NOTE ON THE ASYMPTOTIC DISTRIBUTIONS OF THE FUNCTIONS OF A MULTIVARIATE QUADRATIC FORM IN NORMAL SAMPLE

TAKESI HAYAKAWA

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1. Summary

The exact representations of the probability density function (p.d.f) of an multivariate quadratic form of central case in normal sample were obtained by Hayakawa [3], Khatri [7] and Shah [8] by the use of zonal polynomials and Laguerre polynomials. On the other hand, the exact p.d.f.'s of the latent roots of the multivariate quadratic form in non-central case were also treated by Hayakawa [4], [5] by the use of the new polynomials $P_\kappa(T, A)$.

In this paper we consider the asymptotic distribution of some functions of the multivariate quadratic form in central case under certain conditions.

2. Some useful results

The hypergeometric function of matrix argument is defined by

$$(1)\qquad {}_pF_q^{(m)}(a_1,\cdots,a_p;b_1,\cdots,b_q;S,T)=\sum_{k=0}^{\infty}\sum_{\kappa}\frac{\prod_{i=1}^{p}(a_i)_\kappa}{\prod_{j=1}^{q}(b_j)_\kappa}\frac{C_\kappa(S)C_\kappa(T)}{k!C_\kappa(I_m)}$$

where $C_\kappa(S)$ is a zonal polynomial of a symmetric matrix S corresponding to a partition κ of k, i.e., $k=k_1+\cdots+k_m$, $k_1\geqq k_2\geqq\cdots\geqq k_m\geqq 0$, $\kappa=\{k_1,\cdots,k_m\}$, and

$$(a)_\kappa=\prod_{\alpha=1}^{m}\left(a-\frac{\alpha-1}{2}\right)_{k_\alpha},\qquad (x)_n=x(x+1)\cdots(x+n-1)\,.$$

Special case of ${}_pF_q$ is

$$(2)\qquad {}_1F_0(a,S)=\det(I-S)^{-a}\,,\qquad \text{for } \|S\|<1\,,$$

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where $\|S\|$ denotes the maximum of all the absolute values of the characteristic roots of S.

(3) $$C_\kappa(I_n)=\frac{\chi_{[2\kappa]}(1)}{(2k-1)!!}2^k\left(\frac{n}{2}\right)_\kappa ,$$

where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group, [1], [6], and $(2k-1)!!=1\cdot 3\cdot 5\cdot \cdots \cdot(2k-1)$.

The following lemma is fundamental for our argument.

LEMMA (Fujikoshi [2]). *Let* $\|S\|<1$, *and put* $a_1(\kappa)=\sum_{\alpha=1}^{m}k_\alpha(k_\alpha-\alpha)$, $\kappa=\{k_1,\cdots,k_m\}$, *and* $U=S(I-S)^{-1}$. *Then*

(4.a) $$\sum_{k=1}^{\infty}\sum_{\kappa}\frac{(b)_\kappa}{(k-1)!}C_\kappa(S)=b\operatorname{tr}U\det(I-S)^{-b} ,$$

(4.b) $$\sum_{k=2}^{\infty}\sum_{\kappa}\frac{(b)_\kappa}{(k-2)!}C_\kappa(S)=b\{b(\operatorname{tr}U)^2+\operatorname{tr}U^2\}\det(I-S)^{-b} ,$$

(4.c) $$\sum_{k=3}^{\infty}\sum_{\kappa}\frac{(b)_\kappa}{(k-3)!}C_\kappa(S)$$
$$=b\{b^2(\operatorname{tr}U)^3+3b\operatorname{tr}U\operatorname{tr}U^2+2\operatorname{tr}U^3\}\det(I-S)^{-b} ,$$

(4.d) $$\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(b)_\kappa a_1(\kappa)}{k!}C_\kappa(S)=\frac{b}{2}\{(\operatorname{tr}U)^2+(2b+1)\operatorname{tr}U^2\}\det(I-S)^{-b} .$$

Note. (4.a), (4.b) and (4.d) are given by Fujikoshi and (4.c) is given by the author by the same way as [2].

3. The asymptotic distribution of $\det XAX'$

Let X be an $m\times n$ $(m\leqq n)$ matrix whose columns are normally distributed with mean 0 and covariance matrix Σ, independently. Let A_n be an $n\times n$ positive definite diagonal matrix whose diagonal elements are ordered descendingly, i.e., $a_1\geqq a_2\geqq\cdots\geqq a_n>0$ and let's assume that $\lim_{n\to\infty}a_n=q>0$. The multivariate quadratic form of the normal sample is defined as XA_nX', and the p.d.f. of $nZ=XAX'$ is given by Khatri [7] as follows;

(5) $$\frac{n^{mn/2}}{\Gamma_m(n/2)(\det 2\Sigma)^{n/2}(\det A_n)^{m/2}}\operatorname{etr}\left(-\frac{n}{2q}\Sigma^{-1}Z\right)(\det Z)^{n/2-p}$$
$$\cdot {}_0F_0^{(n)}\left(I-qA_n^{-1},\frac{n}{2q}\Sigma^{-1}Z\right) ,$$

where

$$\Gamma_m(a)=\pi^{m(m-1)/4}\prod_{\alpha=1}^{m}\Gamma\left(a-\frac{\alpha-1}{2}\right)$$

and

$$q>0\ ,\qquad p=\frac{m+1}{2}\ .$$

THEOREM 1. *If for any positive integer* n, $a_n>q/2>0$, *and* $\lim_{n\to\infty}\operatorname{tr}(A_n-qI_n)<\infty$, *then the asymptotic distribution function of*

$$\lambda=\sqrt{\frac{n}{2m}}\log\{\det Z/(\det q\textstyle\sum)\} \tag{6}$$

is given by

$$P\{\lambda\leqq x\}=\Phi(x)+\frac{1}{\sqrt{2mn}}\sum_{\alpha=1}^{2}l_{1\alpha}\Phi^{(2\alpha-1)}(x)+\frac{1}{2mn}\sum_{\alpha=1}^{3}l_{2\alpha}\Phi^{(2\alpha)}(x) \tag{7}$$
$$+\frac{1}{2mn\sqrt{2mn}}\sum_{\alpha=1}^{5}l_{3\alpha}\Phi^{(2\alpha-1)}(x)+O(1/n^2)\ ,$$

where

$$l_{11}=mp-m\operatorname{tr}(A_n/q-I_n)$$

$$l_{12}=\frac{1}{3}$$

$$l_{21}=\frac{1}{2}mp(mp+2)+\frac{m^2}{2}\{\operatorname{tr}(A_n/q-I_n)\}^2+m\operatorname{tr}(A_n/q-I_n)^2$$
$$-m^2p\operatorname{tr}(A_n/q-I_n)$$

$$l_{22}=\frac{1}{3}\{mp+1-m\operatorname{tr}(A_n/q-I_n)\}$$

$$l_{23}=\frac{1}{18}$$

$$l_{31}=\frac{1}{6}m^2(2m^2+3m-1)+m^2\{\operatorname{tr}(A_n/q-I_n)\}^2$$
$$+m^2(m+1)\operatorname{tr}(A_n/q-I_n)^2$$

$$l_{32}=\frac{1}{6}mp(mp+2)(mp+4)-\frac{1}{2}m^2p(mp+2)\operatorname{tr}(A_n/q-I_n)$$
$$+\frac{m^3p}{2}\{\operatorname{tr}(A_n/q-I_n)\}^2+m^2p\operatorname{tr}(A_n/q-I_n)^2-\frac{m^3}{6}\{\operatorname{tr}(A_n/q-I_n)\}^3$$
$$-m^2\operatorname{tr}(A_n/q-I_n)\operatorname{tr}(A_n/q-I_n)^2-\frac{m}{3}\operatorname{tr}(A_n/q-I_n)^3$$

$$l_{33}=\frac{1}{30}(5m^2p^2+20mp+12)-\frac{1}{3}m(mp+1)\operatorname{tr}(A_n/q-I_n)$$
$$+\frac{m^2}{6}\{\operatorname{tr}(A_n/q-I_n)\}^2+\frac{m}{3}\operatorname{tr}(A_n/q-I_n)^2$$

$$l_{34}=\frac{1}{18}(mp+2-m\operatorname{tr}(A_n/q-I_n))$$

$$l_{35}=\frac{1}{162}\ .$$

If we set $A_n=I_n$ and $q=1$, then we have the asymptotic distribution of the generalized variance of normal sample covariance matrix.

PROOF. The characteristic function of λ is expressed as

$$\varphi(t)=E[\exp(it\lambda)]=\varphi_0(t)\varphi_A(t)\ , \tag{9}$$

where

$$\varphi_0(t)=\left(\frac{2}{n}\right)^{it\sqrt{mn/2}}\frac{\Gamma_m(n/2+it\sqrt{n/2m})}{\Gamma_m(n/2)} \tag{10}$$

$$\varphi_A(t)=\frac{q^{mn/2}}{(\det A_n)^{m/2}}\,{}_1F_0^{(n)}\left(\frac{n}{2}+it\sqrt{\frac{n}{2m}}\,;\,I-qA_n^{-1},\,I_m\right)\ . \tag{11}$$

Since $a_n>q/2$, for all n, ${}_1F_0^{(n)}$ converges.

It is well known that $\varphi_0(t)$ is expanded by the use of Stirling's type asymptotic expansion for gamma function as follows;

$$\begin{aligned}\varphi_0(t)=\exp\left(-\frac{t^2}{2}\right)\Bigg[1&-\frac{1}{\sqrt{2mn}}\left\{mp(it)+\frac{1}{3}(it)^3\right\}\\&+\frac{1}{2mn}\left\{\frac{1}{2}mp(mp+2)(it)^2+\frac{1}{3}(mp+1)(it)^4+\frac{1}{18}(it)^6\right\}\\&-\frac{1}{2mn\sqrt{2mn}}\left\{\frac{1}{6}m^2(2m^2+3m-1)(it)\right.\\&+\frac{1}{6}mp(mp+2)(mp+4)(it)^3+\frac{1}{30}(5m^2p^2+20mp+12)(it)^5\\&\left.+\frac{1}{18}(mp+2)(it)^7+\frac{1}{162}(it)^9\right\}+O(1/n^2)\Bigg]\ .\end{aligned} \tag{12}$$

$\varphi_A(t)$ is also expressed as

$$\varphi_A(t)=\frac{q^{mn/2}}{(\det A_n)^{m/2}}\sum_{k=0}^{\infty}\sum_{\kappa}\frac{(m/2)_\kappa}{k!}C_\kappa(I_n-qA_n^{-1})\left\{1+\frac{2it}{\sqrt{2mn}}k\right. \tag{13}$$

$$+\frac{2(it)^2}{2mn}k(k-1)-\frac{1}{2mn\sqrt{2mn}}\left\{4m(it)a_1(\kappa)-\frac{4}{3}(it)^3k(k-1)(k-2)\right\}$$
$$+O(1/n^2)\Big\} ,$$

since $C_\kappa(I_m)/C_\kappa(I_n)=(m/2)_\kappa/(n/2)_\kappa$ by (3). Hence by applying lemma, we have

$$(14)\qquad \varphi_A(t)=1+\frac{it}{\sqrt{2mn}}m\operatorname{tr}(A_n/q-I_n)$$
$$+\frac{(it)^2}{2mn}\left\{\frac{m^2}{2}\{\operatorname{tr}(A_n/q-I_n)\}^2+m\operatorname{tr}(A_n/q-I_n)^2\right\}$$
$$-\frac{1}{2mn\sqrt{2mn}}\Big\{(m^2(\operatorname{tr}(A_n/q-I_n))^2+m^2(m+1)\operatorname{tr}(A_n/q-I_n)^2)(it)$$
$$-\frac{4}{3}(it)^3\Big(\frac{m^3}{8}\{\operatorname{tr}(A_n/q-I_n)\}^3$$
$$+\frac{3}{4}m^2\operatorname{tr}(A_n/q-I_n)\operatorname{tr}(A_n/q-I_n)^2$$
$$+m\operatorname{tr}(A_n/q-I_n)^3\Big)\Big\}+O(1/n^2) .$$

By combining $\varphi_0(t)$ and $\varphi_A(t)$, we have the asymptotic expansion of $\varphi(t)$ as following way.

$$(15)\qquad \varphi(t)=\exp\left(-\frac{t^2}{2}\right)\left[1-\frac{1}{\sqrt{2mn}}\sum_{\alpha=1}^{2}l_{1\alpha}(it)^{2\alpha-1}+\frac{1}{2mn}\sum_{\alpha=1}^{3}l_{2\alpha}(it)^{2\alpha}\right.$$
$$\left.-\frac{1}{2mn\sqrt{2mn}}\sum_{\alpha=1}^{5}l_{3\alpha}(it)^{2\alpha-1}+O(1/n^2)\right] ,$$

where $l_{1\alpha}$, $l_{2\alpha}$, $l_{3\alpha}$'s are given in (8).

Therefore, by inverting (15), we have (7).

4. Asymptotic distribution of $\det XA_NX'/\det(XA_NX'+S)$

Khatri [7] has studied the p.d.f. of $XA_NX'(XA_NX'+S)^{-1}$ and the moment of $\det XA_NX'/\det(XA_NX'+S)$, where S is a central Wishart matrix with b degrees of freedom and S is independent with XA_NX'. In this section, we obtain the asymptotic distribution of $\det XA_NX'/\det(XA_NX'+S)$.

THEOREM 2. *Let $Z=XA_NX'$ be distributed with* p.d.f. (5) *and S be as above.*

Put

$$(16)\qquad \lambda=-\rho N\log\{\det XA_NX'/\det(XA_NX'+S)\} ,$$

where $\rho N = N+(b-m-1)/2 = n$, then the asymptotic distribution of λ under the conditions, $a_N > 1/2$, and $\lim_{N\to\infty} \operatorname{tr}(A_N - I_N) < \infty$, is given as follows.

$$P(\lambda \leqq x) = P(\chi_f^2 \leqq x) + \frac{1}{2n} bm \operatorname{tr}(A_N - I_N)[P(\chi_f^2 \leqq x) - P(\chi_{f+2}^2 \leqq x)] - \frac{bm}{48n^2}\sum_{\alpha=0}^{2} l_\alpha P(\chi_{f+2\alpha}^2 \leqq x) + O(1/n^3) . \tag{17}$$

where

$$\begin{aligned} l_0 &= 6\{(bm-2)\{\operatorname{tr}(A_N - I_N)\}^2 + 2(b-m-1)\operatorname{tr}(A_N - I_N)^2 \\ &\quad + 2(b-m-1)\operatorname{tr}(A_N - I_N)\} + (b^2+m^2-5) , \\ l_1 &= -12b[m\{\operatorname{tr}(A_N - I_N)\}^2 + 2\operatorname{tr}(A_N - I_N)^2 + 2\operatorname{tr}(A_N - I_N)] , \\ l_2 &= 6\{(bm+2)\{\operatorname{tr}(A_N - I_N)\}^2 + 2(b+m+1)\operatorname{tr}(A_N - I_N)^2 \\ &\quad + 2(b+m+1)\operatorname{tr}(A_N - I_N)\} - (b^2+m^2-5) , \end{aligned} \tag{18}$$

where $f = bm/2$, and χ_f^2 is a χ^2-random variable with f degrees of freedom.

PROOF. Since the hth moment of $\det XAX'/\det(XAX'+S)$ was given by Khatri ([7], (47)), which is not valid, the characteristic function of $-\rho N \log\{\det XAX'/\det(XAX'+S)\}$ can be expressed as

$$\varphi(t) = \varphi_0(t)\varphi_A(t) , \tag{19}$$

where

$$\varphi_0(t) = \frac{\Gamma_m(N/2 - it\rho N)\Gamma_m(N/2+b/2)}{\Gamma_m(N/2)\Gamma_m(N/2+b/2-it\rho N)} \tag{20}$$

$$\varphi_A(t) = (\det A_N)^{-m/2} {}_2F_1^{(N)}\left(\frac{N+b}{2}, \frac{N}{2} - it\rho N; \frac{N+b}{2} - it\rho N; I - A_N^{-1}, I_m\right) . \tag{21}$$

Hence, $\varphi_0(t)$ is the c.f. of the usual likelihood ratio criterion, and $\varphi_A(t)$ expresses the certain difference from the usual likelihood ratio criterion. By the same ways as Theorem 1, we can expand $\varphi(t)$ as the following form.

$$\varphi(t) = \frac{1}{(1-2it)^f}\left[1 + \frac{bm}{2n}\operatorname{tr}(A_N - I_N)\left\{1 - \frac{1}{1-2it}\right\} - \frac{bm}{48n^2}\sum_{\alpha=0}^{2} l_\alpha (1-2it)^{-\alpha}\right] + O(1/n^3) , \tag{22}$$

where l_1, l_2, l_3 are given by (18). By inverting $\varphi(t)$, we have (17).

INDIANA UNIVERSITY AND INSTITUTE OF STATISTICAL MATHEMATICS

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