CHARACTERIZATION OF MATUSITA'S MEASURE OF AFFINITY*

A. M. MATHAI AND P. N. RATHIE

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1. Introduction and summary

Matusita ([5]–[8]) introduced and discussed measures of 'affinity' and 'distance' between two statistical populations. This article is mainly concerned with two types of characterizations of 'affinity' and 'distance' when the populations are discrete. One is based on a recurrence relation and the other deals with a maximization principle. By using the main results obtained in this article, characterization theorems are also given for Bhattacharyya's measure of distance ([1], [2]), Jeffreys' measure of invariance ([1], [3]), Pearson's measure of discrepancy [1] and a generalized measure of dispersion introduced by Mathai [4]. Alternate definitions of 'affinity' and 'distance,' as solutions of certain functional equations, are also suggested in this article.

Consider two discrete distributions given by the probabilities,

$$(1.1)\qquad (p_1,\cdots,p_n);\ p_i\geqq 0,\ i=1,2,\cdots,n;\ \sum_{i=1}^{n} p_i=1$$

and

$$(1.2)\qquad (q_1,\cdots,q_n);\ q_i\geqq 0,\ i=1,2,\cdots,n;\ \sum_{i=1}^{n} q_i=1\ .$$

Matusita's measure of 'affinity' between the populations (1.1) and (1.2) is defined as follows.

$$(1.3)\qquad \rho_n=\sum_{i=1}^{n} p_i^{1/2}q_i^{1/2},\ \sum_{i=1}^{n} p_i=1=\sum_{i=1}^{n} q_i\ ,\qquad p_i,q_i\geqq 0,\ i=1,\cdots,n\ .$$

The square of Matusita's distance between (1.1) and (1.2) is given as,

$$(1.4)\qquad D_n=\sum_{i=1}^{n} (p_i^{1/2}-q_i^{1/2})^2=2(1-\rho_n)\ .$$

The aim of this article is to give characterization theorems for ρ_n and D_n. Before discussing the main results, a few other measures will

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also be mentioned here. Bhattacharyya's measure of distance between (1.1) and (1.2) is,

$$\phi=\cos^{-1}\rho_n \,. \tag{1.5}$$

Mathai [4] defined a general measure of dispersion in a statistical population with the help of four axioms. One measure of distance between (1.1) and (1.2) which can be derived from the general measure of dispersion in [4] is,

$$P_{rs}=\left[\sum_{i=1}^{n}|p_i^{1/s}-q_i^{1/s}|^r\right]^{1/r}, \qquad r\geqq 1,\ s\geqq 1\,. \tag{1.6}$$

When $r=s=m$ the mth power of P_{rs} gives Jeffreys' measure of invariance, namely,

$$I_m=\sum_{i=1}^{n}|p_i^{1/m}-q_i^{1/m}|^m\,. \tag{1.7}$$

Pearson's measure of discrepancy between (1.1) and (1.2), denoted by M is,

$$M=4nI_2\,, \tag{1.8}$$

where I_m is given in (1.7), see [1]. With the help of the characterization theorems obtained in this article, the different measures in (1.5), (1.6), (1.7) and (1.8) will also be characterized.

2. Characterization of 'affinity' by recursivity property

Theorem 2.1 deals with the characterization of ρ_n with the help of three postulates. These are recursivity, symmetry and normalization postulates which can also be justified intuitively. Theorem 2.1 will be proved with the help of Lemmas 2.1 and 2.2.

Let $\rho_n\begin{pmatrix}p_1,\cdots,p_n\\q_1,\cdots,q_n\end{pmatrix}$ be a function of $p_1,\cdots,p_n$ and $q_1,\cdots,q_n$, where $p_i, q_i\geqq 0$, $i=1,2,\cdots,n$, $\sum_{i=1}^{n}p_i=1=\sum_{i=1}^{n}q_i$, satisfying the following postulates.

P_1: *Recursivity*

$$\rho_n\begin{pmatrix}p_1,\cdots,p_n\\q_1,\cdots,q_n\end{pmatrix}=\rho_{n-1}\begin{pmatrix}p_1+p_2,\ p_3,\cdots,p_n\\q_1+q_2,\ q_3,\cdots,q_n\end{pmatrix}+(p_1+p_2)^{1/2}(q_1+q_2)^{1/2}$$
$$\cdot\left[\rho_2\begin{pmatrix}p_1/(p_1+p_2),\ p_2/(p_1+p_2)\\q_1/(q_1+q_2),\ q_2/(q_1+q_2)\end{pmatrix}-1\right],$$

for all $n>2$, $p_1+p_2, q_1+q_2>0$.

P_2: *Symmetry*

$$\rho_3 \text{ is symmetric in pairs } \begin{Bmatrix} p_i \\ q_i \end{Bmatrix}, \qquad i=1, 2, 3 .$$

P_3: *Normalization*

$$\rho_2\begin{pmatrix} 1/4, 3/4 \\ 3/4, 1/4 \end{pmatrix} = \cos(\pi/6) .$$

It will be shown that the only function ρ_n satisfying the postulates P_1, P_2 and P_3 is Matusita's measure of 'affinity'. Postulate P_1 is the main postulate and it explains the desired nature of combinations of the measures to be taken when the union of two events are considered. In other words P_1 gives an idea about what happens to the measure when an event is subdivided into two mutually exclusive events. P_2 is a desired property for a measure of 'affinity' and P_3 is only a normalization property.

Let,

$$g(x, y) = \rho_2\begin{pmatrix} x, 1-x \\ y, 1-y \end{pmatrix} - 1 , \qquad x, y \in I = [0, 1] . \tag{2.1}$$

LEMMA 2.1.

$$g(x, y) = g(1-x, 1-y) , \qquad \text{for } x, y \in I . \tag{2.2}$$

PROOF. From postulate P_2 for $n=3$ we have,

$$\rho_3\begin{pmatrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{pmatrix} = \rho_3\begin{pmatrix} p_2, p_1, p_3 \\ q_2, q_1, q_3 \end{pmatrix} , \qquad p_1+p_2+p_3 = q_1+q_2+q_3 = 1 . \tag{2.3}$$

From P_1 we have,

$$\begin{aligned} &\rho_2\begin{pmatrix} p_1+p_2, p_3 \\ q_1+q_2, q_3 \end{pmatrix} + (p_1+p_2)^{1/2}(q_1+q_2)^{1/2}\rho_2\begin{pmatrix} p_1/(p_1+p_2), p_2/(p_1+p_2) \\ q_1/(q_1+q_2), q_2/(q_1+q_2) \end{pmatrix} \\ &= \rho_2\begin{pmatrix} p_2+p_1, p_3 \\ q_2+q_1, q_3 \end{pmatrix} + (p_2+p_1)^{1/2}(q_2+q_1)^{1/2}\rho_2\begin{pmatrix} p_2/(p_2+p_1), p_1/(p_2+p_1) \\ q_2/(q_2+q_1), q_1/(q_2+q_1) \end{pmatrix} . \end{aligned} \tag{2.4}$$

Now by cancelling the first terms on both sides of (2.4) the lemma is proved. Also by taking $x=y=0$ we have,

$$g(0, 0) = g(1, 1) . \tag{2.5}$$

LEMMA 2.2. *$g(x, y)$ satisfies the functional equation*

$$\begin{aligned} &g(x, y) + (1-x)^{1/2}(1-y)^{1/2} g[u/(1-x), v/(1-y)] \\ &= g(u, v) + (1-u)^{1/2}(1-v)^{1/2} g[x/(1-u), y/(1-v)] , \end{aligned} \tag{2.6}$$

for $x, y, u, v \in [0, 1)$, $x+u, y+v \in I$.

PROOF. From P_2 for $n=3$,

$$\rho_3\binom{p_1, p_2, p_3}{q_1, q_2, q_3} = \rho_3\binom{p_3, p_2, p_1}{q_3, q_2, q_1} . \tag{2.7}$$

From (2.7), P_1 and Lemma 2.1, we have

$$\begin{aligned} g(p_3, q_3)&+(p_1+p_2)^{1/2}(q_1+q_2)^{1/2}g[p_1/(p_1+p_2), q_1/(q_1+q_2)] \\ &=g(p_1, q_1)+(p_2+p_3)^{1/2}(q_2+q_3)^{1/2}g[p_3/(p_2+p_3), q_3/(q_2+q_3)] . \end{aligned} \tag{2.8}$$

Since $p_1+p_2+p_3=1=q_1+q_2+q_3$, (2.8) reduces to the following.

$$\begin{aligned} g(p_3, q_3)&+(1-p_3)^{1/2}(1-q_3)^{1/2}g[p_1/(1-p_3), q_1/(1-q_3)] \\ &=g(p_1, q_1)+(1-p_1)^{1/2}(1-q_1)^{1/2}g[p_3/(1-p_1), q_3/(1-q_1)] . \end{aligned} \tag{2.9}$$

Now by putting $p_3=x$, $q_3=y$, $p_1=u$ and $q_1=v$, Lemma 2.2 is proved. When P_1 and Lemma 2.1 are used the conditions $p_1\neq 1$, $q_1\neq 1$, $p_3\neq 1$, $q_3\neq 1$ are automatically satisfied. Now it will be shown that any function satisfying the functional equation (2.6) is Matusita's measure of 'affinity.' That is, this can also be taken as an alternate definition for 'affinity.'

THEOREM 2.1.

$$\rho_n\binom{p_1, \cdots, p_n}{q_1, \cdots, q_n} = \sum_{i=1}^{n} p_i^{1/2} q_i^{1/2}$$

is the only function ρ_n satisfying the postulates P_1, P_2 and P_3. In other words P_1, P_2 and P_3 uniquely determine ρ_n as $\sum_{i=1}^{n} p_i^{1/2} q_i^{1/2}$.

PROOF. This theorem will be proved by showing that $g(x, y)$ defined in (2.2) is of the form $x^{1/2}y^{1/2}+(1-x)^{1/2}(1-y)^{1/2}-1$.

Putting $u/(1-x)=p$, $v/(1-y)=q$, $1-x=r$, $1-y=s$ in (2.6) gives

$$\begin{aligned} g(r, s)&+r^{1/2}s^{1/2}g(p, q) \\ &=g(pr, qs)+(1-pr)^{1/2}(1-qs)^{1/2} \\ &\quad\cdot g[(1-r)/(1-pr), (1-s)/(1-qs)] , \\ &\qquad \text{for } r, s \in (0, 1],\ p, q \in I \text{ with } pr\neq 1,\ qs\neq 1 . \end{aligned} \tag{2.10}$$

Let,

$$\begin{aligned} f(p, q, r, s)&=g(r, s)+[r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2}]g(p, q) , \\ &\qquad \text{for } p, q, r, s \in (0, 1) . \end{aligned} \tag{2.11}$$

We will show that $f(p, q, r, s)$ is symmetric in pairs (p, r) and (q, s). Now by using (2.10) successively we get,

$$(2.12)\quad f(p, q, r, s)=g(pr, qs)+(1-pr)^{1/2}(1-qs)^{1/2} \cdot g[(1-r)/(1-pr), (1-s)/(1-qs)] +(1-r)^{1/2}(1-s)^{1/2}g(p, q)$$

$$(2.13)\quad =g(pr, qs)+(1-pr)^{1/2}(1-qs)^{1/2} \cdot \{g[(1-r)/(1-pr), (1-s)/(1-qs)] +[(1-r)/(1-pr)]^{1/2}[(1-s)/(1-qs)]^{1/2}g(p, q)\}$$

$$(2.14)\quad =g(pr, qs)+(1-pr)^{1/2}(1-qs)^{1/2} \cdot \{g[p(1-r)/(1-pr), q(1-s)/(1-qs)] +[(1-p)/(1-pr)]^{1/2}[(1-q)/(1-qs)]^{1/2}g(r, s)\} .$$

Now by using the result $g(r, s)=g(1-r, 1-s)$ we have,

$$(2.15)\quad f(p, q, r, s)=g(pr, qs)+(1-pr)^{1/2}(1-qs)^{1/2} \cdot \{g[(1-p)/(1-pr), (1-q)/(1-qs)] +[(1-p)/(1-pr)]^{1/2}[(1-q)/(1-qs)]^{1/2}g(r, s)\} .$$

Now comparing (2.13) and (2.15) we see that $f(p, q, r, s)$ is symmetric in (p, r) and (q, s). Therefore, from (2.10) and (2.11),

$$(2.16)\quad f(p, q, r, s)=g(p, q)+[p^{1/2}q^{1/2}+(1-p)^{1/2}(1-q)^{1/2}]g(r, s) =g(r, s)+[r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2}]g(p, q) .$$

That is,

$$(2.17)\quad g(r, s)=[r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2}-1]g(p, q)/ [p^{1/2}q^{1/2}+(1-p)^{1/2}(1-q)^{1/2}-1] .$$

Now p and q are at our choice subject to the condition $p, q \in (0, 1)$. By using the condition given by postulate P_8 the second factor is cancelled and (2.17) yields,

$$(2.18)\quad g(r, s)=r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2}-1 , \qquad r, s \in (0, 1) .$$

That is,

$$(2.19)\quad \rho_2(r, s)=r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2} .$$

It may be noticed that $\rho_2(r, s)=1$ when $r=s$. This agrees with the convention that the measure of affinity is maximum when the vectors $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ coincide. Now we will extend (2.17) to the closed interval $[0, 1]$. To this end we have to show that (2.17) holds for $g(0, y)$ and $g(x, 0)$. Since $g(x, y)=g(1-x, 1-y)$ the other points follow automatically. By putting $p=q=1$ in (2.10) we get,

$$(2.20)\quad [r^{1/2}s^{1/2}-(1-r)^{1/2}(1-s)^{1/2}]g(1, 1)=0 \qquad \text{for all } r, s \in (0, 1) .$$

That is,

$$g(1,1)=0=g(0,0)\,. \tag{2.21}$$

By putting $p=0$ and $s=1$ in (2.10) we get

$$\begin{aligned}(1-r^{1/2})g(0,q)&=g(r-1)-(1-q)^{1/2}g(1-r,0)\\ &=[1-(1-q)^{1/2}]g(r,1)\,, \qquad r\in(0,1],\ q\in[0,1)\,.\end{aligned} \tag{2.22}$$

By putting $p=0$ and $s=1/2$ in (2.10) we get

$$g(r,1/2)+(r/2)^{1/2}g(0,q)=g(0,q/2)+(1-q/2)^{1/2}g[1-r,1/(2-q)] \tag{2.23}$$

for $r\in(0,1]$, $q\in I$. Now by putting $r=1/2$ and using the value of $g(x,y)$ given in (2.18) for points inside (0, 1) we get

$$g(0,q)/2=g(0,q/2)+(1-q/2)^{1/2}[1/(4-2q)^{1/2}+(1-q)^{1/2}/(4-2q)^{1/2}-1] \tag{2.24}$$

for $q\in[0,1)$. Now substituting in (2.24) the values of $g(0,q)$ and $g(0, q/2)$ from (2.22) to (2.24) we get

$$\begin{aligned}&g(r,1)\{[1-(1-q)^{1/2}]/[2(1-r^{1/2})]-[1-(1-q/2)^{1/2}]/(1-r^{1/2})\}\\ &\quad=(1-q/2)^{1/2}[1/(4-2q)^{1/2}+(1-q)^{1/2}/(4-2q)^{1/2}]\,.\end{aligned} \tag{2.25}$$

That is,

$$g(r,1)=r^{1/2}-1 \qquad \text{for } r\in(0,1] \qquad \text{and} \tag{2.26}$$

$$g(0,q)=(1-q)^{1/2}-1 \qquad \text{for } q\in[0,1)\,. \tag{2.27}$$

Now we have $g(0,y)$ for $y\in[0,1)$ and $g(x,1)$ for $x\in(0,1]$. We already have $g(x,y)=g(1-x,1-y)$. So we need only $g(0,1)$ more. Now putting $q=1$ and $r=1/2$ in (2.23) and using (2.26) and (2.27) one gets,

$$g(0,1)=-1\,. \tag{2.28}$$

This completes the proof that,

$$g(r,s)=r^{1/2}s^{1/2}+(1-r)^{1/2}(1-s)^{1/2}-1 \qquad \text{for } r,s\in I\,. \tag{2.29}$$

Now by using the recurrence relation in postulate P_1 successively we have,

$$\begin{aligned}\rho_n\binom{p_1,\cdots,p_n}{q_1,\cdots,q_n}-1&=\rho_{n-1}\binom{p_1+p_2,\,p_3,\cdots,p_n}{q_1+q_2,\,q_3,\cdots,q_n}-1+(p_1+p_2)^{1/2}(q_1+q_2)^{1/2}\\ &\quad\cdot\left\{\rho_2\begin{bmatrix}p_1/(p_1+p_2),\ p_2/(p_1+p_2)\\ q_1/(q_1+q_2),\ q_2/(q_1+q_2)\end{bmatrix}-1\right\}\end{aligned} \tag{2.30}$$

$$=\sum_{i=2}^{n} r_i^{1/2}s_i^{1/2}\left\{\rho_2\begin{bmatrix}p_i/r_i,\ 1-p_i/r_i\\ q_i/s_i,\ 1-q_i/s_i\end{bmatrix}-1\right\} \tag{2.31}$$

where $r_i = p_1 + p_2 + \cdots + p_i$, $s_i = q_1 + q_2 + \cdots + q_i$;

$$= \sum_{i=2}^{n} r_i^{1/2} s_i^{1/2} [(p_i q_i / r_i s_i)^{1/2} + (1 - p_i/r_i)^{1/2} (1 - q_i/s_i)^{1/2} - 1] \tag{2.32}$$

$$= \sum_{i=2}^{n} p_i^{1/2} q_i^{1/2} + \sum_{i=2}^{n} r_{i-1}^{1/2} s_{i-1}^{1/2} - \sum_{i=2}^{n} r_i^{1/2} s_i^{1/2} \tag{2.33}$$

$$= \sum_{i=2}^{n} p_i^{1/2} q_i^{1/2} + r_1^{1/2} s_1^{1/2} - r_n^{1/2} s_n^{1/2} \tag{2.34}$$

$$= \sum_{i=1}^{n} p_i^{1/2} q_i^{1/2} - 1 \ . \tag{2.35}$$

That is,

$$\rho_n = \sum_{i=1}^{n} p_i^{1/2} q_i^{1/2} \ . \tag{2.36}$$

This completes the proof. By using Theorem 2.1 Matusita's measure of distance D_n and Bhattacharyya's measure of distance can be characterized in a similar way. These are stated in Theorems 2.2 and 2.3 without proof.

THEOREM 2.2. *Let*

$$D_n = 2(1 - \rho_n) \tag{2.37}$$

where ρ_n *is any function of* $p_1, \cdots, p_n$ *and* $q_1, \cdots, q_n$, $p_i, q_i \geqq 0$, $i = 1, 2, \cdots, n$, $\sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i$, *satisfying the postulates* P_1, P_2 *and* P_3 *then* D_n *is uniquely determined as* $\sum_{i=1}^{n} (p_i^{1/2} - q_i^{1/2})^2$.

THEOREM 2.3. *Let* $\phi = \cos^{-1} \rho_n$ *where* ρ_n *be as defined in Theorem* 2.2 *and satisfying the postulates* P_1, P_2 *and* P_3. *Then* ϕ *is uniquely determined as Bhattacharyya's measure of distance between the populations* $(p_1, \cdots, p_n)$ *and* $(q_1, \cdots, q_n)$.

It should be remarked that the structure of D_n and ϕ as given in Theorems 2.2 and 2.3 need not be assumed. Theorems 2.2 and 2.3 can be restated by modifying the postulates P_1, P_2 and P_3 in the light of the structures of D_n and ϕ given in Theorems 2.2 and 2.3. Also since the p_i's and q_i's are restricted to be non-negative $0 \leqq \phi \leqq \pi/2$ and ϕ is uniquely determined once $\cos\phi$ is uniquely determined. As stated earlier Jeffreys' invariance I_m for $m = 2$ and Pearson's measure of discrepancy which is $4nI_2$ are also characterized by Theorem 2.2.

3. Characterization of Matusita's measure of distance through a maximization principle

Let,

$$K_n\begin{bmatrix} p_1, \cdots, p_n \\ q_1, \cdots, q_n \end{bmatrix}, \qquad p_i, q_i > 0,\ i=1, \cdots, n;\ \sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i,\ n \geqq 3,$$

be any function of p_i's and q_i's satisfying the following postulates.

Postulates:

Q_1: *Structure*

$$K_n = \sum_{i=1}^{n} p_i^{1/2}(f(p_i) - f(q_i)) \qquad \text{for some function } f(x).$$

Q_2: *Non-negativity*

$$K_n \geqq 0 \qquad \text{for all } n \geqq 3.$$

Q_3: *Normalization*

$$K_2\begin{bmatrix} 1/4, 3/4 \\ 3/4, 1/4 \end{bmatrix} = 4 \sin^2(\pi/12).$$

Here postulates Q_1 and Q_2 give some sort of a maximization principle. In other words the function $\sum_{i=1}^{n} p_i^{1/2} f(q_i)$ has the maximum value when $q_i = p_i$, $i=1, 2, \cdots, n$.

THEOREM 3.1. *K_n satisfying the postulates Q_1, Q_2 and Q_3 is uniquely determined as,*

$$K_n = D_n = 2\left(1 - \sum_{i=1}^{n} p_i^{1/2} q_i^{1/2}\right).$$

PROOF. In the inequality

$$\sum p_i^{1/2} f(q_i) \leqq \sum p_i^{1/2} f(p_i), \qquad p_i, q_i > 0,\ i=1, \cdots, n\ \sum p_i = 1 = \sum q_i,\ n \geqq 3, \tag{3.1}$$

put $p_i = q_i$ for $i=3, 4, \cdots, n$, then one has,

$$p_1^{1/2} f(q_1) + p_2^{1/2} f(q_2) \leqq p_1^{1/2} f(p_1) + p_2^{1/2} f(p_2) \qquad \text{for } p_1, p_2, q_1, q_2 > 0,\ p_1 + p_2 = q_1 + q_2 < 1. \tag{3.2}$$

Now we will show that every solution of (3.2) is differentiable everywhere in (0, 1) and the solutions are given by

$$f(x) = a x^{1/2} + b \tag{3.3}$$

where $a\geqq 0$ and b are real arbitrary constants. First we will prove that $f(x)$ is monotonic increasing. From (3.2) we have

$$(p_1/p_2)^{1/2}[f(q_1)-f(p_1)]\leqq f(p_2)-f(q_2)\ . \tag{3.4}$$

Now interchanging (p_1, q_1) and (p_2, q_2) which is possible due to symmetry in (3.2) and adding the resultant expression to (3.4) we get

$$[(p_1/p_2)^{1/2}-(q_1/q_2)^{1/2}][f(q_1)-f(p_1)]\leqq 0\ . \tag{3.5}$$

Let $q_1>p_1$ then $q_2<p_2$ and thus from (3.5) we have

$$f(q_1)\geqq f(p_1)\ . \tag{3.6}$$

This implies that $f(x)$ is monotonic increasing and therefore $f(x)$ is differentiable almost everywhere in (0, 1). Now we will prove that $f(x)$ is differentiable everywhere in (0, 1). In (3.2) let $q_1=p_1+\delta$, $q_2=p_2-\delta$ with $\delta>0$. Then by rearranging the terms we get

$$p_1^{1/2}[f(p_1+\delta)-f(p_1)]/\delta\leqq p_2^{1/2}[f(p_2)-f(p_2-\delta)]/\delta\ . \tag{3.7}$$

Let p_1, p_2 be points where $f(x)$ is differentiable. From (3.7) we have,

$$p_1^{1/2}f'(p_1)\leqq p_2^{1/2}f'(p_2) \qquad \text{for all } p_1, p_2 \text{ such that } p_1+p_2<1\ . \tag{3.8}$$

Therefore by interchanging p_1 and p_2 and using (3.8) we get

$$p^{1/2}f'(p)=c \tag{3.9}$$

for all p for which $f(p)$ is differentiable where c is a constant. Since $f(x)$ is monotonic increasing $c\geqq 0$. Let p be an arbitrary point in (0, 1). Let p_1 be a point where $f(x)$ is differentiable. Then in (3.7) taking p_2 to be p, taking the infimum as $\delta\to 0$ and by using (3.9) we get

$$c\leqq p^{1/2}D_-f(p)\ . \tag{3.10}$$

In (3.7) putting $p_1=p$, taking the supremum as $\delta\to 0$ and using (3.9) we get

$$p^{1/2}D^+f(p)\leqq c\ . \tag{3.11}$$

Now starting from (3.2), making appropriate substitutions, proceeding as before and combining (3.10) and (3.11) we see that $f(x)$ is differentiable everywhere in (0, 1) with (3.9) valid for all p in (0, 1). Hence (3.3) follows with $c=a/2$. From postulates Q_1 and Q_3 we get $a=2$ and this completes the proof. That is,

$$K_n=2\left(1-\sum_{i=1}^{n} p_i^{1/2}q_i^{1/2}\right)\ . \tag{3.12}$$

It may be noticed that theorems corresponding to Theorem 3.1 can be given for Matusita's measure of affinity ρ_n, Bhattacharyya's measure of distance ϕ and Jeffreys' invariance I_m for $m=2$.

4. Some generalizations

In Section 1 it is mentioned that a general measure of distance between the populations $(p_1, \cdots, p_n)$ and $(q_1, \cdots, q_n)$ can be given as follows.

$$(4.1) \qquad H_n = \left\{ \sum_{i=1}^{n} |p_i^{1/s} - q_i^{1/s}|^r \right\}^{1/r} \qquad \text{for fixed } r, s \geqq 1,$$
$$p_i, q_i \geqq 0, \ i=1, \cdots, n, \ \sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i \ .$$

Thus when r is an even integer, say $r=2t$, one has

$$(4.2) \qquad [H_n]^{2t} = \sum_{j=0}^{2t} \binom{2t}{j} (-1)^j \left[\sum_{i=1}^{n} p_i^{j/s} q_i^{(2t-j)/s} \right] .$$

This show that (4.2) can be characterized by postulates similar to the ones introduced in Section 2. In this section we consider the case when r is an even integer. Other cases of r and s will not be discussed here. The main theorem is as follows.

THEOREM 4.1. *Let*

$$(4.3) \qquad M_{n,r,s} = \sum_{j=0}^{2r} \binom{2r}{j} (-1)^j \phi_n^{(j)} \begin{bmatrix} p_1, \cdots, p_n \\ q_1, \cdots, q_n \end{bmatrix} \qquad \textit{for fixed } r, s \geqq 1, \ s \neq 2r,$$
$$p_i, q_i \geqq 0, \ i=1, \cdots, n; \ \sum_{i=1}^{n} p_i = 1 = \sum_{i=1}^{n} q_i$$

and let

$$(4.4) \qquad M_{n,r,s} = 2 + \sum_{j=1}^{2r-1} \binom{2r}{j} (-1)^j \phi_n^{(j)} \begin{bmatrix} p_1, \cdots, p_n \\ q_1, \cdots, q_n \end{bmatrix} \qquad \textit{when } s=2r \ .$$

Let $\phi_n^{(j)} \begin{bmatrix} p_1, \cdots, p_n \\ q_1, \cdots, q_n \end{bmatrix}$ *satisfy the following postulates* R_1, R_2 *and* R_3 *where*

R_1: *Recursivity*

$$\phi_n^{(j)} \begin{bmatrix} p_1, \cdots, p_n \\ q_1, \cdots, q_n \end{bmatrix} = \phi_{n-1}^{(j)} \begin{bmatrix} p_1+p_2, p_3, \cdots, p_n \\ q_1+q_2, q_3, \cdots, q_n \end{bmatrix} + (p_1+p_2)^{j/s}(q_1+q_2)^{(2r-j)/s}$$
$$\cdot \left\{ \phi_2^{(j)} \begin{bmatrix} p_1/(p_1+p_2), p_2/(p_1+p_2) \\ q_1/(q_1+q_2), q_2/(q_1+q_2) \end{bmatrix} - 1 \right\}$$

for all $n \geqq 3$, $p_1+p_2>0$, $q_1+q_2>0$,
for fixed $s \geqq 1$, $s \neq 2r$ *or* $s=2r$ *and* $j \neq 0$, $j \neq 2r$.

R_2: *Symmetry*

$$\phi_3^{(j)} \textit{ is symmetric in } \begin{Bmatrix} p_i \\ q_i \end{Bmatrix} \qquad i=1, 2, 3 \ .$$

R_3: *Normalization*

$$\phi_2^{(j)}\begin{bmatrix}1/4, 3/4\\ 3/4, 1/4\end{bmatrix}=[3^{j/s}+3^{(2r-j)/s}]2^{-4r/s}-1 .$$

Then $M_{n,r,s}$ is uniquely determined as,

$$M_{n,r,s}=\sum_{i=1}^{n}[p_i^{1/s}-q_i^{1/s}]^{2r} . \tag{4.5}$$

The proof is similar to the proof for Theorem 2.1. By taking,

$$h(x, y)=\phi_2^{(j)}\begin{bmatrix}x, 1-x\\ y, 1-y\end{bmatrix} \tag{4.6}$$

it can be shown that

$$h(x, y)=x^{j/s}y^{(2r-j)/s}+(1-x)^{j/s}(1-y)^{(2r-j)/s}-1 . \tag{4.7}$$

Now by using postulate R_1 it can be easily shown that $M_{n,r,s}$ is uniquuely determined as in (4.5). It should be remarked that Theorem 4.1 is a generalization of Theorem 2.1 and Theorem 4.1 gives characterizations of a number of distance measures including Jeffreys' measure of invariance I_m when m is an even integer.

DEPARTMENT OF MATHEMATICS, MCGILL UNIVERSITY, MONTREAL, CANADA

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