ASYMPTOTIC SOLUTIONS OF THE HYPERGEOMETRIC FUNCTION \(_{1}F_{1}\) OF MATRIX ARGUMENT, USEFUL IN MULTIVARIATE ANALYSIS

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1. Introduction

The purpose of this paper is to show a unified derivation of the asymptotic expansions of the distributions for (i) the generalized variance of the noncentral Wishart matrix, (ii) likelihood ratio statistic for multivariate linear hypothesis, both when the matrix of noncentrality parameters is of the same order as the sample size, and (iii) \(\text{tr} S_{i}S_{i}^{-1}\), when \(S_{i}\) has the Wishart distribution \(W_{p}(\Sigma_{i}, n_{i})\) for large \(n=n_{1}+n_{2}\) with \(\rho_{i}=n_{i}/n>0\) fixed. The unified derivation is given by considering the asymptotic solutions of the system of partial differential equations for hypergeometric function \(_{1}F_{1}\) of matrix argument obtained recently by Muirhead [3].

He obtained an asymptotic solution for \(_{1}F_{1}\) and applied it to the asymptotic expansions of the Hotelling's statistic, Pillai's statistic for central cases and the maximum root of the Wishart matrix, all of which, however, can also be derived by our direct evaluation of \(_{1}F_{1}\) in Sugiura and Fujikoshi [7]. For the above three problems, the direct evaluation is not successful and the method by differential equations is useful. It should be noted that using different technique, the asymptotic expansion for the problem (i) is obtained by Sugiura and Nagao [8], (ii) by Sugiura [6], (iii) by Nagao [5], from which we could conjecture the type of asymptotic solutions for \(_{1}F_{1}\) in general case.

2. Test statistics expressed by \(_{1}F_{1}\)

Let \(S\) have the noncentral Wishart distribution \(W_{p}(\Sigma, n; \Omega)\). Then the characteristic function of \(\sqrt{n} \log |S\Sigma^{-1}/n|\) can be expressed by Constantine [1] as

\[
C_{1}(t) = \left(\frac{2}{n}\right)^{\nu/2} \left[ \Gamma_p\left(\frac{1}{2} n + \sqrt{n} it \right) / \Gamma_p\left(\frac{1}{2} n\right) \right] \\
\cdot _{1}F_{1}\left(-it \sqrt{n} ; \frac{1}{2} n ; -n \theta \right),
\]

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where $\Omega = n\theta$ and $\theta$ does not depend on $n$.

Using the same notation as in Sugiura and Fujikoshi [7], the characteristic function of the likelihood ratio statistic $-2\rho \log \lambda / \sqrt{m}$ with $m = \rho N = N - s + (b - p - 1)/2$ for multivariate linear hypothesis, where $N$, $b$, and $N - s$ mean the sample size, degrees of freedom due to hypothesis and degrees of freedom due to error respectively, can be written by Constantine [1] as

\[
C(t) = \Gamma_p \left( \frac{1}{2} m - it\sqrt{m} - \frac{b - p - 1}{4} \right) \Gamma_p \left( \frac{1}{2} m + \frac{b + p + 1}{4} \right)
\cdot \left[ \Gamma_p \left( \frac{1}{2} m - \frac{b - p - 1}{4} \right) \Gamma_p \left( \frac{1}{2} m - it \sqrt{m} + \frac{b + p + 1}{4} \right) \right]^{-1}
\cdot \Phi \left( -it\sqrt{m}; \frac{1}{2} m - \sqrt{m} it + \frac{b + p + 1}{4}; -m\theta \right),
\]

where the matrix of noncentrality parameters $\Omega$ is assumed to be $m\theta$. The naturality of this assumption was illustrated in Sugiura [6].

Finally let $S_i$ have $W_p(\Sigma_i, n_i)$ for $i = 1, 2$. When $S_1$ and $S_2$ are stochastically independent, the moment generating function of $\sqrt{n} \text{ tr } S_i S_i^{-1}$ for $n = n_1 + n_2$ can be expressed by Muirhead [4] as

\[
M(t) = E[\text{etr} (t \sqrt{n} S_i S_i^{-1})] = \left[ \Gamma_p \left( \frac{1}{2} n_i \right) / \Gamma_p \left( \frac{1}{2} n_2 \right) \right] \phi \left( \frac{1}{2} n_1; -\frac{1}{2} n_2 + \frac{1}{2} (p + 1); -\sqrt{n} t I \right),
\]

where $I = \Sigma_1 \Sigma_1^{-1}$ and

\[
\phi(a; c; I) = [1/\Gamma_p(a)] \int_{S > 0} \text{etr} (-I S) |S|^{a-(p+1)/2} |I + S|^{c-a-(p+1)/2} dS.
\]

This statistic can be used for testing the hypothesis $H_0 : \Sigma_1 = \Sigma_2$ against the one-sided alternatives $H_1 : \Sigma_1 \Sigma_2^{-1} \succ I$, and the asymptotic expansion was obtained by Nagao [5]. It may be remarked that Muirhead [4] derived the expression (2.3) based on the joint distribution of the characteristic roots of $S_i S_i^{-1}$, but it can also be obtained by taking $\Sigma_1 = I$, $\Sigma_2 = I^{-1}$ in two Wishart distributions and making the transformation $(S_1, S_2) \rightarrow (S_1^{1/2} S_1 S_2^{-1/2}, S_2)$.

3. General asymptotic expansions

Generalizing the characteristic function $C_t(t)$ and $C_2(t)$ in (2.1) and (2.2), we shall consider an asymptotic solution of $F_t(a; c; dR)$ for $R = \text{diag} (r_1, \cdots, r_p)$ and $a = c_n + a_0$, $c = b_n + b_0$ ($b_0 \neq 0$), $d = c_n + r_2 \sqrt{n} + r_0$ ($r_2 \neq 0$), which by Muirhead [3], can be characterized as the unique solution of the system of partial differential equations.
\begin{align}
(3.1) \quad r_i \frac{\partial F}{\partial r_i} &= \left\{ c - \frac{1}{2}(p-1) - d r_i + \frac{1}{2} \sum_{j \neq i} \frac{r_i - r_j}{r_i - r_j} \right\} \frac{\partial F}{\partial r_i} - \frac{1}{2} \sum_{j \neq i} \frac{r_i - r_j}{r_i - r_j} \frac{\partial F}{\partial r_j} \\
&= a d F
\end{align}

for \( i = 1, 2, \ldots, p \), subject to the conditions that (a) \( F \) is symmetric with respect to \( r_1, \ldots, r_p \), and (b) \( F \) is analytic about \( R = 0 \) and \( F(0) = 1 \). Putting \( W = I - (I - (r_1/\beta_1) R)^{-1} \) and \( \log F = (\alpha_1 \sqrt{n} + \alpha_3) \log |I - W| + H(W) \) or equivalently, modifying (6.7) in Muirhead [4], the first differential equation in (3.1) can be rewritten as

\begin{align}
(3.2) \quad w_i (1 - w_i)^3 \left[ \frac{\partial^2 H}{\partial w_i^2} + \left( \frac{\partial H}{\partial w_i} \right)^2 \right] + \left[ c - \frac{1}{2}(p-1) \\
+ w_i \left\{ -2a - 2 + \frac{1}{2}(p-1) - c + \beta_2 d/\gamma_1 \right\} + 2(a+1)w_i \\
+ \frac{1}{2} \sum_{j=1}^{p} w_i (1-w_i)(1-w_j) \frac{\partial H}{\partial w_i} - \frac{1}{2} \sum_{j=1}^{p} w_j (1-w_j)^2 \frac{\partial H}{\partial w_j} \right] \\
= a \left\{ - \beta_2 d/\gamma_1 - (a+1)w_i - \frac{1}{2} \sum_{j=1}^{p} w_j \right\},
\end{align}

where \( W = \text{diag}(w_1, \ldots, w_p) \) and \( H(w) \) is symmetric with respect to \( w_1, \ldots, w_p \), subject to \( H(0) = 0 \). We shall look for a solution of (3.2) of the form

\begin{align}
(3.3) \quad H(W) = \sum_{k=0}^{\infty} Q_k(W) n^{-k/2},
\end{align}

under the condition \( Q_0(0) = 0 \). Substituting (3.3) into (3.2) and equating the term of order \( n \), we have \( \frac{\partial Q_k}{\partial w_1} = \alpha_1 (\beta_1 - \gamma_1, \beta_2/\gamma_1 - \alpha_1 w_1)/\beta_2 \), which implies

\begin{align}
(3.4) \quad Q_k = \frac{\alpha_1}{\beta_2} \left( \delta_1, \sigma - \frac{1}{2} \sigma_1 \sigma_1/\beta_2 \right),
\end{align}

where \( \sigma = w_1 + \cdots + w_p \) and \( \delta_1 = \beta_1/\beta_2 - \gamma_1/\gamma_2 \). Equating the term of order \( \sqrt{n} \) yields

\begin{align}
\beta_2 \frac{\partial Q_1}{\partial w_1} + \left\{ \beta_1 - w_1 (\beta_1 + 2\alpha_1) - \beta_2 \gamma_1/\gamma_2 + 2\alpha_1 w_1 \right\} \frac{\partial Q_0}{\partial w_1} \\
= \alpha_1 \left\{ \beta_0 - \beta_2 \gamma_1/\gamma_2 - w_1 (1+\alpha_3) \right\} - \frac{1}{2} \alpha_1 \sum_{j=1}^{p} w_j + \alpha_6 \left\{ \beta_1 - \beta_2 \gamma_1/\gamma_2 - \alpha_1 w_1 \right\},
\end{align}

giving

\begin{align}
(3.5) \quad Q_1 = \left\{ \delta_1 (\alpha_6 - \alpha_1/\beta_2) + \alpha_6 \delta_1 \right\} \sigma_1 + \frac{1}{2} \sigma_1 \left( \alpha_1 \sigma_1 + 2\alpha_1/\beta_2 \right) + \alpha_1 \alpha_6 \delta_1/\beta_2 \\
- \alpha_1 \left[ 2\alpha_6 + \frac{1}{2} \right] / \beta_2 - \frac{1}{3} (\alpha_6/\beta_2) \sigma_1 (3\delta_1 + 2\alpha_1/\beta_2)
\end{align}
\[ + \frac{1}{2} \alpha_i \sigma_i / \beta_i - \frac{1}{4} \alpha_i \sigma_i / \beta_i , \]

where \( \delta_i = \beta_i / \beta_i - \gamma_i / \gamma_i \). Similarly we have

(3.6) \[ Q_i = \sum_{j=1}^{s} g_j \sigma_i + \left( \frac{1}{4} \alpha_i (\alpha_i + \beta_i) \right) \sigma_i + \frac{1}{2} \left( \alpha_i \delta_i - \frac{1}{2} \alpha_i \right) / \beta_i \sigma_i \]

\[ - \alpha_i (\alpha_i + \frac{1}{2} \beta_i \delta_i) \sigma_i / \beta_i + \frac{1}{2} \alpha_i \left( \sigma_i + \frac{1}{4} \sigma_i \right) / \beta_i , \]

where

\[ g_1 = - (\beta_i / \beta_i) \left( (\alpha_i - \alpha_i \beta_i / \beta_i) \delta_i + \alpha_i \delta_0 \right) - \alpha_i \beta_i \delta_i / \beta_i + \alpha_i \delta_0 , \]

\[ g_2 = \frac{1}{2} \left( \alpha_i / \beta_i \right) \left( \beta_i / \beta_i + \frac{1}{2} \beta_i - \delta_i \right) + \frac{1}{2} \left( \beta_i / \beta_i \right) \left( - \alpha_i \delta_i / \beta_i + \left( \frac{1}{2} + 2 \alpha_i \right) \alpha_i / \beta_i \right) \]

\[ + \frac{1}{2} \left( \delta_i + 2 \alpha_i \right) \left( (\alpha_i - 2 \alpha_i \beta_i / \beta_i) \delta_i + \alpha_i \delta_0 \right) \]

\[ + \frac{1}{2} \left( \alpha_i (\delta_i / \beta_i) (\beta_i + \alpha_i + 1 - \beta_i \gamma_i / \gamma_i) - \frac{1}{2} (\alpha_i / \beta_i) \left( \frac{1}{2} + \alpha_i \right) , \right) \]

(3.7) \[ g_3 = \frac{1}{3} \left( \alpha_i / \beta_i \right) \left( - 1 + (2 \alpha_i + 3 \beta_i) \delta_i + 2 \beta_i \delta_i^2 + 2 \alpha_i \beta_i / \beta_i - 2 \alpha_i + \beta_i \gamma_i / \gamma_i \right) \]

\[ - \frac{1}{3} \left( \alpha_i / \beta_i \right) \left( (4 \alpha_i - 2 \alpha_i \beta_i / \beta_i) \delta_i + 2 \alpha_i \delta_0 + \frac{1}{2} \delta_i \right) \]

\[ + \frac{1}{3} \left( \delta_i + 2 \alpha_i \right) \left[ \alpha_i (\delta_i + 2 \alpha_i / \beta_i) + \alpha_i \beta_i / \beta_i - \alpha_i (1 + 2 \alpha_i) / \beta_i \right) , \]

\[ g_4 = - \frac{1}{4} \left( \alpha_i / \beta_i \right) \left[ 6 \delta_i^2 + 16 \alpha_i \delta_i / \beta_i + 5 \left( \alpha_i / \beta_i \right) \left( \frac{6 \alpha_i + \frac{5}{2}} {\beta_i + 4 \alpha_i \beta_i / \beta_i^2} \right) , \right) \]

\[ g_5 = 2 \left( \alpha_i / \beta_i \right) (\alpha_i / \beta_i + \delta_i) , \]

\[ g_6 = - \frac{5}{6} \alpha_i / \beta_i ^2 . \]

Thus we get

Theorem 3.1. An asymptotic solution of the hypergeometric function \( F \) \( (\alpha_i \sqrt{n} + \alpha_i ; \beta_i n + \beta_i \sqrt{n} + \beta_i ; (\gamma_i n + \gamma_i \sqrt{n} + \gamma_i) R) \) for large value of \( n \) with \( \beta_i \neq 0, \gamma_i \neq 0 \), is given by

(3.8) \[ |I - W|^{\gamma_i / \gamma_i + \tau_0} \exp \left[ \alpha_i (\delta_i \sigma_i - \frac{1}{2} \alpha_i \sigma_i / \beta_i) + n^{-1/2} Q_i + n^{-1} Q_i + O(n^{-1/2}) \right) , \]

where \( W = I - (I - (\gamma_i / \beta_i R)^{-1} \sigma_i = \text{tr} W \) and \( \delta_i = \beta_i / \beta_i - \gamma_i / \gamma_i \). The coefficients \( Q_i \) and \( Q_2 \) are given by (3.5) and (3.6) respectively with \( \delta_i = \beta_i / \beta_i - \gamma_i / \gamma_i \).
For problem (iii), it is proved by Muirhead [4] that the function φ(α; c; Γ) given in (2.4) satisfies the same system of differential equations as for \( _{1}F_{1}(α_0; c; Γ) \). Hence generalizing the moment generating function in (2.3), we shall consider an asymptotic solution for \( _{1}F_{1}(α_n + α_0; β_n + β_0; √n tR) \) where \( R = \text{diag} (r_1, \ldots, r_p) \) and \( β_i \neq 0 \). The function \( G = \log _{1}F_{1}(α_n + α_0; β_n + β_0; √n tR) \) is the unique solution for

\[
(3.9) \quad r_1 \left( \frac{∂G}{∂r_1} + \left( \frac{∂G}{∂r_1} \right)^2 + \left( β_n - √n tr_1 + β_0 - \frac{1}{2} (p-1) \right) \right)
\]

subject to the same condition as in (3.1). Substituting an asymptotic series \( \sum_{k=1}^{∞} Q_k n^{-k/2} \) into (3.9), where \( Q_k \) is symmetric with respect to \( r_1, \ldots, r_p \) and \( Q_k(0) = 0 \) for all \( k \), the similar computation as for Theorem 3.1 yields

**THEOREM 3.2.** An asymptotic solution of \( _{1}F_{1}(α_n + α_0; β_n + β_0; √n tR) \)
for large value of \( n \), when \( β_i \neq 0 \), is given by

\[
(3.10) \quad \exp \left[ √n (α_i/β_i) tσ_i + (α_i/2β_i^2) t^2 σ_i (1 - α_i/β_i) + n^{-1/2} Q_i + n^{-1} Q_2 + O(n^{-3/2}) \right],
\]

where \( σ_j = \text{tr} R^j \) and

\[
(3.11) \quad Q_i = (1 - α_i/β_i)(1 - 2α_i/β_i)α_iσ_i t^2 (3β_i^2) + (α_i - α_iβ_i/β_i)tσ_i/β_i,
\]

\[
Q_2 = \frac{1}{4} t^2 σ_i α_i β_i (1 - α_i/β_i)(1 - 5α_i/β_i + 5α_i^2/β_i^2)
\]

\[
+ t^2 \left( \frac{1}{2} α_i β_i (α_i - α_i β_i/β_i)(1 - 2α_i/β_i) - \frac{1}{2} α_i β_i^{-1}(1 + 2β_i)(1 - α_i/β_i) \right)
\]

\[- \frac{1}{4} α_i β_i^3 (1 - α_i/β_i)σ_i^2 \].

4. **Asymptotic expansions of the three statistics**

The first factor in (2.1) can be evaluated as in Fujikoshi [2],

\[
(4.1) \quad \frac{(2/n)^{p+1} Γ_p \left( \frac{1}{2} n + it \sqrt{n} \right)}{Γ_p \left( \frac{1}{2} n \right)} = \exp \left[ -pt^2 - \frac{1}{2} \frac{p(p+1)it}{2} + \frac{2}{3} pt^3 \right]
\]

\[+ n^{-1} \left( \frac{1}{2} p(p+1)it + \frac{2}{3} pt^3 \right) + O(n^{-3/2}) \].

Putting \( α_i = -it \), \( β_i = 1/2 \), \( γ_i = -1 \), \( α_0 = β_1 = β_0 = γ_1 = γ_0 = 0 \) and \( R = θ \) in the-
orem 3.1, gives the asymptotic formula for the second factor in (2.1), which, combined with (4.1), proves the characteristic function of \( \sqrt{n} \cdot \log |S\Sigma^{-1}/n| - \sqrt{n} \log |I+2\theta| \) to be

\[
(4.2) \quad \exp \left[ -\varepsilon t \tau_1 / 2 \right] \left[ 1 - n^{-1/2} \left\{ g_j (it)^j + g_i it \right\} + n^{-1} \sum_{k=1}^3 (it)^k h_{2k} + O(n^{-1/2}) \right],
\]

where \( \tau_1 = 2(p - \sigma_2) \) for \( \sigma_j = \text{tr} \left( I - (I + 2\theta)^{-1} \right)^j \) and

\[
g_i = -\frac{1}{2} \sigma_i^2 + \frac{1}{2} \sigma_2 + \frac{1}{2} p(p+1),
\]

\[
g_i = 2\sigma_i - \frac{8}{3} \sigma_2 + \frac{2}{3} p,
\]

\[
h_2 = \frac{1}{8} \left[ \sigma_1^2 + \sigma_2 + p(p+1) \right] + \frac{5}{2} \sigma_1 + 2\sigma_2 \sigma_3 + \frac{1}{2} \sigma_2^2 - 4\sigma_1 - 4\sigma_2
\]

\[
+ \sigma_2 + \sigma_3 + \frac{1}{2} p(p+1),
\]

\[
h_4 = \frac{1}{8} \left( -\sigma_1 + \frac{4}{3} \sigma_2 - \frac{1}{3} p \right) \left[ \sigma_1^2 + \sigma_2 + p(p+1) \right] - \frac{20}{3} \sigma_6 + 16\sigma_4 - 10\sigma_5 + \frac{2}{3} p,
\]

\[
h_8 = \frac{1}{2} \left( -2\sigma_4 + \frac{8}{3} \sigma_3 - \frac{2}{3} p \right)^2.
\]

Inverting the characteristic function (4.2), we have

\[
(4.4) \quad P(\sqrt{n} \log |S\Sigma^{-1}/n| - \sqrt{n} \log |I+2\theta| < \tau_1 x)
\]

\[
= \Phi(x) + n^{-1/2} \left( g_j \Phi^{(j)}(x) / \tau_1 + g_i \Phi^{(j)}(x) / \tau_i \right)
\]

\[
+ n^{-1} \sum_{k=1}^3 h_{2k} \Phi^{(2k)}(x) \tau_1^{-2k} + O(n^{-1/2}).
\]

After some rearrangement in each coefficient in (4.3), we can see that the result is exactly the same as previously obtained in Sugiura and Nagao [8].

For problem (ii), the first factor in (2.2), namely, products of four gamma functions can be expressed from (2.7) in Sugiura and Fujikoshi [7] as

\[
(4.5) \quad 1 + it m^{-1/2} bp + m^{-1} (it)^j \left[ bp + \frac{1}{2} (bp)^2 \right] + O(m^{-3/2}).
\]

The second factor of \( C_j(t) \) can be evaluated by putting \( n = m, \alpha_i = -it, \beta_i = 1/2, \beta_1 = -it, \beta_6 = (b + p + 1)/4, \gamma_i = -1, \alpha_0 = \gamma_1 = \gamma_0 = 0, \) and \( R = \theta \) in Theorem 3.1. The same argument as for (4.4) yields

\[
(4.6) \quad P(-2\rho \log \lambda / \sqrt{m} - \sqrt{m} \log |I+2\theta| < \tau_2 x)
\]
\[ 
\Phi(x) + m^{-1/2} \left\{ u_1 \Phi^{(1)}(x) \tau_1^{-1} + u_4 \Phi^{(3)}(x) \tau_1^{-4} \right\} 
+ m^{-1} \sum_{k=1}^{3} v_{1k} \Phi^{(2k)}(x) \tau_1^{-2k} + O(m^{-3/2}) ,
\]

where \( \tau_1^2 = 4 \sigma_1 - 2 \sigma_2 \) for \( \sigma_j = \text{tr} \{ I - (I + 2 \theta)^{-1} \} \) and

\[
u_4 = -\frac{20}{3} \sigma_3 - 8 \sigma_2 + 4 \sigma_1,
\]

\[
(4.7) \quad \nu_4 = \frac{5}{2} \sigma_4 + \frac{1}{8} \sigma_4^4 + 2 \sigma_2 \sigma_1 + \frac{5}{8} \sigma_4^2 + \frac{1}{4} \sigma_4 \sigma_2 - (b + p + 7) \sigma_1 \\
- \frac{1}{4} (b + p + 25) \sigma_2 \sigma_4 - \frac{1}{4} (b + p + 1) \sigma_4^2 + \frac{1}{2} \sigma_1 (5b + 5p + pb + 13) \\
+ \sigma_1 \left[ \frac{1}{8} (b + p + 1)^3 + \frac{1}{2} bp + 4 \right] - (p + b + 1) \left( \frac{1}{2} pb + 2 \right) \sigma_1 \\
+ pb \left( \frac{1}{2} pb + 1 \right) ,
\]

\[
(4.8) \quad P(\sqrt{n} (\text{tr } S_2 S_1^{-1} - \rho_1 \rho_1^{-1} \text{tr } \Gamma') < \tau_2 x) \\
= \Phi(x) - n^{-1/2} \left\{ a_4 \Phi^{(1)}(x) \tau_1^{-1} + a_3 \Phi^{(3)}(x) \tau_1^{-3} \right\} \\
+ n^{-1} \sum_{k=1}^{3} b_{1k} \Phi^{(2k)}(x) \tau_1^{-2k} + O(n^{-3/2}) ,
\]

where \( \tau_1^2 = 2 \rho_1 \rho_1^{-1} \sigma_2 \) for \( \sigma_j = \text{tr } \Gamma' \) and

\[
\sigma_1 = \rho_1 \rho_1^{-1} (p + 1) \sigma_1 ,
\]

\[
\sigma_3 = \frac{4}{3} \rho_1 \rho_1^{-1} (1 + 2 \rho_1 / \rho_3) \sigma_1 ,
\]

Again this result agrees with Sugiura [6]. Finally for problem (iii), the moment generating function \( M(t) \) in (2.3) satisfies the same initial condition for \( \_F_1 \), namely, \( M(t) = 1 \) for \( \Gamma = 0 \). Putting \( \alpha_1 = \rho_1 / 2, \alpha_2 = 0, \beta_1 = - \rho_1 / 2, \beta_2 = (p + 1) / 2 \) and \( R = - \Gamma \) in Theorem 3.2, we obtain
\( b_2 = \rho_1 \rho_2^{-3} \left\{ (3p+4) \rho_1 \rho_3^{-1} + (2p+3) \right\} \sigma_1 + \rho_1 \rho_3^{-4} \left\{ 1 + \rho_1 (p+1)^2 / 2 \right\} \sigma_1 \),

\( b_4 = 2 \rho_1 \rho_2^{-3} (1 + 5 \rho_1 \rho_3^{-1} + 5 \rho_2 \rho_3^{-2}) \sigma_1 + \frac{4}{3} (p+1)(1+2 \rho_1 \rho_3^{-1}) \rho_2 \rho_3^{-4} \sigma_1 \sigma_3 ,

\( b_6 = \frac{8}{9} \rho_1 \rho_3^{-5} (1 + 2 \rho_1 \rho_3^{-1}) \sigma_3 \).

It is easily verified that the above result is the same as the Theorem 12.1 in Nagao [5].

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