ON A CLASS OF ASYMPTOTICALLY OPTIMAL NONPARAMETRIC TESTS FOR GROUPED DATA II

MALAY GHOSH

(Received Jan. 22, 1970)

Summary

In a previous paper [5], the author has proposed a class of asymptotically optimal (in the sense of Wald [11]) nonparametric tests for testing the hypothesis of no regression in a multiple linear regression model. In the present paper, we are interested in testing that the intercept in the multiple (linear) regression model is zero along with the absence of regression. A class of permutationally distribution-free tests has been proposed and their asymptotic optimality has been established. These results generalize analogus findings of Puri and Sen [9] for ungrouped data. As an important application, an alternative test to the problem of paired comparison has been considered.

1. Introduction

Consider a (double) sequence of independent and identically distributed random variables (iidrv) $\boldsymbol{X}_\nu=(X_{\nu 1},\cdots,X_{\nu\nu})'$, $\nu\geqq 1$, where $X_{\nu i}$ has a continuous distribution function (df) $F_{\nu i}(x)$ given by

$$F_{\nu i}(x)=F(\sigma^{-1}[x-\boldsymbol{\beta}^{*\prime}\boldsymbol{c}_{\nu i}])\ ,\qquad 1\leqq i\leqq \nu,\ \nu\geqq 1\ , \tag{1.1}$$

where $\boldsymbol{\beta}^*=(\beta_0,\beta_1,\cdots,\beta_p)'$ is the parameter of interest, σ (>0) is the nuisance parameter and $\boldsymbol{c}_{\nu i}=(c_{0\nu i},c_{1\nu i},\cdots,c_{p\nu i})'$, where $c_{0\nu i}=1$, $1\leqq i\leqq\nu$, $\nu\geqq 1$, are vectors of known constants. Let $c_{k\nu}^2=\sum_{i=1}^{\nu} c_{k\nu i}^2$, $c_{k\nu i}^*=c_{k\nu i}/c_{k\nu}$, $k=0,1,\cdots,p$, $\nu\geqq 1$. Note that $c_{0\nu i}^*=\nu^{-1/2}$, $1\leqq i\leqq\nu$, $\nu\geqq 1$. Assume,

$$\max_{1\leq k\leq p}\max_{1\leq i\leq \nu}|c_{k\nu i}^*|=o(1)\ . \tag{1.2}$$

Let

$$\boldsymbol{\Lambda}_{\nu 0}=\left(\left(\sum_{i=1}^{\nu} c_{k\nu i}^* c_{k'\nu i}^*\right)\right)\qquad\text{and}\qquad \boldsymbol{\Lambda}_0=((\lambda_{kk'}))\ , \tag{1.3}$$

Research supported by National Institutes of Health, Institute of General Medical Sciences GM 12868-05.

where $\boldsymbol{A}_{\nu 0}$ and $\boldsymbol{A}_0$ are square matrices of order $(p+1)$ assumed to be positive definite (p.d.). Also, it is assumed that $\boldsymbol{A}_{\nu 0}\rightarrow\boldsymbol{A}_0$ as $\nu\rightarrow\infty$. Further, let

(1.4) $$F(x)+F(-x)=1\ , \qquad \text{for all real } x\ ,$$

(1.5) $$f(x)=F'(x) \text{ exist and } A^2(F)=\int_{-\infty}^{\infty}(f'/f)^2 f dx<\infty\ .$$

Define

(1.6) $$\phi(u)=-f'(F^{-1}(u))/f(F^{-1}(u))\ , \qquad 0<u<1\ ;$$

(1.7) $$\psi(u)=-f'[F^{-1}((1+u)/2)]/f[F^{-1}((1+u)/2)]\ , \qquad 0<u<1\ .$$

Then,

(1.8) $$\int_0^1\phi^2(u)du=\int_0^1\psi^2(u)du=A^2(F)<\infty\ .$$

We want to test

(1.9) $$H_0:\boldsymbol{\beta}^*=\mathbf{0}\ ,$$

against the alternatives $\boldsymbol{\beta}^*\neq\mathbf{0}$. To have an idea where the model (1.1) and the hypothesis (1.9) are realistic, one may refer to Puri and Sen [9].

In our case, the $X_{\nu i}$ $(i=1,\cdots,\nu)$ are not observable. We have a finite set of class intervals.

(1.10) $$I_j : a_{j-1/2}\leqq x<a_{j+1/2} \qquad (j=-l,\ -l+1,\cdots,\ -1,0,1,\cdots,l-1,l)$$

[where $-\infty=a_{-l-1/2}<\cdots<a_{-1/2}<a_{1/2}<\cdots<a_{l+1/2}=\infty$ denote a finite set of ordered points on the real line with $a_{-j}=a_j$; $a_0=0$]. The observable stochastic vector is $\boldsymbol{X}_\nu^*=(X_{\nu 1}^*,\cdots,X_{\nu\nu}^*)'$, where

(1.11) $$X_{\nu i}^*=\sum_{-l}^{l}I_jZ_{\nu ij} \qquad (i=1,\cdots,\nu)$$

and

(1.12) $$Z_{\nu ij}=\begin{cases}1 & \text{if } X_{\nu i}\in I_j\\ 0 & \text{otherwise}\ ,\end{cases}$$

for all $i=1,2,\cdots,\nu$, $\nu\geqq1$ $(j=-l,\cdots,-1,0,1,\cdots,l)$.

A class of asymptotically optimal parametric tests is considered in Section 2, while a class of permutationally distribution-free tests is proposed in Section 3. In this Section, the asymptotic properties of such tests and the allied asymptotic relative efficiency (ARE) results are also studied. The application to the problem of paired comparisons is studied in Section 4.

It is also obvious that the one-sample location problem for grouped data follows as a particular case of our model, since, then, we need only set $\beta_1=\beta_2=\cdots=\beta_p=0$.

2. Asymptotically optimal parametric test

Define

$$F_j=F(a_{j-1/2}/\sigma)\ , \qquad j=-l,\cdots,-1,0,1,\cdots,l+1\ , \tag{2.1}$$

$$P_j=F_{j+1}-F_j\ , \qquad j=-l,\cdots,-1,0,1,\cdots,l\ , \tag{2.2}$$

$$F_j^*=F_j-F_{-j+1}=2F_j-1 \qquad (j=1,2,\cdots,l+1)\ , \tag{2.3}$$

so that, $P_j^*=F_{j+1}^*-F_j^*=2P_j$ $(j=0,1,\cdots,l)$.

$$\Delta_j=\begin{cases}\int_{F_j}^{F_{j+1}}\phi(u)du/P_j\ , & \text{if } P_j\neq 0 \qquad j=-l,\cdots,-1,0,1,\cdots,l\ .\\ 0\ , & \text{otherwise}\ ,\end{cases} \tag{2.4}$$

$$\Delta_j^*=\begin{cases}\int_{F_j^*}^{F_{j+1}^*}\phi(u)du/P_j^*\ , & \text{if } P_j^*\neq 0 \qquad j=1,2,\cdots,l\ .\\ 0\ , & \text{otherwise}\ ,\end{cases} \tag{2.5}$$

When $P_j\neq 0$, we can write,

$$\Delta_j=[f(a_{j-1/2}/\sigma)-f(a_{j+1/2}/\sigma)]/P_j\ , \qquad j=-l,\cdots,-1,0,1,\cdots,l\ . \tag{2.6}$$

It follows immediately that,

$$P_{-j}=P_j\ ,\quad \Delta_{-j}=\Delta_j \qquad (j=0,1,\cdots,l)\ . \tag{2.7}$$

So, $\Delta_0=0$. Also, if $P_j\neq 0$,

$$\begin{aligned}\Delta_j^*&=\left[2\int_{(1+F_j^*)/2}^{(1+F_{j+1}^*)/2}\phi(u)du\Big/2\int_{(1+F_j^*)/2}^{(1+F_{j+1}^*)/2}du\right]\\ &=\int_{F_j}^{F_{j+1}}\phi(u)du\Big/\int_{F_j}^{F_{j+1}}du=\Delta_j \qquad (j=1,2,\cdots,l)\ .\end{aligned} \tag{2.8}$$

Let,

$$A_0^2(F,\{I_j\})=\sum_{j=-l}^{l}\Delta_j^2P_j=2\sum_{j=1}^{l}\Delta_j^2P_j=2\sum_{j\in J_+}\Delta_j^2P_j=2\sum_{j\in J_+}\Delta_j^{*2}P_j\ , \tag{2.9}$$

where $J_+=\{j\in(1,2,\cdots,l)\,|\,P_j>0\}$. Hence,

$$0<A_0^2(F,\{I_j\})\leqq A^2(F)<\infty\ . \tag{2.10}$$

Assume,

$$\max_{j=0,1,\cdots,l} P_j<1 . \tag{2.11}$$

Consider the sequence of alternatives,

$$H_\nu : \beta_k=\beta_{k\nu}=\tau_k/C_{k\nu} \qquad (k=0, 1,\cdots, p) . \tag{2.12}$$

Let,

$$\gamma_k=\sigma^{-1}\tau_k \qquad (k=0, 1,\cdots, p) \text{ and let } \gamma^*=(\gamma_0,\cdots,\gamma_p)' , \tag{2.13}$$

$$h_{\nu i}=\sigma^{-1}\sum_{k=0}^{p} c^*_{k\nu i}\tau_k \qquad (i=1, 2,\cdots, \nu) . \tag{2.14}$$

Then the likelihood function of X^*_ν under H_ν is given by

$$\begin{aligned} L(X^*_\nu\,|\,\gamma^*)&=\prod_{i=1}^{\nu}\left\{\sum_{j=-l}^{l} Z_{\nu ij}P(X_{\nu i}\in I_j)\right\} \\ &=\prod_{i=1}^{\nu}\left\{\sum_{j=-l}^{l} Z_{\nu ij}(F[(a_{j+1}/\sigma)-h_{\nu i}]-F[(a_j/\sigma)-h_{\nu i}])\right\} \\ &=\prod_{i=1}^{\nu}\left\{\sum_{j=-l}^{l} Z_{\nu ij}P_j(1+h_{\nu i}\Delta_j+h_{\nu i}\theta_j g_j(h_{\nu i}))\right\} , \qquad 0<\theta_j<1 , \end{aligned} \tag{2.15}$$

where

$$g_j(h_{\nu i})=P_j^{-1}\int_0^{h_{\nu i}} \{f'[(a_{j+1}/\sigma)-\theta_j y]-f'[(a_j/\sigma)-\theta_j y]\}\,dy , \tag{2.16}$$

for $j\in J_+UJ_-$, where $J_-=\{j\in(-1, -2,\cdots, -l)\,|\,P_j>o\}$. Hence, $g_j(h_{\nu i})=o(1)$, uniformly in $j\in J_+UJ_-$, $i=1, 2,\cdots, \nu$. Let

$$\begin{aligned} T_{k\nu 0}&=[(\partial/\partial\gamma_k)\log L(X^*_\nu\,|\,\gamma)]_{\gamma=0} \\ &=\sum_{i=1}^{\nu} c^*_{k\nu i}\sum_{j=-l}^{l} Z_{\nu ij}\Delta_j \\ &=\sum_{i=1}^{\nu} c^*_{k\nu i}\sum_{j=1}^{l}(Z_{\nu ij}-Z_{\nu i(-j)})\Delta_j , \qquad k=0, 1,\cdots, p . \end{aligned} \tag{2.17}$$

Let

$$U_{\nu ij}=\begin{cases} 1 & \text{if } |X_{\nu i}|\in I_j \text{ i.e. } X_{\nu i}\in I_jUI_{-j} \\ 0 & \text{otherwise} \qquad (i=1,\cdots,\nu;\ j=1, 2,\cdots, l) . \end{cases} \tag{2.18}$$

Then,

$$Z_{\nu ij}-Z_{\nu i(-j)}=U_{\nu ij}\,\mathrm{sgn}\,X_{\nu i} , \qquad j=1, 2,\cdots, l,\ i=1, 2,\cdots, \nu , \tag{2.19}$$

where

$$\mathrm{sgn}\,X_{\nu i}=\begin{cases} 1 & \text{if } X_{\nu i}>0 \\ 0 & \text{if } X_{\nu i}=0 , \qquad i=1, 2,\cdots, \nu . \\ -1 & \text{if } X_{\nu i}<0 \end{cases} \tag{2.20}$$

Thus,

(2.21) $$T_{k\nu 0}=\sum_{i=1}^{\nu} c^*_{k\nu i}\left(\sum_{j=1}^{l} U_{\nu ij}\Delta_j\right)\operatorname{sgn} X_{\nu i}\ , \qquad k=0, 1,\cdots, p\ .$$

Let

(2.22) $$\boldsymbol{T}_{\nu 0}=(T_{0\nu 0},\cdots, T_{p\nu 0})'\ .$$

Next we state three lemmas the proofs of which are exactly the same as Lemmas 2.1–2.3 of [5].

LEMMA 2.1. *Under* H_0, *and* (1.1)–(1.5), $T_{\nu 0}$ *is asymptotically* $N_{p+1}(\mathbf{0}, \boldsymbol{\Lambda}_0 A_0^2(F, \{I_j\}))$.

LEMMA 2.2. *Under* H_0, *and* (1.1)–(1.5), $L_{0\nu}(X_\nu)$ *is asymptotically* $N_1(-1/2\delta_0^2A^2(F), \delta_0^2A^2(F))$, *where*, $\delta_0=\boldsymbol{\gamma}^{*\prime}\boldsymbol{\Lambda}_0\boldsymbol{\gamma}^*$.

LEMMA 2.3. *Under the sequence of alternatives* $\{H_\nu\}$, (1.1)–(1.5), $\boldsymbol{T}_{\nu 0}$ *is asymptotically* $N_{p+1}((\boldsymbol{\Lambda}_0\boldsymbol{\gamma}^*)A_0^2(F, \{I_j\}), \boldsymbol{\Lambda}_0 A_0^2(F, \{I_j\}))$.

If $\{P_\nu\}$ and $\{Q_\nu\}$ denote two sequences of probability measures corresponding to the sequences of null and alternative hypotheses respectively, Lemma 2.2 ensures the contiguity of Q_ν with respect to (wrt) P_ν. Define now,

(2.23) $$S_{\nu 0}=A_0^{-2}(F, \{I_j\})\boldsymbol{T}_{\nu 0}'\boldsymbol{\Lambda}_{\nu 0}^{-1}\boldsymbol{T}_{\nu 0}\ .$$

We describe a parametric test procedure based on the critical function $\phi_1(\boldsymbol{X}_\nu^*)$ for testing H_0 in (1.9) as follows:

(2.24) $$\phi_1(\boldsymbol{X}_\nu^*)=\begin{cases} 1\ , & \text{if } S_{\nu 0}>S_{\nu 0\varepsilon} \\ \delta_{\nu\varepsilon}\ , & \text{if } S_{\nu 0}=S_{\nu 0\varepsilon} \\ 0\ , & \text{if } S_{\nu 0}<S_{\nu 0\varepsilon}\ , \end{cases}$$

where $S_{\nu 0\varepsilon}$ and $\delta_{\nu\varepsilon}$ are chosen in such a way that $E[\phi_1(\boldsymbol{X}_\nu^*)\,|\,P_\nu]=\varepsilon$, $0<\varepsilon<1$, ε being the desired level of significance of the test. Under H_0, $S_{\nu 0}$ is distributed as χ_p^2 (central chi-square with p degrees of freedom), while under H_ν it is distributed as $\chi_p^2(\eta)$ (non-central chi-square with p degrees of freedom and non-centrality parameter η), where $\eta=(\boldsymbol{\gamma}'\boldsymbol{\Lambda}_0\boldsymbol{\gamma})A^2(F, \{I_j\})$.

3. A class of nonparametric tests

We introduce the following notations:

(3.1) $$\sum_{i=1}^{\nu} Z_{\nu ij}=\nu_j\ , \qquad (j=-l,\cdots, -1, 0, 1,\cdots, l)\ .$$

So,

$$\sum_{i=1}^{\nu} U_{\nu ij}=\nu_j+\nu_{-j} \qquad (j=1, 2, \cdots, l) . \tag{3.2}$$

Let

$$F_{\nu,-l}=0 , \quad F_{\nu,j+1}=\sum_{m=-l}^{j} \nu_m/\nu , \qquad j=-l,\cdots, -1, 0, 1,\cdots, l , \tag{3.3}$$

$$F^*_{\nu,j+1}=F_{\nu,j+1}-F_{\nu,-j}=\nu^{-1} \sum_{m=-j}^{j} \nu_m , \qquad j=0, 1, 2\cdots, l , \tag{3.4}$$

$$\hat{\Delta}^*_{\nu j}=\int_{F^*_{\nu,j}}^{F^*_{\nu,j+1}} \phi(u)du \Big/ \int_{F^*_{\nu,j}}^{F^*_{\nu,j+1}} du =\int_{(1+F^*_{\nu,j})/2}^{(1+F^*_{\nu,j+1})/2} \phi(u)du \Big/ \int_{(1+F^*_{\nu,j})/2}^{(1+F^*_{\nu,j+1})/2} du , \tag{3.5}$$

if $\nu_j+\nu_{-j}\neq 0$,

$$\hat{\Delta}^*_{\nu j}=0 , \qquad \text{otherwise } (j=1, 2,\cdots, l) .$$

$$U^*_{k\nu}=\sum_{i=1}^{\nu} c^*_{k\nu i}\Big(\sum_{j=1}^{l} U_{\nu ij}\hat{\Delta}^*_{\nu j}\Big) \operatorname{sgn} X_{\nu i} , \qquad k=0, 1,\cdots, p , \tag{3.6}$$

and $\boldsymbol{U}^*_\nu=(U^*_{0\nu},\cdots, U^*_{p\nu})'$.

We may note that in the case $X_{\nu i} \in I_0$, we have no information regarding the value of $\operatorname{sgn} X_{\nu i}$. This, however, does not effect the test procedure (based on the statistics $U^*_{k\nu}$, $k=0, 1,\cdots, p$), since, if $X_{\nu i} \in I_0$, $U_{\nu ij}=0$ for all $j=1, 2,\cdots, l$ and, as such, contribution of such terms to the calculation of $U^*_{k\nu}$ $(k=0, 1,\cdots, p)$ is zero.

Since, we are dealing with grouped data, even under H_0, the joint distribution of the $U^*_{k\nu}$'s will depend on the Δ^*_j's. But, using a permutation argument, we can get here a permutationally (conditionally) distribution-free test. Under H_0 in (1.9), $|X_{\nu i}|$, $i=1, 2,\cdots, \nu$ are iidrv and so are $X^*_{\nu i0}=\sum_{j=1}^{l} I'_j U_{\nu ij}$, $(I'_j=I_j U I_{-j},\ j=1,\cdots, l,\ i=1,\cdots, \nu)$. Defining now, $W_{\nu i}=\sum_{j=1}^{l} U_{\nu ij}\hat{\Delta}^*_{\nu j}$ $(i=1,\cdots, \nu)$, we see that $\boldsymbol{W}_\nu=(W_{\nu 1},\cdots, W_{\nu\nu})'$ has a joint distribution which remains invariant under any permutation of its ν arguments under H_0. Further, under H_0, $\operatorname{sgn} X_{\nu i}$ can assume two values $+1$ and -1 each with prob. 1/2 independently of the $W_{\nu i}$'s; $\operatorname{sgn} X_{\nu 1},\cdots, \operatorname{sgn} X_{\nu\nu}$ are independently distributed. Use the notations $X^*_{\nu 0}=(X^*_{\nu 10},\cdots, X^*_{\nu\nu 0})'$ and $\operatorname{sgn} \boldsymbol{X}_\nu=(\operatorname{sgn} X_{\nu 1},\cdots, \operatorname{sgn} X_{\nu\nu})'$.

If we now consider a finite group G_ν of transformations $\{g_\nu\}$ which maps the sample space onto itself, where a typical transformation g_ν is such that

$$g_\nu \boldsymbol{W}_\nu=((-1)^{m_1}W_{\nu i_1},\cdots, (-1)^{m_\nu}W_{\nu i_\nu}) , \qquad m_l=(0, 1) , \tag{3.7}$$

$l=1, 2,\cdots, \nu$ and $(i_1,\cdots, i_\nu)$ is any permutation of the integers $(1,\cdots, \nu)$, then, there is a set of $2^{\nu}\nu!$ transformations in G_{ν}. Under H_0, the conditional distribution over the set of $2^{\nu}\nu!$ realizations (generated by G_{ν}) is uniform, each having common conditional probability $(2^{\nu}\nu!)^{-1}$. We denote this conditional probability measure by $\mathcal{P}'_{\nu}$. We may note that under $\mathcal{P}'_{\nu}$, $F^{*}_{\nu,j}$'s and hence $\hat{\Delta}^{*}_{\nu j}$'s remain fixed. It is now easy to verify that

$$(3.8)\qquad E(U^{*}_{k\nu}\,|\,\mathcal{P}'_{\nu})=0 \qquad (k=0, 1, \cdots, p)\ ;$$

$$E(U^{*}_{k\nu}U^{*}_{k'\nu}\,|\,\mathcal{P}'_{\nu})=\left(\sum_{i=1}^{\nu} c^{*}_{k\nu i}c^{*}_{k'\nu i}\right)A_0^2(F^{*}_{\nu}, \{I_j\})\ , \qquad (k, k'=0, 1, \cdots, p)\ ,$$

where

$$(3.9)\qquad A_0^2(F^{*}_{\nu}, \{I_j\})=\sum_{j=1}^{l} \Delta^{*2}_{\nu j}(\nu_j+\nu_{-j})/\nu\ .$$

Consider the quadratic form,

$$(3.10)\qquad M^{*}_{\nu}=(\boldsymbol{U}^{*\prime}_{\nu}\boldsymbol{\Lambda}^{-1}_{\nu}\boldsymbol{U}^{*}_{\nu})A_0^{-2}(F^{*}_{\nu}, \{I_j\})\ .$$

The following nonparametric test is proposed:

$$(3.11)\qquad \phi_4(\boldsymbol{X}^{*}_{\nu 0}, \operatorname{sgn} \boldsymbol{X}_{\nu})=\begin{cases} 1\ , & \text{if } M^{*}_{\nu}>M_{\nu,\varepsilon} \\ \delta_{4\nu\varepsilon}\ , & \text{if } M^{*}_{\nu}=M_{\nu,\varepsilon}\ , \\ 0\ , & \text{if } M^{*}_{\nu}<M_{\nu,\varepsilon} \end{cases}$$

where $M_{\nu,\varepsilon}$ and $\delta_{4\nu\varepsilon}$ are chosen in such a way that $E[\phi_4(\boldsymbol{X}^{*}_{\nu 0}, \operatorname{sgn} \boldsymbol{X}_{\nu})\,|\,\mathcal{P}'_{\nu}]=\varepsilon$, the level of significance. This implies that $E[\phi_4(\boldsymbol{X}^{*}_{\nu 0}, \operatorname{sgn} \boldsymbol{X}_{\nu})\,|\,P_{\nu}]=\varepsilon$ i.e. $\phi_4(\boldsymbol{X}^{*}_{\nu 0}, \operatorname{sgn} \boldsymbol{X}_{\nu})$ is a similar size ε test.

We may remark that it is necessary to introduce the "signed-ranked" statistics instead of the "rank-sum type" statistics because here we are interested in testing for β_0 along with $\beta_1, \cdots, \beta_p$. Had permutation-test statistics as in [5] been introduced without consideration of signs, the one for β_0 would have reduced to a constant under the permutational probability measure $\mathcal{P}'_{\nu}$. A similar situation was faced by Adichie [1] and by Puri and Sen [9] in case of ungrouped data and they introduced "signed-rank" statistics instead of "rank"-statistics for similar testing problems.

To study the asymptotic permutation distribution of the test statistic, we first extend a result of Hájek involving the asymptotic distribution of the "simple linear rank statistic" to that of the "simple linear signed-rank statistic." To formulate the theorem, we proceed as follows:

Consider double sequences $\{b_{\nu i},\ 1\leqq i\leqq \nu,\ \nu\geqq 1\}$ and $\{a_{\nu i},\ 1\leqq i\leqq \nu,\ \nu\geqq 1\}$ of real numbers satisfying

$$\lim_{\nu\to\infty} \max_{1\leq i\leq \nu} b_{\nu i}^2 \Big/ \left(\sum_{i=1}^{\nu} b_{\nu i}^2\right) = 0 , \tag{3.13}$$

$$\lim_{\nu\to\infty} [k_\nu/\nu = 0] \Rightarrow \lim_{\nu\to\infty} \frac{\max\limits_{1\leq i_1<\cdots<i_{k_\nu}\leq \nu} \sum\limits_{\alpha=1}^{k_\nu} (a_{\nu i_\alpha} - \bar{a}_\nu)^2}{\sum\limits_{i=1}^{\nu} (a_{\nu i} - \bar{a}_\nu)^2} = 0 . \tag{3.14}$$

Let $X_{\nu i}, \cdots, X_{\nu\nu}$ be ν i.i.d.r.v. having a continuous d.f. $F(x)$ satisfying $F(x)+F(-x)=1$ for all real x; also let $(R_{\nu 1}, \cdots, R_{\nu\nu})$ be a random vector which takes $\nu!$ possible permutations of $(1, 2, \cdots, \nu)$ with equal probabilities and distributed independently of $(X_{\nu 1}, \cdots, X_{\nu\nu})$. Let

$$S_\nu = \sum_{i=1}^{\nu} b_{\nu i} a_{\nu R_{\nu i}} \operatorname{sgn} X_{\nu i} , \tag{3.15}$$

where $\operatorname{sgn} X_{\nu i}$'s are defined in (2.20).

THEOREM 3.1. *Under* (3.13)–(3.14) *and the assumptions following it* S_ν defined in (3.15) is asymptotically $N_1\left(0, \nu^{-1} \sum_{i=1}^{\nu} b_{\nu i}^2 \sum_{i=1}^{\nu} a_{\nu i}^2\right)$ *a.e.*

PROOF. It can be assumed without any loss of generality that $a_{\nu 1} \leqq a_{\nu 2} \leqq \cdots \leqq a_{\nu\nu}$ and put $a_\nu(\lambda) = a_{\nu i}$ for $(i-1)/\nu < \lambda \leqq i/\nu$, $1 \leqq i \leqq \nu$. Let $T_\nu = \sum_{i=1}^{\nu} b_{\nu i} a_\nu(U_i) \operatorname{sgn} X_{\nu i}$, where $U_1, U_2, \cdots$ is a sequence of independent random variables distributed independently of the $X_{\nu i}$'s, each U_i uniformly distributed over $(0, 1]$, and the rank of U_i in the partial sequense $U_1, \cdots, U_\nu$ is denoted by $R'_{\nu i}$ $(1 \leqq i \leqq \nu,\ \nu \geqq 1)$. Hence, the vector $(R'_{\nu i}, \cdots, R'_{\nu\nu})$ satisfies the same conditions as $(R_{\nu 1}, \cdots, R_{\nu\nu})$ in the theorem, and we get,

$$\begin{aligned} E(T_\nu - S_\nu)^2 &= \left(\sum_{i=1}^{\nu} b_{\nu i}^2\right) E[a_\nu(U_1) - a_\nu(R'_{\nu 1}/\nu)]^2 \\ &\leqq \left(\sum_{i=1}^{\nu} b_{\nu i}^2\right) 2\sqrt{2} \left(\nu^{-1/2} \max_{1\leq i\leq \nu} |a_{\nu i} - \bar{a}_\nu|\right) \left(\nu^{-1} \sum_{i=1}^{\nu} (a_{\nu i} - \bar{a}_\nu^2)\right)^{1/2} , \end{aligned} \tag{3.16}$$

using Lemma 2.1 of [6]. Also, $E(S_\nu)=0$ and

$$\operatorname{Var}(S_\nu) = \nu^{-1} \sum_{i=1}^{\nu} b_{\nu i}^2 \sum_{i=1}^{\nu} a_{\nu i}^2 .$$

Hence,

$$\begin{aligned} E(T_\nu - S_\nu)^2 / \operatorname{Var}(S_\nu) &\leqq 2\sqrt{2} \left[\max_{1\leq i\leq \nu} |a_{\nu i} - \bar{a}_\nu| \Big/ \left(\sum_{1}^{\nu} a_{\nu i}^2\right)^{1/2}\right] \\ &\leqq 2\sqrt{2} \left[\max_{1\leq i\leq \nu} |a_{\nu i} - \bar{a}_\nu| \Big/ \left(\sum_{1}^{\nu} (a_{\nu i} - \bar{a}_\nu)^2\right)^{1/2}\right] \end{aligned} \tag{3.17}$$

$\to 0$ as $\nu \to \infty$ from (3.14) putting $k_\nu = 1$. To prove the theorem, it is sufficient to show that (i) $(T_\nu - E(T_\nu))/\sqrt{\operatorname{Var}(T_\nu)}$ is asymptotically $N_1(0,$

1) a.e. and (ii) $(S_\nu - E(S_\nu))/\sqrt{\mathrm{Var}(S_\nu)} - (T_\nu - E(T_\nu))/\sqrt{\mathrm{Var}(T_\nu)} \to 0$ in probability. The proof of (ii) which is based on (3.17) is given in Hájek [6].

To prove (i), we use Theorems 4.1 and 4.2 of Hájek [6]. We need observe only that the Lindeberg condition

$$\lim_{\nu\to\infty} \frac{1}{\mathrm{Var}(T_\nu)} \sum_{i=1}^{\nu} E[T_{\nu i}^2 I_{(|T_{\nu i}|\geqq\delta)}]=0 \qquad \text{a.e.}\,,$$

where $T_{\nu i}=(b_{\nu i}-\bar{b}_\nu)a_\nu(U_i)\operatorname{sgn} X_{\nu i}$ $(i=1, 2, \cdots, \nu)$, and

$$I_{(|T_{\nu i}|\geqq\delta)}=\begin{cases} 1 & \text{if } |T_{\nu i}|\geqq\delta \\ 0 & \text{otherwise}\,, \end{cases}$$

$1\leqq i\leqq\nu$, is equivalent to the condition

$$\lim_{\nu\to\infty} \frac{1}{\mathrm{Var}(T'_\nu)} \sum_{i=1}^{\nu} E[T'^2_{\nu i} I_{(|T'_{\nu i}|\geqq\delta)}]=0 \qquad \text{a.e.}\,,$$

where $T'_{\nu i}=\sum_{i=1}^{\nu}(b_{\nu i}-\bar{b}_\nu)a_\nu(U_i)$, $T'_\nu=\sum_{i=1}^{\nu}T'_{\nu i}$ and

$$I_{(|T'_{\nu i}|\geqq\delta)}=\begin{cases} 1 & \text{if } |T'_{\nu i}|\geqq\delta \\ 0 & \text{otherwise}\,. \end{cases}$$

The last assertion follows from the fact $|\operatorname{sgn} X_{\nu i}|=1$ with probability 1 for all $i=1, 2,\cdots, \nu$ and $\operatorname{sgn} X_{\nu 1},\cdots, \operatorname{sgn} X_{\nu\nu}, U_1,\cdots, U_\nu$ are mutually independent.

The above theorem will be utilized in deriving the asymptotic permutation distribution of $\boldsymbol{U}_\nu^*$. In order to prove the result, the following lemma is also needed.

LEMMA 3.1. *Under the assumptions* (1.1)-(1.5), $A_0^2(F_\nu^*, \{I_j\})\to A_0^2(F, \{I_j\})$ *as* $\nu\to\infty$ *either in* P_ν- (*or in* Q_ν-) *probability.*

PROOF. $A_0^2(F_\nu^*, \{I_j\})=\sum_{j=1}^{l}\hat{\Delta}_{\nu j}^{*2}(\nu_j+\nu_{-j})/\nu$. Noting that $(\nu_j+\nu_{-j})/\nu\to 2P_j$ in probability as $\nu\to\infty$, we get exactly as in Lemma 3.1 of Sen [10], $A_0^2(F_\nu^*, \{I_j\})\to\sum_{j=1}^{l}\Delta_j^{*2}(2P_j)=\sum_{j=1}^{l}\Delta_j^2(2P_j)=A_0^2(F, \{I_j\})$ in P_ν- (or in Q_ν-) probability as $\nu\to\infty$.

THEOREM 3.2. *Under* $\mathcal{P}'_\nu$, (1.1)-(1.5), $\boldsymbol{U}_\nu^*$ *is asymptotically* $N_{p+1}(\boldsymbol{0}, \boldsymbol{\Lambda}_0 A_0^2(F, \{I_j\}))$ *in probability.*

PROOF. Consider any linear combination $\boldsymbol{d}'\boldsymbol{U}_\nu^*$, $\boldsymbol{d}=(d_0,\cdots, d_p)'\neq\boldsymbol{0}$, $d_0, d_1,\cdots, d_p$ are fixed and finite.

$$\boldsymbol{d}'\boldsymbol{U}_\nu^*=\sum_{i=1}^{\nu}\left(\sum_{k=0}^{p}d_k c_{k\nu i}^*\right)\left(\sum_{j=1}^{l}U_{\nu ij}\hat{\Delta}_{\nu j}^*\right)\operatorname{sgn} X_{\nu i}=\sum_{i=1}^{\nu}m_{\nu i}W_{\nu i}\operatorname{sgn} X_{\nu i}\,,$$

where $m_{\nu i}=\sum_{k=0}^{p} d_k c_{k\nu i}^*$, $i=1, 2, \cdots, \nu$. Now, $\max_{1\leq i\leq\nu}|m_{\nu i}|=o(1)$ and $\sum_{i=1}^{\nu} m_{\nu i}^2=O(1)$, (because of the assumptions (1.2) and (1.3)). So the sequence $\{m_{\nu i}\}$ satisfies the Noether condition. Further, $\nu^{-1}\sum_{i=1}^{\nu} W_{\nu i}^2 = A_0^2(F_\nu^*, \{I_j\})\rightarrow A_0^2(F, \{I_j\})$ in probability. It can also be shown as in [5] that $\max_{1\leq i_1\leq\cdots\leq i_{k_\nu}\leq\nu} \nu^{-1}\sum_{\alpha=1}^{k_\nu} W_{\nu i_\alpha}^2\rightarrow 0$ in $\mathcal{P}_\nu'$-probability. So the sequence $\{W_{\nu i}\}$ satisfies Hájek's condition Q in probability (see [6], p. 519). Now, we refer to Theorem 3.1 to get the result.

LEMMA 3.3. *Under* (1.1)–(1.6) *and* (1.2) $\boldsymbol{U}_\nu^*-\boldsymbol{T}_{\nu 0}\rightarrow \boldsymbol{0}$ *in* P_ν- (*or in* Q_ν-) *probability.*

PROOF. Write

$$U_{k\nu}^* - T_{k\nu 0}=\sum_{j=1}^{l}(\hat{\Delta}_{\nu j}^*-\Delta_j^*)\sum_{i=1}^{\nu} c_{k\nu i}^* U_{\nu ij}\operatorname{sgn} X_{\nu i}\ , \qquad k=0, 1, \cdots, p\ . \tag{3.18}$$

The rest of the proof is the same as in Lemma 3.3 of [5] and, hence, is omitted.

From the above results, it is now obvious that M_ν^* and $S_{\nu 00}$ have asymptotically the same distribution under H_0 or $\{H_\nu\}$. It is immediate on the basis of Lemmas 2.1–2.3 that M_ν^* (or $S_{\nu 00}$) is asymptotically χ_{p+1}^2 under H_0 and $\chi_{p+1}^2(\zeta_0)$ under $\{H_\nu\}$, where,

$$\zeta_0=(\boldsymbol{\gamma}^{*\prime}\boldsymbol{\Lambda}_0\boldsymbol{\gamma}^*)A_0^2(F, \{I_j\})\ . \tag{3.19}$$

The extension of the above results to the case of a countable set of class-intervals is similar as in [5], since the sequence $\{Q_\nu\}$ of probability measures is contiguous to the sequence $\{P_\nu\}$ of probability measures from Lemma 2.2. Again, in the situation where the assumed df F differs from the true df G,since the asymptotically optimal parametric test is similar as in [5] (involving only the additional statistic corresponding to β_0), the ARE of the proposed test as compared to the asymptotically optimal test is $\rho^2(F, G, \{I_j\})$, where $\rho^2(F, G, \{I_j\})$ is defined quite analogous to the corresponding one in [5] (see also [10]).

4. An important application

The test procedure developed in earlier sections can be used in the particular case of ‘paired comparison’ problem. The problem is as follows:

Consider p treatments in a sequence of experiments, the νth sequence yielding paired observations $(Y_{\nu km}, Y_{\nu k'm})$, $m=1, 2, \cdots, n_{\nu kk'}$, $1\leq k<k'\leq p$. $\nu=\sum\sum_{1\leq k<k'\leq p} n_{\nu kk'}$. Assume that the $n_{\nu kk'}$ observations $Y_{\nu m}^{(k,k')}=Y_{\nu km}-Y_{\nu k'm}$ $(m=1, 2, \cdots, n_{\nu kk'})$ have a common continuous d.f. $F_{\nu kk'}(x)$.

This is the situation, for example if in the analysis of an incomplete block experiment, with each block of size two, one makes the assumption of additivity as in the usual analysis of variance model. We want to test the hypothesis

$$H_0 : F_{\nu kk'}(x)+F_{\nu kk'}(-x)=1 \quad \text{and} \quad F_{\nu kk'}(x)=F_{\nu k''k'''}(x) \,, \tag{4.1}$$

where $1<k<k'\leqq p$, $1<k''<k'''\leqq p$, $(k, k')\neq(k'', k''')$, and $\mu_{kk'}$ are certain constants not all zeroes.

Let $n_{\nu k}=\sum_{k'=1}^{p} n_{\nu kk'}$. To reduce the problem to our model, we first set

$$\frac{\mu_{kk'}}{\sqrt{\nu}}=\frac{\gamma_{k'}}{\sqrt{n_{\nu k'}}}-\frac{\gamma_k}{\sqrt{n_{\nu k}}} \,, \qquad 1\leqq k\neq k'\leqq p \,, \tag{4.2}$$

$\gamma=(\gamma_1,\cdots,\gamma_p)'$ being defined in (2.13). We pool all the observations $Y_{\nu m}^{(k,k')}$ $(m=1, 2,\cdots, n_{\nu kk'},\ 1\leqq k<k'\leqq p)$ and denote this pooled set of observations by $X_{\nu 1},\cdots, X_{\nu\nu}$. Then this testing problem belongs to our model with $\beta_0=0$, $c_{k\nu i}=+1$ if $X_{\nu i}$ is from a block where the kth treatment is paired with a treatment k' $(k'=k+1,\cdots, p)$, $c_{k\nu i}=-1$ if $X_{\nu i}$ is from a block were the kth treatment is paired with a treatment k' $(k'=1, 2,\cdots, k-1)$, $c_{k\nu i}=0$, if $X_{\nu i}$ is not involved in the block at all. We consider the situation when the observations are classified in several groups. Define test statistics $T_{k\nu 0}$ as in (2.21) or $U_{k\nu}^*$ as in (3.6), $k=1, 2,\cdots, p$. Then the quadratic form M_ν^* (or $S_{\nu 00}$) is asymptotically χ_{p-1}^2 under H_0 and $\chi_{p-1}^2(\zeta)$ under the sequence of alternatives with

$$\zeta=(\gamma'\Lambda\gamma)A_0^2(F, \{I_j\}) \,, \tag{4.3}$$

Λ and $A_0^2(F, \{I_j\})$ being defined in (1.5) and (2.9) respectively. We may also mention here that since, under the model β_0 is absent, even in the process of finding permutationally distribution-free tests, statistics (as in [5]), regardless of signs, may be used.

We may remark at this stage that the test proposed by Bradley and Terry [3], considered as such has some limitations owing to the fact that it allows a judge only to give two possible verdicts "better" or "worse" in comparing one treatment with another. However, if we want to allow greater flexibility in the judgment of a judge by making the judgment level classified into several ordered categories, the above test procedure will fail. For example, in comparing the performances of two musicians at a music competition, the performance of Musician 1 compared to that of Musician 2 may be classified as "Much Inferior," "Slightly Inferior," "More or less Parallel," "Slightly Better" and "Much Better." In such cases, it will not be possible to use the test of Bradley and Terry [3], while our test procedure can be used if these ordered categories are represented as class-intervals on the real axis symmetric about the origin. We may also remark that in the particular case when we have only two class-intervals below zero and above zero, that d.f. F is symmetric about 0, and also $n_{\nu kk'}=\binom{p}{2}^{-1}\nu$ for all $1\leqq k<k'\leqq p$, our test is asymptotically power-equivalent to the one considered by Bradley and Terry [3]. This can be sketched in the following way: In the latter case (cf Bradley [2]) the non-centrality parameter of the asymptotic chi-square distribution of the statistic turns out to be $\dfrac{4pf^2(0)}{p-1}\sum_{k=1}^{p}(\gamma_k-\bar{\gamma})^2$. In the present case, $n_{\nu kk'/\nu}=\binom{p}{2}^{-1}$, $\mu_{kk'}=\binom{p}{2}^{1/2}$ $(\gamma_{k'}-\gamma_k)$, $1\leqq k<k'\leqq p$, $\sum_{i=1}^{\nu}c_{k\nu i}^2=\sum_{k'=1}^{p}n_{\nu kk'}=2\nu/p$, $\sum_{i=1}^{\nu}c_{k\nu i}c_{k'\nu i}=-n_{\nu kk'}=-\nu\binom{p}{2}^{-1}$, $\sum_{i=1}^{\nu}c_{k\nu i}^*c_{k'\nu i}^*=-(p-1)^{-1}$, $1\leqq k<k'\leqq p$. Let $\rho=-(p-1)^{-1}$. Then $\Lambda=(1-\rho)I_p+\rho J_p$, where I_p is a unit matrix of order p, J_p is a $p\times p$ matrix all elements of which are 1. Note that Λ is not of full rank. But we can take a principal submatrix of it of order $(p-1)$ which is of full rank. Then

$$\zeta=A_0^2(F,\{I_j\})\left(\sum_{k=1}^{p}\gamma_k^2-\frac{1}{p-1}\sum_{k\neq k'}\gamma_k\gamma_{k'}\right)=\frac{pA_0^2(F,\{I_j\})}{p-1}\sum_{k=1}^{p}(\gamma_k-\bar{\gamma})^2 .$$

In the particular case when there are only two class intervals $(-\infty, 0)$ and $(0, \infty)$, $\Delta_2=-\Delta_1=2f(0)$, $P_1=P_2=1/2$. Hence, $A_0^2(F,\{I_j\})=\Delta_1^2P_1+\Delta_2^2P_2=2f^2(0)$. Hence, $\zeta=\dfrac{4pf^2(0)}{p-1}\sum_{k=1}^{p}(\gamma_k-\bar{\gamma})^2$.

It is also possible to introduce statistics analogous to those of Mehra and Puri [8] for grouped data. The test statistic used by Mehra and Puri can be introduced as follows: Let $\{J_{\nu i},\ 1\leqq i\leqq\nu,\ \nu\geqq1\}$ be a double sequence of numbers satisfying either the assumptions (on the "score" functions) by Chernoff and Savage [3a] or by Hájek [7]. Let $R_{\nu,m}^{(k,k')}$ be the rank of $|Y_{\nu m}^{(k,k')}|$ when the $\nu=\sum\sum_{1\leqq k<k'\leqq p}n_{\nu kk'}$ values of $|Y_{\nu m}^{(k,k')}|$, $m=1, 2,\cdots, n_{\nu kk'}$; $1\leqq k<k'\leqq p$, are arranged in ascending order of magnitude in a combined ranking. Define

$$\tau_\nu^{(k,k')}=\sum_{m=1}^{n_{\nu kk'}}J_\nu(R_{\nu,m}^{(k,k')}/(\nu+1))\operatorname{sgn}Y_{\nu m}^{(k,k')} ,$$

where $J_\nu(u)$ is a step function define over $(0, 1]$ taking constant values $J_{\nu i}$ over the interval $((i-1)/\nu, i/\nu]$, i.e. $J_\nu(u)=J_{\nu i}=J(i/(\nu+1))$ for $(i-1)/\nu<u\leqq i/\nu$. Then the test statistics are of the form

$$K_{\nu 0}=\sum_{k=1}^{p}\left\{\sum_{k'\neq k}\tau_\nu^{(k,k')}/\sqrt{n_{\nu kk'}}\right\}^2\left(\nu^{-1}\sum_{i=1}^{\nu}J_{\nu i}^2\right)p . \tag{4.4}$$

It has also been shown that under H_0 defined in (1.9) and under the conditions given in [8], $K_{\nu 0}$ is under H_0 asymptotically χ^2_{p-1} and under the sequence of alternatives defined in (2.12) is asymptotically $\chi^2_{p-1}(\delta)$, where

$$\delta = p^{-1} \sum_{k=1}^{p} \left\{ \sum_{k' \neq k} (\sqrt{\rho_{kk'}}\, \mu_{kk'}) \right\}^2 \int_0^1 J^2(u)du \,, \tag{4.5}$$

where $J_\nu(u) \to J(u)$ as $\nu \to \infty$ ($J(u)$ exists for $0<u<1$ and is square integrable). $\rho_{kk'} = \lim_{\nu\to\infty} n_{\nu kk'}/\nu$ is assumed to exist and is >0 for each pair (k, k'), $(1 \leqq k < k' \leqq p)$. $\rho_{kk'} = \rho_{k'k}$ $(1 \leqq k \neq k' \leqq p)$. If we define statistics analogous to $K_{\nu 0}$ for grouped data, under, the null hypothesis, it is asymptotically χ^2_p, while, under the sequence of alternatives considered in (2.12) is asymptotically $\chi^2_p(\delta)$ with $\delta = \frac{A_0^2(F, \{I_j\})}{p} \sum_{k=1}^{p} \left\{ \sum_{k' \neq k} \sqrt{\rho_{kk'}} \cdot \mu_{kk'} \right\}^2$. In the particular case when each pair of treatments is compared the same number of times, $\delta = \frac{A_0^2(F, \{I_j\})}{p} \sum_{k=1}^{p} \left\{ \sum_{k' \neq k} (\gamma_{k'} - \gamma_k) \right\}^2 = \frac{p}{p-1} \cdot A_0^2(F, \{I_j\}) \sum_{k=1}^{p} (\gamma_k - \bar{\gamma})^2$, so that the test considered by Mehra and Puri is asymptotically power-equivalent to our test. But, in general, this is not so; (the author has a counterexample to this effect (see [4], pp. 121–122)). Hence, unlike our test, the test considered by Mehra and Puri does not possess any asymptotic optimality property.

Acknowledgments

Thanks are due to Prof. P. K. Sen for suggesting the problem and making numerous comments during its progress.

UNIVERSITY OF NORTH CAROLINA AT CHAPEL HILL AND INDIAN STATISTICAL INSTITUTE, CALCUTTA

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