta_j)\ , \qquad j=1,\cdots,k,\ F \text{ is symmetric about } 0\ ,$$

(3.4) $$f_0=\sup_x f(x)<\infty \qquad \text{and } f'(x) \text{ is bounded (a.a.x.)}$$

(3.5) $$\lim_{x\to\infty}\{f(x)J'[2F(x)-1]\} \qquad \text{is finite}\ .$$

For Problem II, the conditions are slightly different and less stringent. We introduce the following notations:

(3.6) $$\bar{J}_n=n^{-1}\sum_{i=1}^{n}J_n(i/(n+1))\ , \qquad A_n^2=n^{-1}\sum_{i=1}^{n}J_n^2(i/(n+1))\ ;$$

(3.7) $$\mu=\int_0^1 J(u)du \qquad \text{and} \qquad A^2=\int_0^1 J^2(u)du \quad (>0)\ .$$

PROBLEM I. When $\boldsymbol{\theta}=\mathbf{0}$, $T_{n1},\cdots,T_{nk}$ are iidrv having a distribution symmetric about $\bar{J}_n/2$, independent of the underlying cdf F. Thus, for every α: $0<\alpha<1$ and n $(\geqq 2)$, there exists a known α_n $(\alpha_n\to\alpha$ as $n\to\infty)$ and an $h_{n,\alpha}$, such that

(3.8) $$P\{\max_{1\leqq j\leqq k}|T_{nj}-\bar{J}_n/2|\leqq h_{n,\alpha}\,|\,\boldsymbol{\theta}=\mathbf{0}\}=1-\alpha_n\to 1-\alpha \qquad \text{as } n\to\infty\ .$$

Note that under $\boldsymbol{\theta}=\mathbf{0}$, $2n^{1/2}(T_{nj}-\bar{J}_n/2)/A_n$ converges in law to a standard normal distribution, while $A_n\to A$ as $n\to\infty$. Hence,

(3.9) $$n^{1/2}h_{n,\alpha}\to AM_{k,\alpha}/2 \qquad \text{as } n\to\infty\ .$$

Denote by $\mathbf{1}_n=(1,\cdots,1)$ an n-component vector with all elements 1. Then $T_{nj}(X_n^{(j)}-a\mathbf{1}_n)$ is $\downarrow$ in a: $-\infty<a<\infty$. Hence, defining for every j $(=1,\cdots,k)$,

$$\hat{\theta}^{(j)}_{L,n}=\sup\{a:\ T_{nj}(X^{(j)}_n-a\mathbf{1}_n)>\bar{J}_n/2+h_{n,\alpha}\}\ , \tag{3.10}$$

$$\hat{\theta}^{(j)}_{U,n}=\inf\{a:\ T_{nj}(X^{(j)}_n-a\mathbf{1}_n)<\bar{J}_n/2-h_{n,\alpha}\}\ ; \tag{3.11}$$

$$I^{(j)}_n=\{\theta_j:\ \hat{\theta}^{(j)}_{L,n}\leqq\theta_j\leqq\hat{\theta}^{(j)}_{U,n}\}\ ,\quad \hat{\delta}^{(j)}_n=(\hat{\theta}^{(j)}_{U,n}-\hat{\theta}^{(j)}_{L,n})\ ,\qquad j=1,\cdots,k\ ; \tag{3.12}$$

we get from (3.8), (3.10) and (3.11) that

$$P\{\theta_j\in I^{(j)}_n,\ j=1,\cdots,k\}=1-\alpha_n\to1-\alpha\ ,\qquad \text{as } n\to\infty\ . \tag{3.13}$$

It is natural to seek a sequential procedure which consists in defining a stopping variable $N^*(d)$ to be the first positive integer m ($\geqq n_0\geqq 2$) for which $\hat{\delta}^{(j)}_m\leqq 2d$ simultaneously for all $j=1,\cdots,k$, and then taking the desired SCI's as

$$I^{(j)}_{N^*(d)}=\{\theta_j:\ \hat{\theta}^{(j)}_{L,N^*(d)}\leqq\theta_j\leqq\hat{\theta}^{(j)}_{U,N^*(d)}\}\ ,\qquad j=1,\cdots,k\ . \tag{3.14}$$

Defining now

$$B(F)=\int_0^\infty (d/dx)J[2F(x)-1]dF(x)\ , \tag{3.15}$$

and proceeding as in Lemma 5.2 of Sen and Ghosh [11], it follows that

$$\lim_{n\to\infty} n^{1/2}\hat{\delta}^{(j)}_n=AM_{k,\alpha}/B(F)\ \text{a.s.}\ ,\qquad \text{for all } j=1,\cdots,k\ . \tag{3.16}$$

The uniform continuity in probability of the $\{n^{1/2}\hat{\delta}^{(j)}_n\}$ (for each j) follows on the same line as in Lemma 5.3 of [11], while the asymptotic (as $n\to\infty$) normality of $[n^{1/2}(\hat{\theta}^{(j)}_{U,n}-\theta_j)-(1/2)AM_{k,\alpha}/B(F),\ j=1,\cdots,k]$ follows along the lines of Lemma 5.4 of [11]. Hence, proceeding as in Section 5 of [11], we arrive at the following theorem.

THEOREM 3.1. *Under* (3.3)-(3.5), $N^*(d)$ *is a non-increasing function of* d, $N^*(d)$ *is finite a.s.,* $\lim_{d\to0}N^*(d)=\infty$ *a.s., and* $\lim_{d\to0}EN^*(d)=\infty$;

$$\lim_{d\to0}N^*(d)/\nu^*(d)=1\ \ a.s.\ , \tag{3.17}$$

$$\lim_{d\to0}P\{\theta_j\in I_{N^*(d)},\ j=1,\cdots,k\}=1-\alpha\ ; \tag{3.18}$$

$$\lim_{d\to0}\{EN^*(d)\}/\nu^*(d)=1\ , \tag{3.19}$$

where

$$\nu^*(d)=\{A^2M^2_{k,\alpha}\}/\{4d^2B^2(F)\}\ . \tag{3.20}$$

PROBLEM II. Here, we require (3.4), while in (3.3), the symmetry of F is not needed. In place of (3.5), we need

$$\lim_{x\to\pm\infty}|J'[F(x)]f(x)|\qquad \text{is finite}\ , \tag{3.21}$$

where $J(u)$: $0<u<1$, is defined by $J(u)=\Psi^{-1}(u)$, and the cdf $\Psi(x)$, defined on $(-\infty, \infty)$, need not be symmetric about 0.

We pool the $2n$ observations $(X_{ij}, X_{ij'}, i=1,\cdots,n)$ into a combined sample of size $2n$ (where $j\neq j'$), and let

$$R_{ni}^{(jj')}=\sum_{i'=1}^{n}[c(X_{ij}-X_{i'j})+c(X_{ij}-X_{i'j'})] , \tag{3.22}$$

be the ranks of the X_{ij} among the $(X_{ij}, X_{ij'})$, $i=1,\cdots,n$, $1\leqq j<j'\leqq k$. Define then

$$T_{njj'}=n^{-1}\sum_{i=1}^{n}J_{2n}(R_{ni}^{(jj')}/(2n+1)) , \qquad 1\leqq j<j'\leqq k , \tag{3.23}$$

where the scores $J_{2n}(i/(2n+1))$ are defined as in after (3.2). We shall find it convenient to write $T_{njj'}$ as

$$T_n(X_n^{(j)}, X_n^{(j')}) , \qquad 1\leqq j<j'\leqq k , \tag{3.24}$$

so that $T_n(X_n^{(j)}+a\mathbf{1}_n, X_n^{(j')})$ is $\uparrow$ in a: $-\infty<a<\infty$, for all $j\neq j'$. We define $\bar{J}_n$ and μ as in (3.6) and (3.7), and further, we let

$$(A_n^*)^2=A_n^2-\bar{J}_n^2 , \qquad (A^*)^2=A^2-\mu^2 ; \tag{3.25}$$

$$B^*(F)=\int_{-\infty}^{\infty}(d/dx)J(F(x))dF(x) . \tag{3.26}$$

Then, it follows from Sen [10] (particularly his Theorem 2.1) that

$$W_n=\max_{1\leqq j<j'\leqq k}[2n^{1/2}|T_n(X_n^{(j)}, X_n^{(j')})-\bar{J}_{2n}|/A_{2n}^*] \tag{3.27}$$

has, under $\boldsymbol{\theta}=\mathbf{0}$, a distribution independent of the underlying $F(x)$, and

$$P\{W_n\leqq W_{n,\alpha}|\boldsymbol{\theta}=\mathbf{0}\}=1-\alpha_n\sim 1-\alpha , \qquad \Rightarrow W_{n,\alpha}\rightarrow R_{k,\alpha} , \tag{3.28}$$

as $n\rightarrow\infty$. Thus, as in Sen [10], we define $\Delta_{jj'}=\theta_j-\theta_{j'}$, $1\leqq j<j'\leqq k$, and

$$\hat{\Delta}_{L,n}^{(jj')}=\inf\{a: T_n(X_n^{(j)}+a\mathbf{1}_n, X_n^{(j')})>\bar{J}_{2n}-A_{2n}^*W_{n,\alpha}/(2n^{1/2})\} , \tag{3.29}$$

$$\hat{\Delta}_{U,n}^{(jj')}=\sup\{a: T_n(X_n^{(j)}+a\mathbf{1}_n, X_n^{(j')})<\bar{J}_{2n}+A_{2n}^*W_{n,\alpha}/(2n^{1/2})\} ; \tag{3.30}$$

$$I_n^{(jj')}=\{\Delta_{jj'}: \hat{\Delta}_{L,n}^{(jj')}\leqq\Delta_{jj'}\leqq\hat{\Delta}_{U,n}^{(jj')}\} ; \tag{3.31}$$

$$\hat{\delta}_n^{(jj')}=(\hat{\Delta}_{U,n}^{(jj')}-\hat{\Delta}_{L,n}^{(jj')}) , \qquad \text{for all } 1\leqq j<j'\leqq k . \tag{3.32}$$

Then, for every n $(\geqq 2)$,

$$P\{\theta_j-\theta_{j'}\in I_n^{(jj')}, 1\leqq j<j'\leqq k\}=1-\alpha_n\sim 1-\alpha . \tag{3.33}$$

Hence, it is natural to consider the following sequential procedure which consists in defining a stopping variable $N^*(d)$ to be the first positive integer m $(\geqq n_0 \geqq 2)$ for which $\hat{\delta}_m^{(jj')} \leqq 2d$, simultaneously for all $1 \leqq j < j' \leqq k$, and then taking the desired SCI's as

$$I_{N^*(d)}^{(jj')} = \{\Delta_{jj'} : \hat{\Delta}_{L,N^*(d)}^{(jj')} \leqq \Delta_{jj'} \leqq \hat{\Delta}_{U,N^*(d)}^{(jj')}\} , \qquad 1 \leqq j < j' \leqq k . \tag{3.34}$$

The almost sure convergence and the uniform continuity in probability of the $\{n^{1/2}\hat{\delta}_n^{(jj')}\}$, $1 \leqq j < j' \leqq k$, follow from the results of Ghosh and Sen [6] where the general case of a single regression rank-statistic (containing the two-sample statistic as a particular case) is studied, and the asymptotic normality of $[n^{1/2}(\hat{\Delta}_{U,n}^{(jj')} - \Delta_{jj'}) - A^* R_{k,\alpha}/2B^*(F),\ 1 \leqq j < j' \leqq k]$ follows as a direct generalization of Theorem 1 of Sen [9]. Hence, we have a theorem very similar to Theorem 3.1, where we need replace $M_{k,\alpha}$ by $R_{k,\alpha}$, $B(F)$ by $B^*(F)$ and A by A^* in the definition of $\nu^*(d)$, and (3.18) by,

$$\lim_{d \to 0} P\{\theta_j - \theta_{j'} \in I_{N^*(d)}^{(jj')},\ \text{for all } 1 \leqq j < j' \leqq k\} = 1 - \alpha . \tag{3.35}$$

We define now (cf. [8] and [10])

$$\begin{aligned} \hat{\Delta}_{n,1}^{(jj')} &= \inf\{a : T_n(\mathbf{X}_n^{(j)} + a\mathbf{1}_n, \mathbf{X}_n^{(j')}) > \bar{J}_{2n}\} , \\ \hat{\Delta}_{n,2}^{(jj')} &= \sup\{a : T_n(\mathbf{X}_n^{(j)} + a\mathbf{1}_n, \mathbf{X}_n^{(j')}) < \bar{J}_{2n}\} ; \end{aligned} \tag{3.36}$$

$$\hat{\Delta}_n^{(jj')} = \frac{1}{2}(\hat{\Delta}_{n,1}^{(jj')} + \hat{\Delta}_{n,2}^{(jj')}) , \qquad \hat{\Delta}_{n\cdot}^{(j)} = k^{-1} \sum_{j'=1}^{k} \hat{\Delta}_n^{(jj')} , \qquad 1 \leqq j < j' \leqq k , \tag{3.37}$$

where $\hat{\Delta}_n^{(jj)} = \Delta_{jj} = 0$, $j = 1, \cdots, k$. Then, the compatible estimators of the $\Delta_{jj'}$ are the

$$Z_n^{(jj')} = \hat{\Delta}_{n\cdot}^{(j)} - \hat{\Delta}_{n\cdot}^{(j')} , \qquad 1 \leqq j < j' \leqq k . \tag{3.38}$$

We let $\hat{\mathbf{\Delta}}_{n\cdot} = (\hat{\Delta}_{n\cdot}^{(1)}, \cdots, \hat{\Delta}_{n\cdot}^{(k)})'$, and

$$H_{n,\alpha} = \max_{1 \leqq j < j' \leqq k} |Z_n^{(jj')} - \Delta_{jj'}| \qquad \text{subject to } \Delta_{jj'} \in I_n^{(jj')},\ 1 \leqq j < j' \leqq k . \tag{3.39}$$

Then, proceeding as in Sen [10], particularly, his (2.40)–(2.43), we have analogous to (3.35)

$$\begin{aligned} \lim_{\delta \to 0} P\Big\{ & \mathbf{l}'\hat{\mathbf{\Delta}}_{N^*(d)} - \frac{1}{2} H_{N^*(d),\alpha} \sum_1^k |l_i| \\ & < \mathbf{l}'\boldsymbol{\theta} < \mathbf{l}'\hat{\mathbf{\Delta}}_{N^*(d)} + \frac{1}{2} H_{N^*(d),\alpha} \sum_1^k |l_i|,\ \text{for all } \mathbf{l} \perp \mathbf{1} \Big\} = 1 - \alpha , \end{aligned} \tag{3.40}$$

where it can be easily shown that under our conditions

$$n^{1/2} H_{n,\alpha} \to A^* R_{k,\alpha}/B^*(F) \text{ a.s.} , \qquad \text{as } n \to \infty . \tag{3.41}$$

PROBLEM III. Here, we utilize (3.16) to estimate the unknown $B(F)$. The following pooled sample estimator is proposed:

$$\hat{B}_n = kAM_{k,\alpha} \Big/ \left\{ n^{1/2} \sum_{j=1}^{k} \hat{\delta}_n^{(j)} \right\} . \tag{3.42}$$

Also, we let

$$\begin{aligned} \hat{\theta}_{n,1}^{(j)} &= \sup \{a:\ T_{nj}(X_n^{(j)} - a\mathbf{1}_n) > \bar{J}_n/2\} , \\ \hat{\theta}_{n,2}^{(j)} &= \inf \{a:\ T_{nj}(X_n^{(j)} - a\mathbf{1}_n) < \bar{J}_n/2\} , \end{aligned} \tag{3.43}$$

$$\hat{\theta}_n^{(j)} = (\hat{\theta}_{n,1}^{(j)} + \hat{\theta}_{n,2}^{(j)})/2 , \qquad j=1,\cdots,k ;\ \hat{\boldsymbol{\theta}}_n = (\hat{\theta}_n^{(1)}, \cdots, \hat{\theta}_n^{(k)})' . \tag{3.44}$$

Then, noting that $\{\sqrt{n}(\hat{\theta}_n^{(j)} - \theta_j),\ j=1,\cdots,k\}$ are iidrv and are asymptotically normally distributed (cf. [8]) with zero mean and variance $A^2/4B^2(F)$, where $B(F)$ can be estimated (a.s.) by $\hat{B}_n$ in (3.42), and

$$\begin{aligned} \sup_{\boldsymbol{l}:\ \boldsymbol{l}'\boldsymbol{l}=1} & n[\boldsymbol{l}'(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta})]^2 (4\hat{B}_n^2/A^2) \\ &= (4n\hat{B}_n^2/A^2) \sum_{j=1}^{k} (\hat{\theta}^{(j)} - \theta_j)^2 \\ &= [\hat{B}_n^2/B^2(F)](4nB^2(F)/A^2) \sum_{j=1}^{k} (\hat{\theta}^{(j)} - \theta_j)^2 \sim \chi_k^2 , \end{aligned} \tag{3.45}$$

we proceed as in Problem III of Section 2 and consider the following sequential procedure. The stopping variable $N^*(d)$ is defined to be the first positive integer m ($\geqq n_0 \geqq 2$) for which $m \geqq \chi_{k,\alpha}^2 A^2/(4\hat{B}_m^2 d^2)$, and then the desired SCI is

$$\boldsymbol{l}'\hat{\boldsymbol{\theta}}_{N^*(d)} - d \leqq \boldsymbol{l}'\boldsymbol{\theta} \leqq \boldsymbol{l}'\hat{\boldsymbol{\theta}}_{N^*(d)} + d , \qquad \text{for all } \boldsymbol{l}:\ \boldsymbol{l}'\boldsymbol{l}=1 . \tag{3.46}$$

Again, the uniform continuity in probability of the $\{n^{1/2}(\hat{\theta}^{(j)} - \theta_j),\ j=1,\cdots,k\}$ follows from the results of Sen and Ghosh [11], and hence, by (3.42), we get a theorem parallel to Theorem 3.1, where in the definition of $\nu^*(d)$, $\chi_{k,\alpha}^2$ replaces $M_{k,\alpha}^2$ and (3.18) is replaced by

$$\lim_{d\to 0} P\{\boldsymbol{l}'\hat{\boldsymbol{\theta}}_{N^*(d)} - d \leqq \boldsymbol{l}'\boldsymbol{\theta} \leqq \boldsymbol{l}'\hat{\boldsymbol{\theta}}_{N^*(d)} + d,\ \text{for all } \boldsymbol{l}:\ \boldsymbol{l}'\boldsymbol{l}=1\} = 1-\alpha . \tag{3.47}$$

4. Asymptotic efficiency results

Suppose we have two SCIP (say, A and B) with the same prescribed bound and the same asymptotic coverage probability, and let $N_A(d)$ and $N_B(d)$ be the corresponding stopping variables. Then, the A.R.E. of the procedure A with respect to the procedure B is defined by

$$e_{A,B} = \lim_{d\to 0} [\{EN_B(d)\}/\{EN_A(d)\}] . \tag{4.1}$$

Let M stand for the SCIP based on the sample means and variances (considered in Section 2), and let R stand for the parallel procedures based on rank statistics (considered in Section 3). Then we have from

Theorems 2.1 and 3.1 that for the Problems I and III

$$e_{R,M}=4\sigma^2B^2(F)/A^2\,, \tag{4.2}$$

while for the Problem II

$$e_{R,M}=\sigma^2[B^*(F)/A^*]^2\,. \tag{4.3}$$

Since (4.2) agrees with the corresponding A.R.E. in the single sample case, treated in detail in [11] [see their (6.2)], we refer to Section 6 of [11] for details. (4.3) agrees with (4.2) when $J(u)$ is skew-symmetric [i.e., $J(u)+J(1-u)=\text{constant}$, $0<u<1$] and hence, for both the Wilcoxon scores and the normal scores statistics, the values are the same as under (4.2). Hence, here also, we refer to Section 6 of [11] for details.

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