ASYMPTOTIC SOLUTIONS OF THE HYPERGEOMETRIC FUNCTION ${}_1F_1$ OF MATRIX ARGUMENT, USEFUL IN MULTIVARIATE ANALYSIS

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1. Introduction

The purpose of this paper is to show a unified derivation of the asymptotic expansions of the distributions for (i) the generalized variance of the noncentral Wishart matrix, (ii) likelihood ratio statistic for multivariate linear hypothesis, both when the matrix of noncentrality parameters is of the same order as the sample size, and (iii) $\operatorname{tr} S_1S_2^{-1}$, when S_i has the Wishart distribution $W_p(\Sigma_i, n_i)$ for large $n=n_1+n_2$ with $\rho_i=n_i/n>0$ fixed. The unified derivation is given by considering the asymptotic solutions of the system of partial differential equations for hypergeometric function ${}_1F_1$ of matrix argument obtained recently by Muirhead [3].

He obtained an asymptotic solution for ${}_1F_1$ and applied it to the asymptotic expansions of the Hotelling's statistic, Pillai's statistic for central cases and the maximum root of the Wishart matrix, all of which, however, can also be derived by our direct evaluation of ${}_1F_1$ in Sugiura and Fujikoshi [7]. For the above three problems, the direct evaluation is not successful and the method by differential equations is useful. It should be noted that using different technique, the asymptotic expansion for the problem (i) is obtained by Sugiura and Nagao [8], (ii) by Sugiura [6], (iii) by Nagao [5], from which we could conjecture the type of asymptotic solutions for ${}_1F_1$ in general case.

2. Test statistics expressed by ${}_1F_1$

Let S have the noncentral Wishart distribution $W_p(\Sigma, n; \Omega)$. Then the characteristic function of $\sqrt{n}\log|S\Sigma^{-1}/n|$ can be expressed by Constantine [1] as

$$(2.1)\qquad C_1(t)=\left(\frac{2}{n}\right)^{itp\sqrt{n}}\left[\Gamma_p\left(\frac{1}{2}n+\sqrt{n}\,it\right)\Big/\Gamma_p\left(\frac{1}{2}n\right)\right] \cdot {}_1F_1\left(-it\sqrt{n}\,;\frac{1}{2}n;\,-n\theta\right),$$

where $\Omega=n\theta$ and θ does not depend on n.

Using the same notation as in Sugiura and Fujikoshi [7], the characteristic function of the likelihood ratio statistic $-2\rho\log\lambda/\sqrt{m}$ with $m=\rho N=N-s+(b-p-1)/2$ for multivariate linear hypothesis, where N, b, and $N-s$ mean the sample size, degrees of freedom due to hypothesis and degrees of freedom due to error respectively, can be written by Constantine [1] as

$$(2.2)\qquad \begin{aligned} C_2(t)&=\Gamma_p\Big(\frac{1}{2}m-it\sqrt{m}-\frac{b-p-1}{4}\Big)\Gamma_p\Big(\frac{1}{2}m+\frac{b+p+1}{4}\Big)\\ &\cdot\Big[\Gamma_p\Big(\frac{1}{2}m-\frac{b-p-1}{4}\Big)\Gamma_p\Big(\frac{1}{2}m-it\sqrt{m}+\frac{b+p+1}{4}\Big)\Big]^{-1}\\ &\cdot{}_1F_1\Big(-it\sqrt{m}\,;\frac{1}{2}m-\sqrt{m}it+\frac{b+p+1}{4}\,;\,-m\theta\Big)\,, \end{aligned}$$

where the matrix of noncentrality parameters Ω is assumed to be mθ. The naturality of this assumption was illustrated in Sugiura [6].

Finally let S_i have $W_p(\Sigma_i, n_i)$ for $i=1, 2$. When S_1 and S_2 are stochastically independent, the moment generating function of $\sqrt{n}\,\mathrm{tr}\, S_1S_2^{-1}$ for $n=n_1+n_2$ can be expressed by Muirhead [4] as

$$(2.3)\qquad \begin{aligned} M(t)&=E[\mathrm{etr}\,(t\sqrt{n}S_1S_2^{-1})]\\ &=\Big[\Gamma_p\Big(\frac{1}{2}n\Big)\Big/\Gamma_p\Big(\frac{1}{2}n_2\Big)\Big]\psi\Big(\frac{1}{2}n_1;\,-\frac{1}{2}n_2+\frac{1}{2}(p+1);\,-\sqrt{n}t\Gamma\Big)\,, \end{aligned}$$

where $\Gamma=\Sigma_1\Sigma_2^{-1}$ and

$$(2.4)\qquad \psi(a\,;c\,;\Gamma)=[1/\Gamma_p(a)]\int_{S>0}\mathrm{etr}\,(-\Gamma S)|S|^{a-(p+1)/2}|I+S|^{c-a-(p+1)/2}dS\,.$$

This statistic can be used for testing the hypothesis $H_0:\Sigma_1=\Sigma_2$ against the one-sided alternatives $H_1:\Sigma_1\Sigma_2^{-1}>I$, and the asymptotic expansion was obtained by Nagao [5]. It may be remarked that Muirhead [4] derived the expression (2.3) based on the joint distribution of the characteristic roots of $S_1S_2^{-1}$, but it can also be obtained by taking $\Sigma_1=I$, $\Sigma_2=\Gamma^{-1}$ in two Wishart distributions and making the transformation $(S_1, S_2)\to(S_2^{-1/2}S_1S_2^{-1/2}, S_2)$.

3. General asymptotic expansions

Generalizing the characteristic function $C_1(t)$ and $C_2(t)$ in (2.1) and (2.2), we shall consider an asymptotic solution of ${}_1F_1(a\,;c\,;dR)$ for $R=\mathrm{diag}\,(r_1,\cdots,r_p)$ and $a=\alpha_1\sqrt{n}+\alpha_0$, $c=\beta_2n+\beta_1\sqrt{n}+\beta_0$ $(\beta_2\neq0)$, $d=\gamma_2n+\gamma_1\sqrt{n}+\gamma_0$ $(\gamma_2\neq0)$, which by Muirhead [3], can be characterized as the unique solution of the system of partial differential equations

$$(3.1)\quad r_i\frac{\partial^2 F}{\partial r_i^2}+\left\{c-\frac{1}{2}(p-1)-dr_i+\frac{1}{2}\sum_{j\neq i}\frac{r_i}{r_i-r_j}\right\}\frac{\partial F}{\partial r_i}-\frac{1}{2}\sum_{j\neq i}\frac{r_i}{r_i-r_j}\frac{\partial F}{\partial r_j}=adF$$

for $i=1, 2, \cdots, p$, subject to the conditions that (a) F is symmetric with respect to $r_1, \cdots, r_p$ and (b) F is analytic about $R=0$ and $F(0)=1$. Putting $W=I-(I-(\gamma_2/\beta_2)R)^{-1}$ and $\log F=(\alpha_1\sqrt{n}+\alpha_0)\log|I-W|+H(W)$ or equivalently, modifying (6.7) in Muirhead [4], the first differential equation in (3.1) can be rewritten as

$$(3.2)\quad w_1(1-w_1)^2\left\{\frac{\partial^2 H}{\partial w_1^2}+\left(\frac{\partial H}{\partial w_1}\right)^2\right\}+\left[c-\frac{1}{2}(p-1)\right.$$
$$+w_1\left\{-2a-2+\frac{1}{2}(p-1)-c+\beta_2 d/\gamma_2\right\}+2(a+1)w_1^2$$
$$\left.+\frac{1}{2}\sum_{j=2}^{p}\frac{w_1(1-w_1)(1-w_j)}{w_1-w_j}\right]\frac{\partial H}{\partial w_1}-\frac{1}{2}\sum_{j=2}^{p}\frac{w_j(1-w_j)^2}{w_1-w_j}\frac{\partial H}{\partial w_j}$$
$$=a\left\{c-\beta_2 d/\gamma_2-(a+1)w_1-\frac{1}{2}\sum_{j=2}^{p}w_j\right\},$$

where $W=\operatorname{diag}(w_1, \cdots, w_p)$ and $H(w)$ is symmetric with respect to $w_1, \cdots, w_p$ subject to $H(0)=0$. We shall look for a solution of (3.2) of the form

$$(3.3)\quad H(W)=\sum_{k=0}^{\infty}Q_k(W)n^{-k/2},$$

under the condition $Q_k(0)=0$. Substituting (3.3) into (3.2) and equating the term of order n, we have $\partial Q_0/\partial w_1=\alpha_1(\beta_1-\gamma_1\beta_2/\gamma_2-\alpha_1 w_1)/\beta_2$, which implies

$$(3.4)\quad Q_0=\alpha_1\left(\delta_1\sigma_1-\frac{1}{2}\alpha_1\sigma_2/\beta_2\right),$$

where $\sigma_j=w_1^j+\cdots+w_p^j$ and $\delta_1=\beta_1/\beta_2-\gamma_1/\gamma_2$. Equating the term of order $\sqrt{n}$ yields

$$\beta_2\frac{\partial Q_1}{\partial w_1}+\{\beta_1-w_1(\beta_1+2\alpha_1-\beta_2\gamma_1/\gamma_2+2\alpha_1 w_1^2\}\frac{\partial Q_0}{\partial w_1}$$
$$=\alpha_1\{\beta_0-\beta_2\gamma_0/\gamma_2-w_1(1+\alpha_0)\}-\frac{1}{2}\alpha_1\sum_{j=2}^{p}w_j+\alpha_0\{\beta_1-\beta_2\gamma_1/\gamma_2-\alpha_1 w_1\},$$

giving

$$(3.5)\quad Q_1=\{\delta_1(\alpha_0-\alpha_1\beta_1/\beta_2)+\alpha_1\delta_0\}\sigma_1+\frac{1}{2}\sigma_2\left[\alpha_1\delta_1(\delta_1+2\alpha_1/\beta_2)+\alpha_1^2\beta_1/\beta_2^2\right.$$
$$\left.-\alpha_1\left(2\alpha_0+\frac{1}{2}\right)\Big/\beta_2\right]-\frac{1}{3}(\alpha_1^2/\beta_2)\sigma_3(3\delta_1+2\alpha_1/\beta_2)$$

$$+\frac{1}{2}\alpha_1^3\sigma_4/\beta_2^2-\frac{1}{4}\alpha_1\sigma_1^2/\beta_2 ,$$

where $\delta_0=\beta_0/\beta_2-\gamma_0/\gamma_2$. Similarly we have

$$Q_2=\sum_{j=1}^{6}g_j\sigma_j+\left\{\frac{1}{4}\alpha_1(\alpha_1+\beta_1)/\beta_2^2+\frac{1}{2}\left(\alpha_1\delta_1-\frac{1}{2}\alpha_0\right)\Big/\beta_2\right\}\sigma_1^2 \tag{3.6}$$
$$-\alpha_1\left(\alpha_1+\frac{1}{2}\beta_2\delta_1\right)\sigma_1\sigma_2/\beta_2^2+\frac{1}{2}\alpha_1^2\left(\sigma_1\sigma_3+\frac{1}{4}\sigma_2^2\right)\Big/\beta_2^2 ,$$

where

$$g_1=-(\beta_1/\beta_2)\{(\alpha_0-\alpha_1\beta_1/\beta_2)\delta_1+\alpha_1\delta_0\}-\alpha_1\beta_0\delta_1/\beta_2+\alpha_0\delta_0 ,$$

$$g_2=\frac{1}{2}(\alpha_1^2/\beta_2)\left(\beta_0/\beta_2+\frac{1}{2}\beta_2^{-1}-\delta_1^2\right)+\frac{1}{2}(\beta_1/\beta_2)\left\{-\alpha_1^2\beta_1/\beta_2^2+\left(\frac{1}{2}+2\alpha_0\right)\alpha_1/\beta_2\right\}$$
$$+\frac{1}{2}(\delta_1+2\alpha_1/\beta_2)\{(\alpha_0-2\alpha_1\beta_1/\beta_2)\delta_1+\alpha_1\delta_0\}$$
$$+\frac{1}{2}(\alpha_1\delta_1/\beta_2)(\beta_0+2\alpha_0+1-\beta_2\gamma_0/\gamma_2)-\frac{1}{2}(\alpha_0/\beta_2)\left(\frac{1}{2}+\alpha_0\right) ,$$

$$g_3=\frac{1}{3}(\alpha_1^2/\beta_2^2)\{-1+(2\alpha_1+3\beta_1)\delta_1+2\beta_2\delta_1^2+2\alpha_1\beta_1/\beta_2-2\alpha_0+\beta_2\gamma_0/\gamma_2-\beta_0\} \tag{3.7}$$
$$-\frac{1}{3}(\alpha_1/\beta_2)\left\{(4\alpha_0-2\alpha_1\beta_1/\beta_2)\delta_1+2\alpha_1\delta_0+\frac{1}{2}\delta_1\right\}$$
$$+\frac{1}{3}(\delta_1+2\alpha_1/\beta_2)\{\alpha_1\delta_1(\delta_1+2\alpha_1/\beta_2)+\alpha_1^2\beta_1/\beta_2^2-\alpha_1(1+2\alpha_0)/\beta_2\} ,$$

$$g_4=-\frac{1}{4}(\alpha_1^2/\beta_2)\left\{6\delta_1^2+16\alpha_1\delta_1/\beta_2+5(\alpha_1/\beta_2)^2-\left(6\alpha_0+\frac{5}{2}\right)\Big/\beta_2+4\alpha_1\beta_1/\beta_2^2\right\} ,$$

$$g_5=2(\alpha_1^3/\beta_2^2)(\alpha_1/\beta_2+\delta_1) ,$$

$$g_6=-\frac{5}{6}\alpha_1^4/\beta_2^3 .$$

Thus we get

THEOREM 3.1. *An asymptotic solution of the hypergeometric function* ${}_1F_1(\alpha_1\sqrt{n}+\alpha_0; \beta_2 n+\beta_1\sqrt{n}+\beta_0; (\gamma_2 n+\gamma_1\sqrt{n}+\gamma_0)R)$ *for large value of* n *with* $\beta_2\neq 0$, $\gamma_2\neq 0$, *is given by*

$$|I-W|^{\alpha_1\sqrt{n}+\alpha_0}\exp\left[\alpha_1\left(\delta_1\sigma_1-\frac{1}{2}\alpha_1\sigma_2/\beta_2\right)+n^{-1/2}Q_1+n^{-1}Q_2+O(n^{-3/2})\right] , \tag{3.8}$$

where $W=I-(I-(\gamma_2/\beta_2)R)^{-1}$, $\sigma_j=\mathrm{tr}\,W^j$ *and* $\delta_1=\beta_1/\beta_2-\gamma_1/\gamma_2$. *The coefficients* Q_1 *and* Q_2 *are given by* (3.5) *and* (3.6) *respectively with* $\delta_0=\beta_0/\beta_2-\gamma_0/\gamma_2$.

For problem (iii), it is proved by Muirhead [4] that the function $\psi(a;c;\Gamma)$ given in (2.4) satisfies the same system of differential equations as for ${}_1F_1(a;c;\Gamma)$. Hence generalizing the moment generating function in (2.3), we shall consider an asymptotic solution for ${}_1F_1(\alpha_1 n+\alpha_0;\beta_1 n+\beta_0;\sqrt{n}tR)$ where $R=\mathrm{diag}(r_1,\cdots,r_p)$ and $\beta_1\neq 0$. The function $G=\log{}_1F_1(\alpha_1 n+\alpha_0;\beta_1 n+\beta_0;\sqrt{n}tR)$ is the unique solution for

$$\tag{3.9} \begin{aligned} r_1\left\{\frac{\partial^2 G}{\partial r_1^2}+\left(\frac{\partial G}{\partial r_1}\right)^2\right\}&+\left\{\beta_1 n-\sqrt{n}tr_1+\beta_0-\frac{1}{2}(p-1)\right.\\ &\left.+\frac{1}{2}\sum_{j=2}^{p}\frac{r_1}{r_1-r_j}\right\}\frac{\partial G}{\partial r_1}-\frac{1}{2}\sum_{j=2}^{p}\frac{r_j}{r_1-r_j}\frac{\partial G}{\partial r_j}=\sqrt{n}t(\alpha_1 n+\alpha_0)\ , \end{aligned}$$

subject to the same condition as in (3.1). Substituting an asymptotic series $\sum_{k=-1}^{\infty}Q_k n^{-k/2}$ into (3.9), where Q_k is symmetric with respect to $r_1,\cdots,r_p$ and $Q_k(0)=0$ for all k, the similar computation as for Theorem 3.1 yields

THEOREM 3.2. *An asymptotic solution of* ${}_1F_1(\alpha_1 n+\alpha_0;\beta_1 n+\beta_0;\sqrt{n}tR)$ *for large value of* n, *when* $\beta_1\neq 0$, *is given by*

$$\tag{3.10} \exp\left[\sqrt{n}(\alpha_1/\beta_1)t\sigma_1+(\alpha_1/2\beta_1^2)t^2\sigma_2(1-\alpha_1/\beta_1)+n^{-1/2}Q_1+n^{-1}Q_2+O(n^{-3/2})\right]\ ,$$

where $\sigma_j=\mathrm{tr}\,R^j$ *and*

$$\tag{3.11} \begin{aligned} Q_1&=(1-\alpha_1/\beta_1)(1-2\alpha_1/\beta_1)\alpha_1\sigma_3 t^3/(3\beta_1^3)+(\alpha_0-\alpha_1\beta_0/\beta_1)t\sigma_1/\beta_1\ ,\\ Q_2&=\frac{1}{4}t^4\sigma_4\alpha_1\beta_1^{-4}(1-\alpha_1/\beta_1)(1-5\alpha_1/\beta_1+5\alpha_1^2/\beta_1^2)\\ &\quad+t^2\left[\frac{1}{2}\sigma_2\beta_1^{-2}\left\{(\alpha_0-\alpha_1\beta_0/\beta_1)(1-2\alpha_1/\beta_1)-\frac{1}{2}\alpha_1\beta_1^{-1}(1+2\beta_0)(1-\alpha_1/\beta_1)\right\}\right.\\ &\quad\left.-\frac{1}{4}\alpha_1\beta_1^{-3}(1-\alpha_1/\beta_1)\sigma_1^2\right]\ . \end{aligned}$$

4. Asymptotic expansions of the three statistics

The first factor in (2.1) can be evaluated as in Fujikoshi [2],

$$\tag{4.1} \begin{aligned} (2/n)^{itp\sqrt{n}}\,&\Gamma_p\left(\frac{1}{2}n+it\sqrt{n}\right)\Big/\Gamma_p\left(\frac{1}{2}n\right)\\ &=\exp\left[-pt^2-\frac{1}{\sqrt{n}}\left\{\frac{1}{2}p(p+1)it+\frac{2}{3}p(it)^3\right\}\right.\\ &\qquad\left.+n^{-1}\left\{\frac{1}{2}p(p+1)it+\frac{2}{3}p(it)^4\right\}+O(n^{-3/2})\right]\ . \end{aligned}$$

Putting $\alpha_1=-it$, $\beta_2=1/2$, $\gamma_2=-1$, $\alpha_0=\beta_1=\beta_0=\gamma_1=\gamma_0=0$ and $R=\theta$ in Theorem 3.1, gives the asymptotic formula for the second factor in (2.1), which, combined with (4.1), proves the characteristic function of $\sqrt{n}\cdot\log|S\Sigma^{-1}/n|-\sqrt{n}\log|I+2\theta|$ to be

$$\exp[-t^2\tau_1^2/2]\Big[1-n^{-1/2}\{g_3(it)^3+g_1 it\}+n^{-1}\sum_{k=1}^{3}(it)^{2k}h_{2k}+O(n^{-3/2})\Big]\,, \tag{4.2}$$

where $\tau_1^2=2(p-\sigma_2)$ for $\sigma_j=\mathrm{tr}\,(I-(I+2\theta)^{-1})^j$ and

$$g_1=-\frac{1}{2}\sigma_1^2-\frac{1}{2}\sigma_2+\frac{1}{2}p(p+1)\,,$$

$$g_3=2\sigma_4-\frac{8}{3}\sigma_3+\frac{2}{3}p\,,$$

$$\begin{aligned} h_2&=\frac{1}{8}\{\sigma_1^2+\sigma_2-p(p+1)\}^2+\frac{5}{2}\sigma_4+2\sigma_1\sigma_3+\frac{1}{2}\sigma_2^2-4\sigma_3-4\sigma_1\sigma_2\\ &\quad+\sigma_2+\sigma_1^2+\frac{1}{2}p(p+1)\,,\end{aligned} \tag{4.3}$$

$$h_4=\left(-\sigma_4+\frac{4}{3}\sigma_3-\frac{1}{3}p\right)\{\sigma_1^2+\sigma_2-p(p+1)\}-\frac{20}{3}\sigma_6+16\sigma_5-10\sigma_4+\frac{2}{3}p\,,$$

$$h_6=\frac{1}{2}\left(-2\sigma_4+\frac{8}{3}\sigma_3-\frac{2}{3}p\right)^2.$$

Inverting the characteristic function (4.2), we have

$$\begin{aligned} &P(\sqrt{n}\log|S\Sigma^{-1}/n|-\sqrt{n}\log|I+2\theta|<\tau_1 x)\\ &\quad=\Phi(x)+n^{-1/2}(g_1\Phi^{(1)}(x)/\tau_1+g_3\Phi^{(3)}(x)/\tau_1^3)\\ &\qquad+n^{-1}\sum_{k=1}^{3}h_{2k}\Phi^{(2k)}(x)\tau_1^{-2k}+O(n^{-3/2})\,.\end{aligned} \tag{4.4}$$

After some rearrangement in each coefficient in (4.3), we can see that the result is exactly the same as previously obtained in Sugiura and Nagao [8].

For problem (ii), the first factor in (2.2), namely, products of four gamma functions can be expressed from (2.7) in Sugiura and Fujikoshi [7] as

$$1+itm^{-1/2}bp+m^{-1}(it)^2\left\{bp+\frac{1}{2}(bp)^2\right\}+O(m^{-3/2})\,. \tag{4.5}$$

The second factor of $C_2(t)$ can be evaluated by putting $n=m$, $\alpha_1=-it$, $\beta_2=1/2$, $\beta_1=-it$, $\beta_0=(b+p+1)/4$, $\gamma_2=-1$, $\alpha_0=\gamma_1=\gamma_0=0$, and $R=\theta$ in Theorem 3.1. The same argument as for (4.4) yields

$$P(-2\rho\log\lambda/\sqrt{m}-\sqrt{m}\log|I+2\theta|<\tau_2 x) \tag{4.6}$$

$$=\Phi(x)+m^{-1/2}\{u_1\Phi^{(1)}(x)\tau_2^{-1}+u_3\Phi^{(3)}(x)\tau_2^{-3}\}$$
$$+m^{-1}\sum_{k=1}^{3} v_{2k}\Phi^{(2k)}(x)\tau_2^{-2k}+O(m^{-3/2}) ,$$

where $\tau_2^2=4\sigma_1-2\sigma_2$ for $\sigma_j=\mathrm{tr}\,\{I-(I+2\theta)^{-1}\}^j$ and

$$u_1=\frac{1}{2}\{\sigma_1^2+\sigma_2-(b+p+1)\sigma_1\}+pb ,$$

$$u_3=-2\sigma_4+\frac{20}{3}\sigma_3-8\sigma_2+4\sigma_1 ,$$

$$\begin{aligned}(4.7)\qquad v_2=&\frac{5}{2}\sigma_4+\frac{1}{8}\sigma_1^4+2\sigma_1\sigma_3+\frac{5}{8}\sigma_2^2+\frac{1}{4}\sigma_2\sigma_1^2-(b+p+7)\sigma_3\\ &-\frac{1}{4}(b+p+25)\sigma_1\sigma_2-\frac{1}{4}(b+p+1)\sigma_1^3+\frac{1}{2}\sigma_2(5b+5p+pb+13)\\ &+\sigma_1^2\left\{\frac{1}{8}(b+p+1)^2+\frac{1}{2}bp+4\right\}-(p+b+1)\left(\frac{1}{2}pb+2\right)\sigma_1\\ &+pb\left(\frac{1}{2}pb+1\right) ,\end{aligned}$$

$$\begin{aligned}v_4=&-\sigma_2\sigma_4-\sigma_4\sigma_1^2-\frac{20}{3}\sigma_6+32\sigma_5+(b+p+1)\sigma_1\sigma_4+\frac{10}{3}\sigma_1^2\sigma_3+\frac{10}{3}\sigma_2\sigma_3\\ &-(62+2pb)\sigma_4-\frac{10}{3}(p+b+1)\sigma_3\sigma_1-4\sigma_2\sigma_1^2-4\sigma_2^2+\frac{4}{3}(5pb+46)\sigma_3\\ &+2(2p+2b+3)\sigma_1\sigma_2+2\sigma_1^3-8(pb+4)\sigma_2-2(b+p+1)\sigma_1^2+4(pb+2)\sigma_1 ,\end{aligned}$$

$$v_6=2\left(-\sigma_4+\frac{10}{3}\sigma_3-4\sigma_2+2\sigma_1\right)^2 .$$

Again this result agrees with Sugiura [6]. Finally for problem (iii), the moment generating function $M(t)$ in (2.3) satisfies the same initial condition for ${}_1F_1$, namely, $M(t)=1$ for $\Gamma=0$. Putting $\alpha_1=\rho_1/2$, $\alpha_0=0$, $\beta_1=-\rho_2/2$, $\beta_0=(p+1)/2$ and $R=-\Gamma$ in Theorem 3.2, we obtain

$$\begin{aligned}(4.8)\qquad &P(\sqrt{n}(\mathrm{tr}\,S_1S_2^{-1}-\rho_1\rho_2^{-1}\,\mathrm{tr}\,\Gamma)<\tau_3x)\\ &=\Phi(x)-n^{-1/2}\{a_1\Phi^{(1)}(x)\tau_3^{-1}+a_3\Phi^{(3)}(x)\tau_3^{-3}\}\\ &\quad+n^{-1}\sum_{k=1}^{3} b_{2k}\Phi^{(2k)}(x)\tau_3^{-2k}+O(n^{-3/2}) ,\end{aligned}$$

where $\tau_3^2=2\rho_1\rho_2^{-3}\sigma_2$ for $\sigma_j=\mathrm{tr}\,\Gamma^j$ and

$$a_1=\rho_1\rho_2^{-2}(p+1)\sigma_1 ,$$

$$a_3=\frac{4}{3}\rho_1\rho_2^{-4}(1+2\rho_1/\rho_2)\sigma_3 ,$$

$$(4.9)\qquad b_2=\rho_1\rho_2^{-3}\{(3p+4)\rho_1\rho_2^{-1}+(2p+3)\}\sigma_2+\rho_1\rho_2^{-4}\{1+\rho_1(p+1)^2/2\}\sigma_1^2\ ,$$

$$b_4=2\rho_1\rho_2^{-5}(1+5\rho_1\rho_2^{-1}+5\rho_1^2\rho_2^{-2})\sigma_4+\frac{4}{3}(p+1)(1+2\rho_1\rho_2^{-1})\rho_1^2\rho_2^{-6}\sigma_1\sigma_3\ ,$$

$$b_6=\frac{8}{9}\rho_1^2\rho_2^{-8}(1+2\rho_1\rho_2^{-1})^2\sigma_3^2\ .$$

It is easily verified that the above result is the same as the Theorem 12.1 in Nagao [5].

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